The higher order Carathéodory–Julia theorem and related boundary interpolation problems

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Abstract. The higher order analogue of the classical Carathéodory-Julia theorem on boundary angular derivatives has been obtained in [7]. Here we study boundary interpolation problems for Schur class functions (analytic and bounded by one in the open unit disk) motivated by that result.

Mathematics Subject Classification (2000). 47A57, 47A20, 47A48.

Keywords. boundary interpolation, angular derivatives, unitary extensions, characteristic function of a unitary colligation.

1. Introduction

We denote by $\mathcal{S}$ the Schur class of analytic functions mapping the open unit disk $\mathbb{D}$ into its closure. A well known property of Schur functions w is that the kernel

$$K_w(z,\zeta) = \frac{1 - w(z)\overline{w(\zeta)}}{1 - z\bar{\zeta}} \tag{1.1}$$

is positive on $\mathbb{D} \times \mathbb{D}$ and therefore, that the matrix

$$\mathbf{P}_n^w(z) := \left[\frac{1}{i!j!}\,\frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j}\,\frac{1 - |w(z)|^2}{1 - |z|^2}\right]_{i,j=0}^n \tag{1.2}$$

which will be referred to as to a *Schwarz-Pick matrix*, is positive semidefinite for every $n \geq 0$ and $z \in \mathbb{D}$. We extend this notion to boundary points as follows: *given a point* $t_0 \in \mathbb{T}$*, the boundary Schwarz-Pick matrix is*

$$\mathbf{P}_n^w(t_0) = \lim_{z \to t_0} \mathbf{P}_n^w(z), \tag{1.3}$$

provided the limit in (1.3) *exists.* It is clear that once the boundary Schwarz-Pick matrix $\mathbf{P}_n^w(t_0)$ exists for $w \in \mathcal{S}$, it is positive semidefinite. In (1.3) and in what follows, all the limits are nontangential, i.e., $z \in \mathbb{D}$ tends to a boundary point nontangentially. Let us assume that $w \in \mathcal{S}$ possesses nontangential boundary limits

$$w_j(t_0) := \lim_{z\to t_0} \frac{w^{(j)}(z)}{j!} \quad \text{for } \ j = 0, \ldots, 2n+1 \tag{1.4}$$

and let

$$\mathbb{P}_n^w(t_0) := \begin{bmatrix} w_1(t_0) & \cdots & w_{n+1}(t_0) \\ \vdots & & \vdots \\ w_{n+1}(t_0) & \cdots & w_{2n+1}(t_0) \end{bmatrix} \boldsymbol{\Psi}_n(t_0) \begin{bmatrix} w_0(t_0)^* & \ldots & w_n(t_0)^* \\ & \ddots & \vdots \\ 0 & & w_0(t_0)^* \end{bmatrix}, \tag{1.5}$$

where the first factor is a Hankel matrix, the third factor is an upper triangular Toeplitz matrix and where $\boldsymbol{\Psi}_n(t_0) = [\Psi_{j\ell}]_{j,\ell=0}^n$ is the upper triangular matrix

$$\boldsymbol{\Psi}_n(t_0) = \begin{bmatrix} t_0 & -t_0^2 & t_0^3 & \cdots & (-1)^n \binom{n}{0} t_0^{n+1} \\ 0 & -t_0^3 & 2t_0^4 & \cdots & (-1)^n \binom{n}{1} t_0^{n+2} \\ \vdots & & t_0^5 & \cdots & (-1)^n \binom{n}{2} t_0^{n+3} \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & (-1)^n \binom{n}{n} t_0^{2n+1} \end{bmatrix}, \tag{1.6}$$

with entries

$$\Psi_{j\ell} = (-1)^\ell \binom{\ell}{j} t_0^{\ell+j+1}, \quad 0 \le j \le \ell \le n. \tag{1.7}$$

For notational convenience, in (1.5) and in what follows we use the symbol a^* for the complex conjugate of $a \in \mathbb{C}$.

We denote the lower diagonal entry in the Schwarz-Pick matrix $\mathbf{P}_n^w(z)$ by

$$d_{w,n}(z) := \frac{1}{(n!)^2} \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \frac{1-|w(z)|^2}{1-|z|^2}. \tag{1.8}$$

The following theorem was obtained in [7].

Theorem 1.1. *For $w \in \mathcal{S}$, $t_0 \in \mathbb{T}$ and $n \in \mathbb{Z}_+$, the following are equivalent:*

1. *The following limit inferior is finite*

$$\liminf_{z\to t_0} d_{w,n}(z) < \infty \tag{1.9}$$

   where $z \in \mathbb{D}$ approaches t_0 unrestrictedly.
2. *The following nontangential boundary limit exists and is finite:*

$$d_{w,n}(t_0) := \lim_{z\to t_0} d_{w,n}(z) < \infty. \tag{1.10}$$

3. *The boundary Schwarz-Pick matrix $\mathbf{P}_n^w(t_0)$ defined via the nontangential boundary limit* (1.3) *exists.*

4. *The nontangential boundary limits* (1.4) *exist and satisfy*
$$|w_0(t_0)| = 1 \quad \textit{and} \quad \mathbb{P}_n^w(t_0) \geq 0, \tag{1.11}$$
where $\mathbb{P}_n^w(t_0)$ *is the matrix defined in* (1.5).

Moreover, when these conditions hold, then
$$\mathbf{P}_n^w(t_0) = \mathbb{P}_n^w(t_0). \tag{1.12}$$

In case $n = 0$, Theorem 1.1 reduces to the classical Carathéodory-Julia theorem [8, 9]; this has been discussed in detail in [7]. The relation
$$d_{w,n}(t_0) = \begin{bmatrix} w_{n+1}(t_0) & \cdots & w_{2n+1}(t_0) \end{bmatrix} \mathbf{\Psi}_n(t_0) \begin{bmatrix} w_n(t_0)^* \\ \vdots \\ w_0(t_0)^* \end{bmatrix}$$
expresses equality of the lower diagonal entries in (1.12); upon separating the term corresponding to $i = n$ it can be written as
$$d_{w,n}(t_0) = \sum_{i=0}^{n-1} \sum_{j=0}^{n} w_{n+i+1}(t_0)\Psi_{ij}(t_0)w_{n-j}(t_0)^* + (-1)^n t_0^{2n+1} w_{2n+1}(t_0)w_0(t_0)^*. \tag{1.13}$$
Theorem 1.1 motivates the following interpolation problem:

Problem 1.2. *Given points* $t_1, \ldots, t_k \in \mathbb{T}$, *given integers* $n_1, \ldots, n_k \geq 0$ *and given numbers* $c_{i,j}$ $(j = 0, \ldots, 2n_i+1;\ i = 1, \ldots, k)$, *find all Schur functions* w *such that*
$$\liminf_{z \to t_i} d_{w,n_i}(z) < \infty \quad (i = 1, \ldots, k) \tag{1.14}$$
and
$$w_j(t_i) := \lim_{z \to t_i} \frac{w^{(j)}(z)}{j!} = c_{i,j} \quad (i = 1, \ldots, k;\ j = 0, \ldots, 2n_i + 1). \tag{1.15}$$

The problem makes sense since conditions (1.14) guarantee the existence of the nontangential limits (1.15); upon preassigning the values $w_j(t_i)$ for $i = 1, \ldots, k$ and $j = 0, \ldots, 2n_i+1$, we come up with interpolation Problem 1.2. It is convenient to reformulate Problem 1.2 in the following form:

Problem 1.3. *Given points* $t_1, \ldots, t_k \in \mathbb{T}$, *given integers* $n_1, \ldots, n_k \geq 0$ *and given numbers*
$$c_{i,j} \quad \textit{and} \quad \gamma_i \quad (j = 0, \ldots, 2n_i;\ i = 1, \ldots, k),$$
find all Schur functions w *such that*
$$d_{w,n_i}(t_i) := \frac{1}{(n_i!)^2} \lim_{z \to t_i} \frac{\partial^{2n_i}}{\partial z^{n_i} \partial \bar{z}^{n_i}} \frac{1 - |w(z)|^2}{1 - |z|^2} = \gamma_i \tag{1.16}$$
and
$$w_j(t_i) = c_{i,j} \quad (i = 1, \ldots, k;\ j = 0, \ldots, 2n_i). \tag{1.17}$$

If w is a solution to Problem 1.2, then conditions (1.14) guarantee the existence of the nontangential limits (1.16) and by a virtue of (1.13),

$$\begin{aligned} d_{w,n_i}(t_i) &= \sum_{\ell=0}^{n_i-1}\sum_{j=0}^{n_i} w_{n_i+\ell+1}(t_i)\Psi_{\ell j}(t_0)w_{n_i-j}(t_i)^* \\ &\quad +(-1)^{n_i}t_i^{2n_i+1}w_{2n_i+1}(t_i)w_0(t_i)^*. \end{aligned} \tag{1.18}$$

Thus, for every Schur function w, satisfying (1.14) and (1.15), conditions (1.16) hold with

$$\gamma_i = \sum_{\ell=0}^{n-1}\sum_{j=0}^{n_i} c_{i,n_i+\ell+1}\Psi_{\ell j}(t_0)c_{i,n_i-j}^* + (-1)^{n_i}t_i^{2n_i+1}c_{i,2n_i+1}c_{i,0}^*. \tag{1.19}$$

Conversely, if w is a solution of Problem 1.3, then it clearly satisfies (1.14) and by Theorem 1.1, all the limits in (1.15) exist and satisfy relation (1.18). Since $w_0(t_i)$ is *unimodular*, the equation (1.18) can be solved for $w_{2n_i+1}(t_i)$; on account of interpolation conditions (1.17), we have

$$w_{2n_i+1}(t_i) = (-1)^n\bar{t}_i^{\,2n_i+1}\left(d_{w,n_i}(t_i) - \sum_{\ell=0}^{n_i-1}\sum_{j=0}^{n_i} c_{i,n_i+\ell+1}\Psi_{\ell j}(t_i)c_{i,n_i-j}^*\right)c_{i,0}. \tag{1.20}$$

It is readily seen now that w is a solution of Problem 1.2 with the data $c_{i,2n_i+1}$ chosen by

$$c_{i,2n_i+1} = (-1)^n\bar{t}_i^{\,2n_i+1}\left(\gamma_i - \sum_{\ell=0}^{n_i-1}\sum_{j=0}^{n_i} c_{i,n_i+\ell+1}\Psi_{\ell j}(t_i)c_{i,n_i-j}^*\right)c_{i,0}. \tag{1.21}$$

It is known that boundary interpolation problems become more tractable if they involve inequalities. Such a relaxed problem is formulated below; besides of certain independent interest it will serve as an important intermediate step in solving Problem 1.2.

