Problem 1. Consider the linear system $Ax = 0$, where

$$A = \begin{pmatrix} 3 & 1 & -2 \\ 2 & 0 & -2 \\ 4 & 0 & -4 \end{pmatrix}.$$ 

a) Determine whether the homogeneous linear system $Ax = 0$ possesses a nontrivial solution and describe the solution-set in parametric vector form.

b) If the columns of $A$ are denoted by $v_1, v_2, v_3$, is the collection of vectors $\{v_1, v_2, v_3\}$ linearly independent in $\mathbb{R}^3$?

c) Is the collection $\{v_1, v_3\}$ linearly independent? Why?

Solution:
Row-reducing $A$ yields

$$A \sim \begin{pmatrix} 3 & -1 & 2 \\ 2 & 0 & -2 \\ 0 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which shows that $x_3$ is a free variable and

$$x_1 = x_3$$
$$x_2 = -x_3$$
$$x_3 = x_3.$$

In parametric form, solutions of the homogeneous system $Ax = 0$ can be expressed as

$$x = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

where $x_3$ can be any real number. Thus, there are infinitely many non-trivial solutions of the homogeneous system.

The columns of $A$ are linearly dependent, since $Ax = 0$ has non-trivial solutions.
Problem 2.

a) Is the set \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \} \), where
\[
\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ -5 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -5 \\ -16 \\ -8 \end{pmatrix}, \quad \text{and}
\]
\[
\mathbf{v}_4 = \begin{pmatrix} 2 \\ 5 \\ -19 \\ 0 \end{pmatrix}
\]
linearly independent in \( \mathbb{R}^4 \)? If not, provide a linear dependence relation.

b) Is the set \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) as in part (a) linearly independent? Why?

Solution: The matrix
\[
A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -5 & 5 \\ 2 & -5 & -16 & -19 \\ 4 & 2 & -8 & 0 \end{pmatrix},
\]
formed by the vectors \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \} \), is row-equivalent to the matrix
\[
\begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]
This is enough to show that the vectors \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \} \) are linearly dependent, since there is not a pivot in every column, hence there is a free variable. More particularly, the solutions of the homogeneous system are given by
\[
\begin{align*}
x_1 &= 3x_3 \\
x_2 &= -2x_3 \\
x_3 &= x_3 \\
x_4 &= 0.
\end{align*}
\]
A linear dependence relation can be giving as
\[
x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = 3x_3\mathbf{v}_1 - 2x_3\mathbf{v}_2 + x_3\mathbf{v}_3 + 0\mathbf{v}_4 = 0,
\]
or more simply, as
\[
3\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = 0.
\]
You can verify that \[ \mathbf{v}_3 = 2\mathbf{v}_2 - 3\mathbf{v}_1. \]

**Problem 3.** Suppose \( A \) is a \( 3 \times 3 \) matrix that is row-equivalent to

\[
U = \begin{bmatrix}
\text{■} & \text{★} & \text{★} \\
0 & \text{■} & \text{★} \\
0 & 0 & \text{■}
\end{bmatrix},
\]

where \( \text{■} \) is nonzero and \( \text{★} \) is real. Please answer and provide adequate justification for the following questions:

a) Does \( A\mathbf{x} = \mathbf{0} \) possess a nontrivial solution?

b) Are the columns of \( A \) linearly independent?

c) Do the columns of \( A \) span \( \mathbb{R}^3 \)?

**Solution:**
Since \( \text{■} \) is nonzero, each of the diagonal terms can be made 1. Then, the \( \text{★} \) entries can be made 0 by the appropriate elementary row operation. Thus, \( A \) is row-equivalent to the identity, which has linearly independent columns which span \( \mathbb{R}^3 \). Also, the matrix \( U \) has a pivot in every row and column, so its columns are linearly independent and span \( \mathbb{R}^3 \). Since the columns of \( A \) are linearly independent, \( A\mathbf{x} = \mathbf{0} \) has only the trivial solution.