Problem 1.4. *Given points $t_1,\ldots,t_k \in \mathbb{T}$, given integers $n_1,\ldots,n_k \ge 0$ and given numbers $c_{i,j}$ and γ_i $(j = 0,\ldots,2n_i;\ i = 1,\ldots,k)$, find all Schur functions w such that*

$$d_{w,n_i}(t_i) \le \gamma_i, \tag{1.22}$$

$$w_j(t_i) = c_{i,j} \quad (i = 1,\ldots,k;\ j = 0,\ldots,2n_i). \tag{1.23}$$

By Theorem 1.1, for every solution w of Problem 1.3 there exists the limit $w_{2n_i+1}(t_i) := \lim_{z\to t_i}\frac{w^{(2n_i+1)}(z)}{(2n_i+1)!}$ which satisfies (1.20). Let $c_{i,2n_i+1}$ be defined as in (1.21). Then it follows from (1.20), (1.21) and (1.22) that

$$0 \le \gamma_i - d_{w,n_i}(t_i) = (-1)^{n_i}t_i^{2n_i+1}\left(c_{i,2n_i+1} - w_{2n_i+1}(t_i)\right)c_{i,0}^*, \tag{1.24}$$

It is convenient to reformulate Problem 1.4 in the following equivalent form.

Problem 1.5. *Given the data*

$$t_i \in \mathbb{T} \quad and \quad c_{i,j} \in \mathbb{C} \quad (j = 0, \ldots, 2n_i + 1;\ i = 1, \ldots, k), \tag{1.25}$$

find all Schur functions w *such that*

$$d_{w,n_i}(t_i) \leq \gamma_i\ , \tag{1.26}$$

$$w_j(t_i) = c_{i,j} \quad (i = 1, \ldots, k;\ j = 0, \ldots, 2n_i) \tag{1.27}$$

and

$$(-1)^{n_i} t_i^{2n_i+1} \left(c_{i,2n_i+1} - w_{2n_i+1}(t_i)\right) c_{i,0}^* \geq 0 \quad (i = 1, \ldots, k). \tag{1.28}$$

where γ_i*'s are defined by* (1.19).

In Section 3 we will construct the Pick matrix P in terms of the interpolation data (1.25) (see formulas (3.1)–(3.2) below). Then we will show that Problem 1.5 has a solution if and only if $|c_{i,0}| = 1$ for $i = 1, \ldots, k$ and $P \geq 0$. In case P is singular, Problem 1.5 has a unique solution w which is a finite Blaschke product of degree $r \leq \operatorname{rank} P$. This unique w may or may not be a solution of Problem 1.2. The case when P is positive definite is more interesting.

Theorem 1.6. *Let* $|c_{i,0}| = 1$ *for* $i = 1, \ldots, k$ *and* $P > 0$*. Then*

1. *Problem* 1.5 *has infinitely many solutions which are parametrized by the linear fractional transformation*

$$w(z) = s_0(z) + s_2(z)\left(1 - \mathcal{E}(z)s(z)\right)^{-1} \mathcal{E}(z)s_1(z) \tag{1.29}$$

   where $\mathcal{E}$ *is a free parameter running over the Schur class* $\mathcal{S}$ *and where the coefficient matrix*

$$\mathbf{S}(z) = \begin{bmatrix} s_0(z) & s_2(z) \\ s_1(z) & s(z) \end{bmatrix} \tag{1.30}$$

   is rational and inner in $\mathbb{D}$*.*

2. *A function* w *of the form* (1.29) *is a solution of Problem* 1.2 *if and only if either*

$$\liminf_{z \to t_i} \frac{1 - |\mathcal{E}(z)|^2}{1 - |z|^2} = \infty \quad or \quad \lim_{z \to t_i} \mathcal{E}(z) \neq s(t_i)^* \tag{1.31}$$

   for $i = 1, \ldots, k$*, where the latter limit is understood as nontangential, and* s *is the right bottom entry of the coefficient matrix* $\mathbf{S}(z)$*.*

Boundary interpolation problems for Schur class functions closely related to Problem 1.5 were studied previously in [3]–[6], [15]. Interpolation conditions (1.27) and (1.28) there were accompanied by various additional restrictions that in fact are equivalent to our conditions (1.26). Establishing these equivalences is a special issue which will be presented elsewhere. A version of Problem 1.2 (with certain assumptions on the data that guarantee (1.14) to be in force) was studied in [4] for rational matrix valued Schur functions. In this case, the parameters $\mathcal{E}$ in the parametrization formula (1.29) are also rational and therefore, the situation expressed by the first relation in (1.31) does not come into play. A similar matrix valued problem was considered in [6] where the solvability criteria was established rather than the description of all solutions. Problem 1.3 was considered in [20] in case $n_1 = \ldots = n_k = 0$; the second part in Theorem 1.6 can be considered as a higher order generalization of some results in [20].

The paper is organized as follows. In Section 2 we recall some needed results from [7] and present some consequences of conditions (1.26) holding for a Schur class function. In Section 3 we introduce the Pick matrix P in terms of the interpolation data and establish the Stein equality this matrix satisfies. In Section 4 we imbed Problem 1.5 in the general scheme of the Abstract Interpolation Problem (AIP) developed in [10, 13, 14]. In Section 5 we recall some needed results on AIP and then prove the first part of Theorem 1.6 in Section 6. Explicit formulas for the coefficients in the parametrization formula (1.29) are derived in Theorem 6.3. Explicit formula for the unique solution of Problem 1.5 in case P is singular is given in Theorem 6.2. In Section 6 we also prove certain properties of the coefficient matrix (1.30) which enable us to prove the second part of Theorem 1.6 in Section 7.

2. Preliminaries

The proof of Theorem 1.1 presented in [7] relies on the de Branges-Rovnyak spaces L^w and H^w associated to a Schur function w. In this section we recall some needed definitions and results. We use the standard notation L_2 for the Lebesgue space of square integrable functions on the unit circle $\mathbb{T}$; the symbols H_2^+ and H_2^- stand for the Hardy spaces of functions with vanishing negative (respectively, nonnegative) Fourier coefficients. The elements in H_2^+ and H_2^- will be identified with their unique analytic (resp., conjugate-analytic) continuations inside the unit disk and consequently H_2^+ and H_2^- will be identified with the Hardy spaces of the unit disk.

Let w be a Schur function. The nontangential boundary limits $w(t)$ exist and are bounded by one at a.e. $t \in \mathbb{T}$ and the matrix valued function $\left[\begin{smallmatrix} 1 & w(t) \\ w(t)^* & 1 \end{smallmatrix}\right]$ is defined and positive semidefinite almost everywhere on $\mathbb{T}$. The space L^w is the the range space $\left[\begin{smallmatrix} 1 & w \\ w^* & 1 \end{smallmatrix}\right]^{1/2}(L_2 \oplus L_2)$ endowed with the range norm. The set of functions $\left[\begin{smallmatrix} 1 & w \\ w^* & 1 \end{smallmatrix}\right] f$ where $f \in L_2 \oplus L_2$ is dense in L^w and

$$\left\| \begin{bmatrix} 1 & w \\ w^* & 1 \end{bmatrix} f \right\|_{L^w}^2 = \left\langle \begin{bmatrix} 1 & w \\ w^* & 1 \end{bmatrix} f, \ f \right\rangle_{L_2 \oplus L_2}. \tag{2.1}$$

Definition 2.1. A function $f = \left[\begin{smallmatrix} f_+ \\ f_- \end{smallmatrix}\right]$ is said to belong to the de Branges–Rovnyak space H^w if it belongs to L^w and if $f_+ \in H_2^+$ and $f_- \in H_2^-$.

As it was shown in [7], the vector valued functions

$$K_z^{(j)}(t) = \frac{1}{j!}\frac{\partial^j}{\partial \bar{z}^j}\left(\begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -w(z)^* \end{bmatrix} \cdot \frac{1}{1-t\bar{z}}\right) \tag{2.2}$$

defined for $z \in \mathbb{D}$, $t \in \mathbb{T}$ and $j \in \mathbb{Z}_+$, belong to the space H^w and furthermore, for every $z \in \mathbb{D}$ and every $f = \begin{bmatrix} f_+ \\ f_- \end{bmatrix} \in H^w$,

$$\left\langle f,\ K_z^{(j)}\right\rangle_{H^w} = \frac{1}{j!}\frac{\partial^j}{\partial z^j} f_+(z). \tag{2.3}$$

Setting $f = K_\zeta^{(i)}$ in (2.3), we get

$$\left\langle K_\zeta^{(i)},\ K_z^{(j)}\right\rangle_{H^w} = \frac{1}{j!i!}\frac{\partial^{j+i}}{\partial z^j \partial \bar{\zeta}^i}\left(\frac{1-w(z)\overline{w(\zeta)}}{1-z\bar{\zeta}}\right). \tag{2.4}$$

Upon differentiating in (2.2) and taking into account that $|t| = 1$, we come to the following explicit formulas for $K_z^{(j)}$:

$$K_z^{(j)}(t) = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix}\begin{bmatrix} t^j(1-t\bar{z})^{-j-1} \\ -\sum_{\ell=0}^{j} w_\ell(z)^* t^{j-\ell}(1-t\bar{z})^{\ell-j-1} \end{bmatrix}, \tag{2.5}$$

where $w_\ell(z)$ are the Taylor coefficients from the expansion

$$w(\zeta) = \sum_{\ell=0}^{\infty} w_\ell(z)(\zeta - z)^\ell, \quad w_\ell(z) = \frac{w^{(\ell)}(z)}{\ell!}.$$

The two next theorems (also proved in [7]) explain the role of condition (1.9).

Theorem 2.2. *Let $w \in \mathcal{S}$, $t_0 \in \mathbb{T}$, $n \in \mathbb{Z}_+$ and let*

$$\liminf_{z \to t_0} d_{w,n}(z) < \infty \tag{2.6}$$

Then the nontangential boundary limits

$$w_j(t_0) := \lim_{z \to t_0} \frac{w^{(j)}(z)}{j!} \quad \text{exist for } j = 0, \ldots, n \tag{2.7}$$

and the functions

$$K_{t_0}^{(j)}(t) = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix}\begin{bmatrix} t^j(1-t\bar{t}_0)^{-j-1} \\ -\sum_{\ell=0}^{j} w_\ell(t_0)^* t^{j-\ell}(1-t\bar{t}_0)^{\ell-j-1} \end{bmatrix} \tag{2.8}$$

belong to the space H^w for $j = 0, \ldots, n$. Moreover, the kernels $K_z^{(j)}$ defined in (2.5) converge to $K_{t_0}^{(j)}$ for $j = 1, \ldots, n$ in norm of H^w as $z \in \mathbb{D}$ approaches t_0 nontangentially:

$$K_z^{(j)} \xrightarrow{H^w} K_{t_0}^{(j)} \quad \text{for} \quad j = 1, \ldots, n \quad \text{as } (z \to t_0).$$

Theorem 2.3. *Let $w \in \mathcal{S}$, $t_0 \in \mathbb{T}$, $n \in \mathbb{Z}_+$. If the numbers $c_0, \dots, c_n$ are such that the function*

$$F(t) = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} t^n(1-t\bar{t}_0)^{-n-1} \\ -\sum_{\ell=0}^{n} c_\ell^* t^{n-\ell}(1-t\bar{t}_0)^{\ell-n-1} \end{bmatrix}$$

belongs to H^w, then condition (2.6) *holds, the limits* (2.7) *exist and $w_j(t_0) = c_j$ for $j = 0, \dots, n$; consequently, F coincides with $K_{t_0}^{(n)}$.*

Now the preceding analysis can be easily extended to a multi-point setting. Given a Schur function w and k-tuples $\mathbf{z} = (z_1, \dots, z_k)$ of points in $\mathbb{D}$ and $\mathbf{n} = (n_1, \dots, n_k)$ of nonnegative integers, define the *generalized Schwarz-Pick matrix*

$$\mathbf{P}_{\mathbf{n}}^w(\mathbf{z}) := \left[\left[\frac{1}{\ell! r!} \frac{\partial^{\ell+r}}{\partial z^\ell \partial \bar{\zeta}^r} \left(\frac{1 - w(z)\overline{w(\zeta)}}{1 - z\bar{\zeta}} \right) \Bigg|_{\substack{z = z_i, \\ \zeta = z_j}} \right]_{\substack{\ell = 0, \dots, n_i \\ r = 0, \dots, n_j}} \right]_{i,j=1}^{k}. \tag{2.9}$$

Given a tuple $\mathbf{t} = (t_1, \dots, t_k)$ of distinct points $t_i \in \mathbb{T}$, define the boundary generalized Schwarz-Pick matrix

$$\mathbf{P}_{\mathbf{n}}^w(\mathbf{t}) := \lim_{\mathbf{z} \to \mathbf{t}} \mathbf{P}_{\mathbf{n}}^w(\mathbf{z}) \tag{2.10}$$

provided the latter limit exists, where $\mathbf{z} \to \mathbf{t}$ means that $z_i \in \mathbb{D}$ approaches t_i for $i = 1, \dots, k$ nontangentially. It is readily seen that conditions

$$\liminf_{z \to t_i} d_{w,n_i}(z) < \infty \quad \text{for } \ i = 1, \dots, k, \tag{2.11}$$

(where d_{w,n_i} is defined via formula (1.8)) are necessary for the limit (2.10) to exist. They are also sufficient as the next theorem shows.

Theorem 2.4. *Let $\mathbf{t} = (t_1, \dots, t_k)$ be a tuple of distinct points $t_i \in \mathbb{T}$, let $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{Z}_+^k$ and let w be a Schur function satisfying conditions* (2.11). *Then*

1. *The following nontangential boundary limits exist:*

$$w_j(t_i) := \lim_{z \to t_i} \frac{w^{(j)}(z)}{j!} \quad (j = 0, \dots, 2n_i + 1; \ \ i = 1, \dots, k). \tag{2.12}$$

2. *The functions*

$$K_{t_i}^{(j)}(t) = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} t^j(1-t\bar{t}_i)^{-j-1} \\ -\sum_{\ell=0}^{j} w_\ell(t_i)^* t^{j-\ell}(1-t\bar{t}_i)^{\ell-j-1} \end{bmatrix} \tag{2.13}$$

belong to the space H^w for $j = 0, \dots, n_i$ and $i = 1, \dots, k$.

3. *The boundary generalized Schwarz-Pick matrix* $\mathbf{P}^w_{\mathbf{n}}(\mathbf{t})$ *defined via the nontangential limit* (2.10) *exists and is equal to the Gram matrix of the set* $\{K^{(j)}_{t_i}\,:\; j=0,\ldots,n_i;\; i=1,\ldots,k\}$*:*

$$\mathbf{P}^w_{\mathbf{n}}(\mathbf{t}) := \left[\left[\left\langle K^{(r)}_{t_j},\; K^{(\ell)}_{t_i}\right\rangle_{H^w}\right]_{\substack{\ell=0,\ldots,n_i\\ r=0,\ldots,n_j}}\right]_{i,j=1}^k. \tag{2.14}$$

4. *The matrix* $\mathbf{P}^w_{\mathbf{n}}(\mathbf{t})$ *can be expressed in terms of the nontangential limits* (2.12) *as follows:*

$$\mathbf{P}^w_{\mathbf{n}}(\mathbf{t}) = \left[\mathbf{P}^w_{ij}\right]_{i,j=1}^k \tag{2.15}$$

where $\mathbf{P}^w_{ij}$ *is the* $(n_i+1)\times(n_j+1)$ *matrix defined by*

$$\mathbf{P}^w_{ij} = \mathbf{H}_{ij}\boldsymbol{\Psi}_{n_j}(t_j)\mathbf{W}^*_j, \tag{2.16}$$

where $\boldsymbol{\Psi}_{n_j}(t_j)$ *is defined as in* (1.6), $\mathbf{W}_j$ *is the lower triangular Toeplitz matrix given by*

$$\mathbf{W}_j = \begin{bmatrix} w_0(t_j) & 0 & \ldots & 0 \\ w_1(t_j) & w_0(t_i) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ w_{n_j}(t_j) & \ldots & w_1(t_j) & w_0(t_j)\end{bmatrix} \tag{2.17}$$

and where $\mathbf{H}_{ij}$ *is the matrix with the entries*

$$\begin{aligned}[\mathbf{H}_{ij}]_{r,s} \;=\; & \sum_{\ell=0}^{r}(-1)^{r-\ell}\begin{pmatrix} s+r-\ell \\ s\end{pmatrix}\frac{w_\ell(t_i)}{(t_i-t_j)^{s+r-\ell+1}} \\ & -\sum_{\ell=0}^{s}(-1)^{r}\begin{pmatrix} s+r-\ell \\ r\end{pmatrix}\frac{w_\ell(t_j)}{(t_i-t_j)^{s+r-\ell+1}}\,. \end{aligned}\tag{2.18}$$

if $i\neq j$, *and it is the Hankel matrix*

$$\mathbf{H}_{jj} = \begin{bmatrix} w_1(t_j) & w_2(t_j) & \cdots & w_{n_j+1}(t_j) \\ w_2(t_j) & w_3(t_j) & \cdots & w_{n+2}(t_j) \\ \vdots & \vdots & & \vdots \\ w_{n_j+1}(t_j) & w_{n_j+2}(t_i) & \cdots & w_{2n_j+1}(t_j)\end{bmatrix} \tag{2.19}$$

otherwise.

Proof. The two first statements follow by Theorems 1.1 and 2.2. Due to relation (2.4), the matrix in (2.9) can be written as

$$\mathbf{P}^w_{\mathbf{n}}(\mathbf{z}) := \left[\left[\left\langle K^{(r)}_{z_j},\; K^{(\ell)}_{z_i}\right\rangle_{H^w}\right]_{\substack{\ell=0,\ldots,n_i\\ r=0,\ldots,n_j}}\right]_{i,j=1}^k. \tag{2.20}$$

By Statement 3 in Theorem 2.2,

$$K^{(j)}_{z_i} \overset{H^w}{\longrightarrow} K^{(j)}_{t_i} \quad \text{for} \quad j=1,\ldots,n_i;\;\; i=1,\ldots,k$$

as z_i approaches t_i nontangentially. Passing to the limit in (2.20) we get the existence of the boundary generalized Schwarz-Pick matrix $\mathbf{P}^w_{\mathbf{n}}(\mathbf{t})$ and obtain its representation (2.14). Let us consider the block partitioning

$$\mathbf{P}^w_{\mathbf{n}}(\mathbf{z}) = \left[\mathbf{P}^w_{ij}(z_i, z_j)\right]_{i,j=1}^k$$

conformal with that in (2.15) so that

$$\mathbf{P}^w_{ij}(z_i, z_j) := \left[\frac{1}{\ell! r!} \frac{\partial^{\ell+r}}{\partial z^\ell \partial \bar{\zeta}^r} \left(\frac{1 - w(z)\overline{w(\zeta)}}{1 - z\bar{\zeta}}\right)\Bigg|_{\substack{z = z_i,\\ \zeta = z_j}}\right]_{\substack{\ell = 0,\dots,n_i\\ r = 0,\dots,n_j}}. \tag{2.21}$$

The direct differentiation in (2.21) gives

$$\begin{aligned}\left[\mathbf{P}^w_{ij}(z_i, z_j)\right]_{\ell,r} &= \sum_{s=0}^{\min\{\ell,r\}} \frac{(\ell+r-s)!}{(\ell-s)!(r-s)!} \frac{z_i^{r-s}\bar{z}_j^{\ell-s}}{(1-z_i\bar{z}_j)^{\ell+r-s+1}} \\ &- \sum_{\alpha=0}^{\ell}\sum_{\beta=0}^{r}\sum_{s=0}^{\min\{\alpha,\beta\}} \frac{(\alpha+\beta-s)!}{(\alpha-s)!(\beta-s)!} \frac{z_i^{\beta-s}\bar{z}_j^{\alpha-s} w_{\ell-\alpha}(z_i) w_{r-\beta}(z_j)^*}{(1-z_i\bar{z}_j)^{\alpha+\beta-s+1}}.\end{aligned}$$

For $i \neq j$, we pass to the limit in the latter equality as $z_i \to t_i$ and $z_j \to t_j$ and take into account (2.12):

$$\begin{aligned}\left[\mathbf{P}^w_{ij}\right]_{\ell,r} &= \sum_{s=0}^{\min\{\ell,r\}} \frac{(\ell+r-s)!}{(\ell-s)!(r-s)!} \frac{t_i^{r-s}\bar{t}_j^{\ell-s}}{(1-t_i\bar{t}_j)^{\ell+r-s+1}} \\ &- \sum_{\alpha=0}^{\ell}\sum_{\beta=0}^{r}\sum_{s=0}^{\min\{\alpha,\beta\}} \frac{(\alpha+\beta-s)!}{(\alpha-s)!(\beta-s)!} \frac{t_i^{\beta-s}\bar{t}_j^{\alpha-s} w_{\ell-\alpha}(t_i) w_{r-\beta}(t_j)^*}{(1-t_i\bar{t}_j)^{\alpha+\beta-s+1}}.\end{aligned}$$

Verification of the fact that the product on the right hand side of (2.15) gives the matrix with the same entries, is straightforward and will be omitted. Finally, it is readily seen from (2.21) and (1.2) that the j-th diagonal block $\mathbf{P}^w_{jj}(z_j, z_j)$ coincides with the Schwarz-Pick matrix $\mathbf{P}^w_{n_j}(z_j)$. Therefore, by Theorem 1.1 and formula (1.5), its nontangential boundary limit equals

$$\begin{aligned}\mathbf{P}^w_{jj} &= \mathbb{P}^w_{n_j}(t_j) \qquad (2.22)\\ &= \begin{bmatrix} w_1(t_j) & \cdots & w_{n_j+1}(t_j) \\ \vdots & & \vdots \\ w_{n_j+1}(t_j) & \cdots & w_{2n_j+1}(t_j)\end{bmatrix} \mathbf{\Psi}_{n_j}(t_j) \begin{bmatrix} w_0(t_j)^* & \dots & w_{n_j}(t_j)^* \\ & \ddots & \vdots \\ 0 & & w_0(t_j)^*\end{bmatrix},\end{aligned}$$

which coincides with (2.16) for $j = i$. □

3. The Pick matrix and the Stein identity

The Pick matrix P defined and studied in this section is important for formulating a solvability criterion for Problem 1.5 and for parametrizing its solution set. The definition of the Pick matrix is motivated by the formulas for the matrix $\mathbf{P}_{\mathbf{n}}^{w}(\mathbf{t})$ discussed in the previous section. Namely,

$$P = [P_{ij}]_{i,j=1}^{k} \in \mathbb{C}^{N\times N} \quad \text{where} \quad N = \sum_{i=1}^{k}(n_i+1), \tag{3.1}$$

and the block entries $P_{ij} \in \mathbb{C}^{(n_i+1)\times(n_j+1)}$ are defined by

$$P_{ij} = H_{ij} \cdot \mathbf{\Psi}_{n_j}(t_j) \cdot W_j^*, \tag{3.2}$$

where $\mathbf{\Psi}_{n_j}(t_j)$ is defined as in (1.6), where

$$W_i = \begin{bmatrix} c_{i,0} & 0 & \dots & 0 \\ c_{i,1} & c_{i,0} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ c_{i,n_i} & \dots & c_{i,1} & c_{i,0} \end{bmatrix}, \tag{3.3}$$

$$H_{ii} = \begin{bmatrix} c_{i,1} & c_{i,2} & \cdots & c_{i,n_i+1} \\ c_{i,2} & c_{i,3} & \cdots & c_{i,n+2} \\ \vdots & \vdots & & \vdots \\ c_{i,n_i+1} & c_{i,n_i+2} & \cdots & c_{i,2n_i+1} \end{bmatrix} \tag{3.4}$$

for $i = 1, \dots, k$ and where the matrices H_{ij} (for $i \neq j$) are defined entrywise by

$$\begin{aligned} [H_{ij}]_{r,s} &= \sum_{\ell=0}^{r}(-1)^{r-\ell}\begin{pmatrix} s+r-\ell \\ s \end{pmatrix}\frac{c_{i,\ell}}{(t_i-t_j)^{s+r-\ell+1}} \\ &\quad -\sum_{\ell=0}^{s}(-1)^{r}\begin{pmatrix} s+r-\ell \\ r \end{pmatrix}\frac{c_{j,\ell}}{(t_i-t_j)^{s+r-\ell+1}}\,. \end{aligned} \tag{3.5}$$

for $r = 0, \dots, n_i$ and $s = 0, \dots, n_j$. The latter formulas define P exclusively in terms of the interpolation data of (1.25). We also associate with the same data the following matrices:

$$T = \begin{bmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_k \end{bmatrix}, \qquad \text{where} \quad T_i = \begin{bmatrix} \bar{t}_i & 1 & \dots & 0 \\ 0 & \bar{t}_i & & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \bar{t}_i \end{bmatrix}, \tag{3.6}$$

$$E = \begin{bmatrix} E_1 & \dots & E_k \end{bmatrix}, \qquad \text{where} \quad E_i = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}, \tag{3.7}$$

$$M = \begin{bmatrix} M_1 & \dots & M_k \end{bmatrix}, \qquad \text{where} \quad M_i = \begin{bmatrix} c_{i,0}^* & \dots & c_{i,n_i}^* \end{bmatrix}. \tag{3.8}$$

Note that $T_i \in \mathbb{C}^{(n_i+1)\times(n_i+1)}$ and E_i, $M_i \in \mathbb{C}^{1\times(n_i+1)}$. The main result of this section is:

Theorem 3.1. *Let* $|c_{i,0}| = 1$ *for* $i = 1, \dots, k$ *and let us assume that the diagonal blocks* P_{ii} *of the matrix* P *defined in* (3.1)–(3.5) *are Hermitian for* $i = 1, \dots, k$.

Then the matrix P is Hermitian and satisfies the Stein identity

$$P - T^*PT = E^*E - M^*M, \tag{3.9}$$

where the matrices T, E and M are defined in (3.6)–(3.8).

In view of (3.6)–(3.8), verifying (3.9) is equivalent to verifying

$$P_{ij} - T_i^*P_{ij}T_j = E_i^*E_j - M_i^*M_j \quad (i,j = 1,\dots,k). \tag{3.10}$$

The identity like that is not totally surprising due to a special (Hankel and Toeplitz) structure of the factors H_{ij} and W_j in (3.2). Indeed, the identity verified in the next lemma (though, not exactly of the form (3.10)) follows from the structure of P_{ij} only (without any symmetry assumptions). Note that the righthand side in (3.9) as well as the one in (3.10) is of rank 2 .

Lemma 3.2. *Let P_{ij} be defined as in* (3.2). *Then*

$$P_{ij} - T_i^*P_{ij}T_j = E_i^*\overline{M}_j\boldsymbol{\Psi}_{n_j}(t_j)W_j^*T_j - M_i^*M_j \tag{3.11}$$

where according to (3.8),

$$\overline{M}_j = \begin{bmatrix} c_{j,0} & c_{j,1} & \dots & c_{j,n_j} \end{bmatrix} = E_jW_j^\top.$$

Proof. We shall make use of the following equalities

$$W_j^*T_j = T_jW_j^*, \quad \overline{T}_j\boldsymbol{\Psi}_{n_j}(t_j)T_j = \boldsymbol{\Psi}_{n_j}(t_j), \quad E_j\boldsymbol{\Psi}_{n_j}(t_j)T_j = E_j. \tag{3.12}$$

The first equality follows by the Toeplitz triangular structure of W_j^* and T_j. The matrix $\overline{T}_j\boldsymbol{\Psi}_{n_j}(t_j)T_j$ is upper triangular as the product of upper triangular matrices and due to (1.7) and (3.6), its $s\ell$-th entry (for $\ell \geq s$) equals

$$\begin{aligned}
&\left[\overline{T}_j\boldsymbol{\Psi}_{n_j}(t_j)T_j\right]_{s,\ell} = \Psi_{s,\ell} + t_j\Psi_{s,\ell-1} + \bar{t}_j\Psi_{s+1,\ell} + \Psi_{s+1,\ell-1}\\
&= (-1)^\ell t_j^{s+\ell+1}\left[\begin{pmatrix} \ell \\ s \end{pmatrix} - \begin{pmatrix} \ell-1 \\ s \end{pmatrix} + \begin{pmatrix} \ell \\ s+1 \end{pmatrix} - \begin{pmatrix} \ell-1 \\ s+1 \end{pmatrix}\right]\\
&= (-1)^\ell t_j^{s+\ell+1}\begin{pmatrix} \ell \\ s \end{pmatrix} = \Psi_{s,\ell}.
\end{aligned}$$

This completes the verification of the second equality in (3.12). The last relation in (3.12) follows by (1.7) and (3.6) and (3.7):

$$E_j\boldsymbol{\Psi}_{n_j}(t_j)T_j = \begin{bmatrix} t_j & -t_j^2 & \dots & (-1)^{n_j}t_j^{n_j+1} \end{bmatrix} T_j = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} = E_j.$$

We will also use the identity

$$H_{ij}\overline{T}_j - T_i^*H_{ij} = E_i^*\overline{M}_j - M_i^*E_j \tag{3.13}$$

which holds for every $i, j = 1, \dots, k$ and is verified by straightforward calculations (separately for the cases $i = j$ and $i \neq j$). We have

$$\begin{aligned} P_{ij} - T_i^* P_{ij} T_j &= H_{ij} \boldsymbol{\Psi}_{n_j}(t_j) W_i^* - T_i^* H_{ij} \boldsymbol{\Psi}_{n_j}(t_j) T_j W_j^* \\ &= \left(H_{ij} \overline{T}_j - T_j^* H_{ij} \right) \boldsymbol{\Psi}_{n_j}(t_j) T_j W_j^* \\ &= \left(E_i^* \overline{M}_j - M_i^* E_j \right) \boldsymbol{\Psi}_{n_j}(t_j) T_j W_j^*, \end{aligned} \tag{3.14}$$

where the first equality follows by (3.2) and the first relation in (3.12), the second equality relies on the second relation in (3.12) and the last equality is a consequence of (3.13). Combining the third relation in (3.12) with formulas (3.3) and (3.8) we get

$$E_j \boldsymbol{\Psi}_{n_i}(t_i) T_j W_j^* = E_j W_j^* = M_j$$

which being substituted into (3.14) leads us to (3.11). □

Proof of Theorem 3.1**:** By Lemma 3.2 the structure of P implies (3.11). First we consider the case when $j = i$. Since, by assumption, matrices P_{ii} ($i = 1, \dots, k$) are Hermitian, the left hand sides in (3.11) are Hermitian, and hence the right hand sides in (3.11) must be Hermitian. In other words

$$E_i^* \overline{M}_i \boldsymbol{\Psi}_{n_i}(t_i) W_i^* T_i = (\overline{M}_i \boldsymbol{\Psi}_{n_i}(t_i) W_i^* T_i)^* E_i \quad \text{for } i = 1, \dots, k.$$

Multiplying the latter relation by E_i from the left and taking into account that

$$E_i E_i^* = 1 \quad \text{and} \quad E_i (\overline{M}_i \boldsymbol{\Psi}_{n_i}(t_i) W_i^* T_i)^* = (c_{i,0} t_i c_{i,0}^* \bar{t}_i)^* = 1,$$

we get

$$\overline{M}_i \boldsymbol{\Psi}_{n_i}(t_i) W_i^* T_i = E_i \quad \text{for } i = 1, \dots, k.$$

Therefore, relations (3.11) turn into (3.10), which is equivalent to (3.9). Furthermore, for $i \neq j$, the Stein equation

$$X - T_i^* X T_j = E_i^* E_j - M_i^* M_j$$

has a unique solution X. Taking adjoint of both sides in (3.10) we conclude that the matrix P_{ij}^* satisfies the same Stein equation as P_{ji} does and then, by the above uniqueness, $P_{ij}^* = P_{ji}$ for $i \neq j$. It follows now that P is Hermitian. □

Theorem 3.3. *Let $t_1, \dots, t_k \in \mathbb{T}$, $n_1, \dots, n_k \in \mathbb{Z}_+$, $N = \sum_{i=1}^k (n_1 + 1)$ and let us assume that a Schur function w satisfies conditions* (2.11)*. Then the matrix $\mathbf{P}_{\mathbf{n}}^w(\mathbf{t})$ defined via the limit* (2.10) *(that exists by Theorem* 3.1*) satisfies the Stein identity*

$$\mathbf{P}_{\mathbf{n}}^w(\mathbf{t}) - T^* \mathbf{P}_{\mathbf{n}}^w(\mathbf{t}) T = E^* E - (M^w)^* M^w, \tag{3.15}$$

where the matrices T and E are defined in (3.6), (3.7) *and*

$$M^w = \begin{bmatrix} M_1 & \dots & M_k \end{bmatrix}, \quad \textit{where} \quad M_i^w = \begin{bmatrix} w_0(t_i)^* & \dots & w_{n_i}(t_i)^* \end{bmatrix}. \tag{3.16}$$

Proof. By Theorem 2.2, the matrix $\mathbf{P}^w_{\mathbf{n}}(\mathbf{t})$ admits the representation (2.15)-(2.18), that has the same structure as the Pick matrix P constructed in (3.1)–(3.5) but with parameters c_{ij} replaced by $w_j(t_i)$. Furthermore, it is positive semidefinite (and therefore, its diagonal blocks are Hermitian) due to representation (2.14), whereas $|w_0(t_i)| = 1$ for $i = 1,\dots,k$, by Theorem 1.1. Upon applying Theorem 3.1 we conclude that $\mathbf{P}^w_{\mathbf{n}}(\mathbf{t})$ satisfies the same Stein identity as P but with M^w instead of M, i.e., the Stein identity (3.15). □

4. Reformulation of Problem 1.5

The formula (2.14) for $\mathbf{P}^w_{\mathbf{n}}(\mathbf{t})$ motivates us to introduce the matrix function

$$\widetilde{\mathbf{F}}^w(t) = \begin{bmatrix} \widetilde{\mathbf{F}}^w_1(t) & \dots & \widetilde{\mathbf{F}}^w_k(t) \end{bmatrix}, \tag{4.1}$$

where

$$\widetilde{\mathbf{F}}^w_i(t) := \begin{bmatrix} K^{(0)}_{t_i}(t) & K^{(1)}_{t_i}(t) & \dots & K^{(n_i)}_{t_i}(t) \end{bmatrix} \quad (i = 1,\dots,k), \tag{4.2}$$

and $K^{(j)}_{t_i}(t)$ $(j = 0,\dots,n_i)$ are the functions defined in (2.13).

Theorem 4.1. *Let $t_1,\dots,t_k \in \mathbb{T}$, $n_1,\dots,n_k \in \mathbb{Z}_+$ and let us assume that a Schur function w satisfies conditions* (2.11)*. Then for $\widetilde{\mathbf{F}}^w$ defined in* (4.2), (4.3) *we have*

1. *The function $\widetilde{\mathbf{F}}^w x$ belongs to the de Branges-Rovnyak space H^w for every vector $x \in \mathbb{C}^N$ and*
$$\|\widetilde{\mathbf{F}}^w x\|^2_{H^w} = x^*\mathbf{P}^w_{\mathbf{n}}(\mathbf{t})x \tag{4.3}$$
where $\mathbf{P}^w_{\mathbf{n}}(\mathbf{t})$ is the boundary generalized Schwarz-Pick matrix (that exists due to conditions (2.11)*) and $N := \sum_{i=1}^k (n_i + 1)$.*
2. *$\widetilde{\mathbf{F}}^w$ admits the representation*
$$\widetilde{\mathbf{F}}^w(t) = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} E \\ -M^w \end{bmatrix} (\mathrm{I} - tT)^{-1}, \tag{4.4}$$

where the matrices T, E and M^w are defined in (3.6), (3.7) *and* (3.16)*, respectively.*

Proof. By Theorem 1.1, conditions (1.26) guarantee that the functions $K^{(j)}_{t_i}$ defined in (2.13) belong to H^w and the boundary Schwarz-Pick matrix $\mathbf{P}^w_{\mathbf{n}}(\mathbf{t})$ exists and admits a representation (2.14). Now it follows from (4.1) and (4.2) that for every $x \in \mathbb{C}^n$, the function $\mathbf{F}^w x$ belongs to H^w as a linear combination of the kernels $K^{(j)}_{t_i} \in H^w$, while relation (4.3) is an immediate consequence of (2.14). Furthermore, by definitions (3.6), (3.7) and (3.16) of T_i, E_i and M^w_i,

$$\begin{bmatrix} E_i \\ -M^w_i \end{bmatrix} (\mathrm{I} - tT_i)^{-1} = \begin{bmatrix} \dfrac{1}{1-t\bar{t}_i} & \dots & \dfrac{t^{n_i}}{(1-t\bar{t}_i)^{n_i+1}} \\ -\dfrac{w_0(t_i)^*}{1-t\bar{t}_i} & \dots & -\displaystyle\sum_{\ell=0}^{n_i} \dfrac{w_\ell(t_i)^* t^{n_i-\ell}}{(1-t\bar{t}_i)^{n_i+1-\ell}} \end{bmatrix}. \tag{4.5}$$

Multiplying both sides of (4.5) by the matrix $\begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix}$ on the left and taking into account (2.13) and (4.2) we get

$$\begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} E \\ -M^w \end{bmatrix} (\mathrm{I} - tT)^{-1} = \begin{bmatrix} K_{t_i}^{(0)}(t) & K_{t_i}^{(1)}(t) & \dots & K_{t_i}^{(n_i)}(t) \end{bmatrix}$$
$$=: \widetilde{\mathbf{F}}_i^w(t) \qquad (i = 1, \dots, k). \tag{4.6}$$

Now representation formula (4.4) follows by definitions (block partitionings) (4.1), (3.6), (3.7) and (3.16) of $\widetilde{\mathbf{F}}^w$, T, E and M^w. □

Now we modify $\widetilde{\mathbf{F}}^w$ replacing M^w by M in (4.4): we introduce the function

$$\mathbf{F}^w(t) := \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} E \\ -M \end{bmatrix} (\mathrm{I} - tT)^{-1} \tag{4.7}$$

with T, E and M defined in (3.6)–(3.8). The two next theorems show that Problem 1.5 can be reformulated in terms of this function and of the Pick matrix P.

Theorem 4.2. *Assume that w solves Problem* 1.5 *(i.e., $w \in \mathcal{S}$ and satisfies interpolation conditions* (1.26)*–*(1.28)*) and let $\mathbf{F}^w$ be defined as in* (4.7)*. Then*

1. *The function $\mathbf{F}^w x$ belongs to H^w for every vector $x \in \mathbb{C}^N$ and*
$$\|\mathbf{F}^w x\|_{H^w}^2 \le x^* P x \tag{4.8}$$
where P is the Pick matrix defined in (3.1)*–*(3.5)*.*
2. *The numbers $c_{i,0}$ are unimodular for $i = 1, \dots, k$ and the matrix P is positive semidefinite*
$$|c_{i,0}| = 1 \ \ (i = 1, \dots, k) \quad \text{and} \quad P \ge 0. \tag{4.9}$$
3. *P satisfies the Stein identity* (3.9)*.*

Furthermore, if w is a solution of Problem 1.2*, then*

$$\|\mathbf{F}^w x\|_{H^w}^2 = x^* P x \quad \text{for every } x \in \mathbb{C}^N. \tag{4.10}$$

Proof. Conditions (1.26) guarantee (by Theorem 1.1) that the limits $w_0(t_i)$ are unimodular for $i = 1, \dots, k$; since $w_0(t_i) = c_{i,0}$ (according to (1.27)), the first condition in (4.9) follows.

Conditions (1.26) also guarantee (by Theorem 4.1), that for every $x \in \mathbb{C}^N$, the function $\widetilde{\mathbf{F}}^w x$ belongs to H^w for every vector $x \in \mathbb{C}^N$ and equality (4.3) holds, where $\widetilde{\mathbf{F}}^w$ is defined by the representation formula (4.4). On account of interpolation conditions (1.27) (only for $j = 0, \dots, n_i$ and for every $i = 1, \dots, k$) and by definitions (3.8) and (3.16), it follows that $M = M^w$. Then the formulas (4.4) and (4.7) show that $\mathbf{F}^w \equiv \widetilde{\mathbf{F}}^w$, so that equality (4.3) holds with $\mathbf{F}^w$ instead of $\widetilde{\mathbf{F}}^w$:

$$\|\mathbf{F}^w x\|_{H^w}^2 = x^* \mathbf{P}_{\mathbf{n}}^w(\mathbf{t}) x. \tag{4.11}$$

Thus, to prove (4.8), it suffices to show that $\mathbf{P}_{\mathbf{n}}^w(\mathbf{t}) \le P$. We will use formulas (2.15)–(2.19) defining $\mathbf{P}_{\mathbf{n}}^w(\mathbf{t})$ in terms of the boundary limits $w_j(t_i)$. In view of these formulas and due to interpolation conditions (1.27), $\mathbf{P}^w_{\mathbf{n}}(\mathbf{t})$ can be expressed in terms of the interpolation data (1.25). Indeed, comparing (3.2)–(3.5) and (2.15)–(2.18) we conclude that

$$\mathbf{P}^w_{ij} = P_{ij} \quad (i \neq j) \tag{4.12}$$

and that formula (2.22) for the diagonal blocks of $\mathbf{P}^w$ turns into

$$\mathbf{P}^w_{ii} = \begin{bmatrix} c_{i,1} & \cdots & c_{i,n_i+1} \\ c_{i,2} & \cdots & c_{i,n_i+2} \\ \vdots & & \vdots \\ c_{i,n_i+1} & \cdots & w_{2n_i+1}(t_i) \end{bmatrix} \mathbf{\Psi}_{n_i}(t_i) \begin{bmatrix} c^*_{i,0} & c^*_{i,1} & \cdots & c^*_{i,n_i} \\ 0 & c^*_{i,0} & \cdots & c^*_{i,n_i-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c^*_{i,0} \end{bmatrix}. \tag{4.13}$$

Taking into account the upper triangular structure of $\mathbf{\Psi}_{n_i}(t_i)$, we conclude from (3.2), (3.3) and (4.13) that all the corresponding entries in P_{ii} and $\mathbf{P}^w_{ii}$ are equal except for the rightmost bottom entries that are equal to γ_i and to $d_{w,n_i}(t_i)$, respectively. Thus, by condition (1.26),

$$P_{ii} - \mathbf{P}^w_{ii} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \gamma_i - d_{w,n_i}(t_i) \end{bmatrix} \geq 0, \tag{4.14}$$

for $i = 1, \dots, k$ which together with (4.12) imply $P \geq \mathbf{P}^w$ and therefore, relation (4.8). If w is a solution of Problem 1.2 (or equivalently, of Problem 1.3), then $\gamma_i - d_{w,n_i}(t_i) = 0$ for $i = 1, \dots, k$ in (4.14) which proves the final statement in the theorem. Since $\mathbf{P}^w \geq 0$, we conclude from the inequality $P \geq \mathbf{P}^w$ that $P \geq 0$ which completes the proof of the second statement of the theorem. The third statement follows from (4.9) by Theorem 3.1. □

The next theorem is the converse to Theorem 4.2.

Theorem 4.3. *Let P, T, E and M be the matrices given by* (3.1)–(3.8). *Let* $|c_{i,0}| = 1$ *and* $P \geq 0$. *Let w be a Schur function such that*

$$\mathbf{F}^w x := \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} E \\ -M \end{bmatrix} (\mathrm{I} - tT)^{-1} x \quad \text{belongs to } H^w \tag{4.15}$$

for every $x \in \mathbb{C}^N$ and satisfies (4.8). *Then w is a solution of Problem* 1.5. *If moreover,* (4.10) *holds, then w is a solution of Problem* 1.2.

Proof. By the definitions (3.6)–(3.8) of T, E and M, the columns of the $2 \times N$ matrix $\mathbf{F}^w$ defined in (4.7), are of the form

$$\begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} t^j(1 - t\bar{t}_i)^{-j-1} \\ -\sum_{\ell=0}^{j} c^*_{i,\ell} t^{j-\ell}(1 - t\bar{t}_i)^{\ell-j-1} \end{bmatrix}$$

for $j = 1, \dots, n_i$ and $i = 1, \dots, k$, and all of them belong to H^w by the assumption (4.15) of the theorem. In particular, the functions

$$F_i(t) = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} t^{n_i}(1-t\bar{t}_i)^{-n_i-1} \\ -\sum_{\ell=0}^{n_i} c_{i,\ell}^* t^{n_i-\ell}(1-t\bar{t}_i)^{\ell-n_i-1} \end{bmatrix}$$

belong to H^w, which implies, by Theorems 3.2 and (2.4) that

$$\liminf_{z\to t_i} d_{w,n_i}(z) < \infty \quad \text{for } \ i = 1, \dots, k, \tag{4.16}$$

and that the nontangential limits (2.12) exist and satisfy

$$w_j(t_i) = c_{ij} \quad \text{for} \quad j = 1, \dots, n_i \quad \text{and} \quad i = 1, \dots, k. \tag{4.17}$$

Therefore, w meets conditions (1.27) for $i = 1, \dots, k$ and $\ell_i = 0, \dots, n_i$. By Theorem 4.1, conditions (4.16) guarantee that the boundary generalized Schwarz-Pick matrix $\mathbf{P}_{\mathbf{n}}^w(\mathbf{t})$ exists and that

$$\|\widetilde{\mathbf{F}}^w x\|_{H^w}^2 = x^*\mathbf{P}_{\mathbf{n}}^w(\mathbf{t})x \quad \text{for every } \ x \in \mathbb{C}^N, \tag{4.18}$$

where $\widetilde{\mathbf{F}}^w$ is the $2 \times N$ matrix function defined in (4.4). By Theorem 2.4, $\mathbf{P}_{\mathbf{n}}^w(\mathbf{t})$ is represented in terms of the boundary limits (2.12) by formulas (2.15)–(2.18). Equalities (4.17) along with definitions (3.8) and (3.16) of M and M^w show that the two latter matrices are equal and thus $\mathbf{F}^w \equiv \widetilde{\mathbf{F}}^w$, by (4.4) and (4.7). Now combining (4.18) and (4.15) gives $\mathbf{P}_{\mathbf{n}}^w(\mathbf{t}) \le P$ which implies inequalities for the diagonal blocks

$$\mathbf{P}_{ii}^w \le P_{ii} \quad (i = 1, \dots, k). \tag{4.19}$$

Since $d_{w,n_i}(t_i)$ and γ_i are (the lower) diagonal entries in $\mathbf{P}_{ii}^w$ and P_{ii} respectively, the latter inequality implies (1.28).

By Theorems 3.1 and 3.3, the matrices P and $\mathbf{P}_{\mathbf{n}}^w(\mathbf{t})$ possess the Stein identities (3.9) and (3.15), respectively; since $M = M^w$, the matrix $\widetilde{P} := P - \mathbf{P}_{\mathbf{n}}^w(\mathbf{t})$ satisfies the homogeneous Stein identity

$$\widetilde{P} - T^*\widetilde{P}T = 0.$$

By the diagonal structure (3.6) of T and in view of (4.19) we have for the diagonal blocks $\widetilde{P}_{ii}$ of $\widetilde{P}$,

$$\widetilde{P}_{ii} - T_i^*\widetilde{P}_{ii}T_i = 0 \quad \text{and} \quad \widetilde{P}_{ii} \ge 0 \quad (i = 1, \dots, k). \tag{4.20}$$

By the Jordan structure (3.6) of T_i, it follows from (4.20) that $\widetilde{P}_{ii}$ is necessarily of the form

$$\widetilde{P}_{ii} = P_{ii} - \mathbf{P}_{ii}^w = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \delta_i \end{bmatrix} \quad \text{with} \quad \delta_i \ge 0 \tag{4.21}$$

(for a simple proof see e.g., [6, Corollary 10.7]). On the other hand, by the representations (2.16) and (3.2),

$$\mathbf{P}_{ii}^w = \mathbf{H}_{ii}\boldsymbol{\Psi}_{n_i}(t_i)\mathbf{W}_i^* \quad \text{and} \quad P_{ii} = H_{ii}\boldsymbol{\Psi}_{n_i}(t_i)W_i^*$$

and since by (4.17), $\mathbf{W}_i = W_i$ (which is readily seen from the definitions (2.17) and (3.3)), we conclude that

$$P_{ii} - \mathbf{P}_{ii}^w = (H_{ii} - \mathbf{H}_{ii})\, \mathbf{\Psi}_{n_i}(t_i) W_i^*.$$

Combining the last equality with (4.22) gives

$$H_{ii} - \mathbf{H}_{ii} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \delta_i \end{bmatrix} \left(\mathbf{\Psi}_{n_i}(t_i) W_i^*\right)^{-1}. \tag{4.22}$$

Since $|c_{i,0}| = 1$, it is seen from definitions (1.6) and (3.3) that the matrix $\mathbf{\Psi}_{n_i}(t_i)W_i^*$ is upper triangular and invertible and that its lower diagonal entry equals

$$g_i := (-1)^{n_i} t_i^{2n_i+1} c_{i,0}^*. \tag{4.23}$$

Therefore, the inverse matrix $\left(\mathbf{\Psi}_{n_i}(t_i)W_i^*\right)^{-1}$ is upper triangular with the lower diagonal entry equal g_i^{-1} so that the matrix on the right hand side in (4.22) has all the entries equal to zero except the lower diagonal entry which is equal to $\delta_i g_i^{-1}$. Taking into account the definitions (2.19) and (3.4) we write (4.22) more explicitly as

$$\left[c_{i,j+k+1} - w_{j+k+1}(t_i)\right]_{j,k=0}^{n_i} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \delta_i g_i^{-1} \end{bmatrix}.$$

Upon equating the corresponding entries in the latter equality we arrive at

$$w_j(t_i) = c_{i,j} \quad (j = 1, \dots, 2n_i)$$

and

$$c_{i,2n_i+1} - w_{2n_i+1}(t_i) = \delta_i g_i^{-1}.$$

The first line (together with (4.17)) proves (1.27). The second one can be written as

$$\left(c_{i,2n_i+1} - w_{2n_i+1}(t_i)\right) g_i = \delta_i \geq 0,$$

which implies (1.28), due to (4.23). In the case when equality (4.10) holds, we get from (4.21) that $\delta_i = 0$ for $i = 1, \dots, k$ and, therefore, that w is a solution of Problem 1.3 (or equivalently, of Problem 1.2). □

We recall now briefly the setting of the Abstract Interpolation Problem **AIP** (in a generality we need) for the Schur class $\mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ of functions analytic on $\mathbb{D}$ whose values are contractive operators mapping a Hilbert space $\mathcal{E}$ into another Hilbert space $\mathcal{E}_*$. The data of the problem consists of Hilbert spaces $\mathcal{E}$, $\mathcal{E}_*$ and X, a positive semidefinite linear operator P on X, an operator T on X such that the operator $(I - zT)$ has a bounded inverse at every point $z \in \overline{\mathbb{D}}$ except for a finitely many points, and two linear operators $M : X \to \mathcal{E}$ and $E : X \to \mathcal{E}_*$ satisfying the identity

$$P - T^*PT = E^*E - M^*M. \tag{4.24}$$

Definition 4.4. A function $w \in \mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ is said to be a solution of the **AIP** with the data

$$\{P, T, E, M\} \tag{4.25}$$

subject to above assumptions, if the function

$$(\mathbf{F}^w x)(t) := \begin{bmatrix} \mathrm{I}_{\mathcal{E}_*} & w(t) \\ w(t)^* & \mathrm{I}_{\mathcal{E}} \end{bmatrix} \begin{bmatrix} E \\ -M \end{bmatrix} (\mathrm{I} - tT)^{-1} x. \tag{4.26}$$

belongs to the space H^w and

$$\|\mathbf{F}^w x\|_{H^w} \le \|P^{\frac{1}{2}} x\|_X \quad \text{for every} \quad x \in X.$$

The main conclusion of this section is that Problem 1.5 can be included into the **AIP** upon specifying the data in (4.24) in terms of the data (1.25) of Problem 1.5. Let $X = \mathbb{C}^N$ and $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$ and let us identify the matrices P, T, E and M defined in (3.1)–(3.8) with operators acting between the corresponding finite dimensional spaces. For T of the form (3.6), the operator $(\mathrm{I} - tT)^{-1}$ is well defined on X for all $t \in \mathbb{T} \setminus \{t_1, \ldots, t_k\}$. Also we note that when $X = \mathbb{C}^N$,

$$\|P^{\frac{1}{2}} x\|_X^2 = x^* P x.$$

Now Theorems 4.2 and 4.3 lead us to the following result.

Theorem 4.5. *Let the matrices P, T, E and M be given by* (3.1)–(3.8) *and let conditions* (4.9) *be satisfied. Then a Schur function w is a solution of Problem* 1.5 *if and only if it is a solution of the* **AIP** *with the data* (4.25).

Corollary 4.6. *Conditions $P \ge 0$ and $|c_{i,0}| = 1$ for $i = 1, \ldots, k$ are necessary and sufficient for Problem* 1.5 *to have a solution.*

Proof: Necessity of the conditions was proved in Theorem 4.2. Sufficiency follows from Theorem 4.5 and from a general result [10] stating that **AIP** always has a solution.

5. On the Abstract Interpolation Problem (AIP)

In this section we recall some results on the **AIP** formulated in Definition 4.4. Then in the next section we will specify these results for the setting of Problem 1.5, when $X = \mathbb{C}^N$, $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$ and operators T, E, M and $P \ge 0$ are just matrices defined in terms of the data of Problem 1.5 via formulas (3.1)–(3.8). In this section they are assumed to be operators satisfying the Stein identity (4.24) for every $x \in X$. This identity means that the formula

$$\mathbf{V} : \begin{bmatrix} P^{\frac{1}{2}} x \\ Mx \end{bmatrix} \to \begin{bmatrix} P^{\frac{1}{2}} T x \\ Ex \end{bmatrix}, \quad x \in X \tag{5.1}$$

defines a linear map that can be extended by continuity to an isometry $\mathbf{V}$ acting from

$$\mathcal{D}_{\mathbf{V}} = \text{Clos}\left\{ \begin{bmatrix} P^{\frac{1}{2}}x \\ Mx \end{bmatrix}, \ x \in X \right\} \subseteq [X] \oplus \mathcal{E} \tag{5.2}$$

onto

$$\mathcal{R}_{\mathbf{V}} = \text{Clos}\left\{ \begin{bmatrix} P^{\frac{1}{2}}Tx \\ Ex \end{bmatrix}, \ x \in X \right\} \subseteq [X] \oplus \mathcal{E}_*, \tag{5.3}$$

where $[X] = \text{Clos}\{P^{\frac{1}{2}}X\}$. One of the main results concerning the **AIP** is the characterization of the set of all solutions in terms of minimal unitary extensions of $\mathbf{V}$: let $\mathcal{H}$ be a Hilbert spaces containing $[X]$ and let

$$\mathbf{U}: \ \mathcal{H} \oplus \mathcal{E} \to \mathcal{H} \oplus \mathcal{E}_* \quad (\mathcal{H} \supset X) \tag{5.4}$$

be a unitary operator such that $\mathbf{U}|_{\mathcal{D}_{\mathbf{V}}} = \mathbf{V}$ and having no nonzero reducing subspaces in $\mathcal{H} \ominus [X]$. Then the *characteristic function* of $\mathbf{U}$ defined as

$$w(z) = \mathbf{P}_{\mathcal{E}_*} \mathbf{U} \left(I - z\mathbf{P}_{\mathcal{H}}\mathbf{U}\right)^{-1}|_{\mathcal{E}} \quad (z \in \mathbb{D}), \tag{5.5}$$

is a solution of the **AIP** and all the solutions to the **AIP** can be obtained in this way.

A parametrization of all the solutions can be obtained as follows: introduce the defect spaces

$$\Delta := \begin{bmatrix} [X] \\ \mathcal{E} \end{bmatrix} \ominus \mathcal{D}_{\mathbf{V}} \quad \text{and} \quad \Delta_* := \begin{bmatrix} [X] \\ \mathcal{E}_* \end{bmatrix} \ominus \mathcal{R}_{\mathbf{V}} \tag{5.6}$$

and let $\widetilde{\Delta}$ and $\widetilde{\Delta}_*$ be isomorphic copies of Δ and Δ_*, respectively, with unitary identification maps

$$i: \ \Delta \to \widetilde{\Delta} \quad \text{and} \quad i_*: \ \Delta_* \to \widetilde{\Delta}_*.$$

Define a unitary operator $\mathbf{U}_0$ from $\mathcal{D}_{\mathbf{V}} \oplus \Delta \oplus \widetilde{\Delta}_*$ onto $\mathcal{R}_{\mathbf{V}} \oplus \Delta_* \oplus \widetilde{\Delta}$ by the rule

$$\mathbf{U}_0|_{\mathcal{D}_{\mathbf{V}}} = \mathbf{V}, \quad \mathbf{U}_0|_{\Delta} = i, \quad \mathbf{U}_0|_{\widetilde{\Delta}_*} = i_*^{-1}. \tag{5.7}$$

This operator is called *the universal unitary colligation* associated to the Stein identity (4.24). Since $\mathcal{D}_{\mathbf{V}} \oplus \Delta = [X] \oplus \mathcal{E}$ and $\mathcal{R}_{\mathbf{V}} \oplus \Delta_* = [X] \oplus \mathcal{E}_*$, we can decompose $\mathbf{U}_0$ defined by (5.7) as follows

$$\mathbf{U}_0 = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & 0 \end{bmatrix}: \quad \begin{bmatrix} [X] \\ \mathcal{E} \\ \widetilde{\Delta}_* \end{bmatrix} \to \begin{bmatrix} [X] \\ \mathcal{E}_* \\ \widetilde{\Delta} \end{bmatrix}. \tag{5.8}$$

Note that $U_{33} = 0$, since (by definition (5.7)), for every $\widetilde{\delta}_* \in \widetilde{\Delta}_*$, the vector $\mathbf{U}_0\delta_*$ belongs to Δ_*, which is a subspace of $[X] \oplus \mathcal{E}_*$ and therefore, is orthogonal to $\widetilde{\Delta}$. The *characteristic function* of $\mathbf{U}_0$ is defined as follows:

$$\mathbf{S}(z) = \mathbf{P}_{\mathcal{E}_* \oplus \widetilde{\Delta}} \mathbf{U}_0 \left(I - z\mathbf{P}_{[X]}\mathbf{U}_0\right)^{-1}|_{\mathcal{E} \oplus \widetilde{\Delta}_*} \quad (z \in \mathbb{D}), \tag{5.9}$$

where $\mathbf{P}_{\mathcal{E}_*\oplus\widetilde{\Delta}}$ and $\mathbf{P}_{[X]}$ are the orthogonal projections of the space $[X]\oplus\mathcal{E}_*\oplus\widetilde{\Delta}$ onto $\mathcal{E}_*\oplus\widetilde{\Delta}$ and $[X]$, respectively. Upon substituting (5.8) into (5.9) we get a representation of the function $\mathbf{S}$ in terms of the block entries of $\mathbf{U}_0$:

$$\begin{aligned}\mathbf{S}(z) &= \begin{bmatrix} s_0(z) & s_2(z) \\ s_1(z) & s(z)\end{bmatrix} \\ &= \begin{bmatrix} U_{22} & U_{23} \\ U_{32} & 0\end{bmatrix} + z\begin{bmatrix} U_{21} \\ U_{31}\end{bmatrix}(I_n - zU_{11})^{-1}\begin{bmatrix} U_{12} & U_{13}\end{bmatrix}.\end{aligned} \tag{5.10}$$

The next theorem was proved in [10].

Theorem 5.1. *Let $\mathbf{S}$ be the characteristic function of the universal unitary colligation partitioned as in* (5.10). *Then all the solutions w of the* **AIP** *are parametrized by the formula*

$$w(z) = s_0(z) + s_2(z)\left(1-\mathcal{E}(z)s(z)\right)^{-1}\mathcal{E}(z)s_1(z), \tag{5.11}$$

where $\mathcal{E}$ runs over the Schur class $\mathcal{S}(\widetilde{\Delta},\,\widetilde{\Delta}_*)$.

Since $\mathbf{S}$ is the characteristic function of a unitary colligation, it belongs to the Schur class $\mathcal{S}(\mathcal{E}\oplus\widetilde{\Delta}_*,\ \mathcal{E}_*\oplus\widetilde{\Delta})$ (see [17], [1], [2]) and therefore, one can introduce the corresponding de Branges–Rovnyak space $H^{\mathbf{S}}$ as it was explained in Section 2. The next result about realization of a unitary colligation in a function model space goes back to M. Livsits, B. Sz.-Nagy, C. Foias, L. de Branges and J. Rovnyak. In its present formulation it appears in [10]–[14].

Theorem 5.2. *Let $\mathbf{U_0}$ be a unitary colligation of the form* (5.8) *and let $\mathbf{S}$ be its characteristic function defined in* (5.9). *Then the transformation $\mathcal{F}_{\mathbf{U_0}}$ defined as*

$$\left(\mathcal{F}_{\mathbf{U_0}}[x]\right)(z) = \begin{bmatrix}\left(\mathcal{F}^+_{\mathbf{U_0}}[x]\right)(z) \\ \left(\mathcal{F}^-_{\mathbf{U_0}}[x]\right)(z)\end{bmatrix} := \begin{bmatrix}\mathbf{P}_{\mathcal{E}_*\oplus\widetilde{\Delta}}\mathbf{U}_0\left(I - z\mathbf{P}_{[X]}\mathbf{U}_0\right)^{-1}[x] \\ \bar{z}\mathbf{P}_{\mathcal{E}\oplus\widetilde{\Delta}_*}\mathbf{U}_0^*\left(I-\bar{z}\mathbf{P}_{[X]}\mathbf{U}_0^*\right)^{-1}[x]\end{bmatrix} \tag{5.12}$$

maps $[X]$ onto the de Branges–Rovnyak space $H^{\mathbf{S}}$ and is a partial isometry.

The transformation $\mathcal{F}_{\mathbf{U_0}}$ is called *the Fourier representation* of the space $[X]$ associated with the unitary colligation $\mathbf{U_0}$. Note that the last theorem does not assume any special structure for $\mathbf{U}_0$. However, if $\mathbf{U_0}$ is the universal unitary colligation (5.7) associated to the partially defined isometry $\mathbf{V}$ given in (5.1), then $\mathcal{F}_{\mathbf{U_0}}$ can be expressed in terms of P, T, E and M. The formulation of the following theorem can be found (in a more general setting) in [11], [14]; the proof is contained in [12]. We reproduce it here since the source is hardly available.

Theorem 5.3. *Let $\mathbf{U_0}$ be the universal unitary colligation* (5.7) *associated to the isometry $\mathbf{V}$ given by* (5.1) *and let $\mathbf{S}$ be its characteristic function given by* (5.9). *Then*

$$\left(\mathcal{F}_{\mathbf{U_0}}P^{\frac{1}{2}}x\right)(t) = \begin{bmatrix} I_{\mathcal{E}_*\oplus\widetilde{\Delta}} & \mathbf{S}(t) \\ \mathbf{S}(t)^* & I_{\mathcal{E}\oplus\widetilde{\Delta}_*}\end{bmatrix}\begin{bmatrix} E(I-tT)^{-1} \\ 0 \\ -M(I-tT)^{-1} \\ 0\end{bmatrix}x \tag{5.13}$$

for almost every point $t\in\mathbb{T}$ and for every $x\in X$.

Proof. We will verify (5.13) for "plus" and "minus" components separately, i.e., we will verify the relations

$$\begin{aligned}\left(\mathcal{F}^{+}_{\mathbf{U_0}}P^{\frac{1}{2}}x\right)(t) &= \left(\begin{bmatrix} E \\ 0\end{bmatrix} - \mathbf{S}(t)\begin{bmatrix} M \\ 0\end{bmatrix}\right)(I-tT)^{-1}x,\\ \left(\mathcal{F}^{-}_{\mathbf{U_0}}P^{\frac{1}{2}}x\right)(t) &= \left(\mathbf{S}(t)^*\begin{bmatrix} E \\ 0\end{bmatrix} - \begin{bmatrix} M \\ 0\end{bmatrix}\right)(I-tT)^{-1}x,\end{aligned}$$

which are equivalent (upon analytic and conjugate-analytic continuations inside $\mathbb{D}$, respectively) to

$$\left(\mathcal{F}^{+}_{\mathbf{U_0}}P^{\frac{1}{2}}x\right)(z) = \left(\begin{bmatrix} E \\ 0\end{bmatrix} - \mathbf{S}(z)\begin{bmatrix} M \\ 0\end{bmatrix}\right)(I-zT)^{-1}x, \tag{5.14}$$

$$\left(\mathcal{F}^{-}_{\mathbf{U_0}}P^{\frac{1}{2}}x\right)(z) = \bar{z}\left(\mathbf{S}(z)^*\begin{bmatrix} E \\ 0\end{bmatrix} - \begin{bmatrix} M \\ 0\end{bmatrix}\right)(\bar{z}I-T)^{-1}x. \tag{5.15}$$

To prove (5.14), we pick an arbitrary vector

$$v=\begin{bmatrix} y \\ e \\ \delta_*\end{bmatrix}\in\begin{bmatrix} [X] \\ \mathcal{E} \\ \widetilde{\Delta}_*\end{bmatrix}$$

and note that by definitions (5.9) and (5.12),

$$\mathbf{P}_{\mathcal{E}_*\oplus\widetilde{\Delta}}\mathbf{U}_0\left(I-z\mathbf{P}_{[X]}\mathbf{U}_0\right)^{-1}\begin{bmatrix} y \\ e \\ \delta_*\end{bmatrix} = \left(\mathcal{F}^{+}_{\mathbf{U_0}}y\right)(z)+\mathbf{S}(z)\begin{bmatrix} e \\ \delta_*\end{bmatrix}. \tag{5.16}$$

Introduce the vector

$$v'=\begin{bmatrix} y' \\ e' \\ \delta'_*\end{bmatrix} := \left(I-z\mathbf{P}_{[X]}\mathbf{U}_0\right)^{-1}\begin{bmatrix} y \\ e \\ \delta_*\end{bmatrix}$$

so that $\left(I-z\mathbf{P}_{[X]}\mathbf{U}_0\right)v'=v$. Comparing the corresponding components in the latter equality we conclude that $e=e'$, $\delta_*=\delta'_*$ and

$$y=y'-z\mathbf{P}_{[X]}\mathbf{U}_0\begin{bmatrix} y' \\ e' \\ \delta'_*\end{bmatrix}=y'-z\mathbf{P}_{[X]}\mathbf{U}_0\begin{bmatrix} y' \\ e \\ \delta_*\end{bmatrix}, \tag{5.17}$$

so that

$$v'=\begin{bmatrix} y' \\ e \\ \delta_*\end{bmatrix}=\left(I-z\mathbf{P}_{[X]}\mathbf{U}_0\right)^{-1}\begin{bmatrix} y \\ e \\ \delta_*\end{bmatrix}. \tag{5.18}$$

Substituting (5.17) and (5.18) respectively into the right and the left hand side expressions in (5.16) we arrive at

$$\mathbf{P}_{\mathcal{E}_*\oplus\widetilde{\Delta}}\mathbf{U}_0\begin{bmatrix} y' \\ e \\ \delta_* \end{bmatrix} = \left(\mathcal{F}^+_{\mathbf{U_0}}y'\right)(z) - z\left(\mathcal{F}^+_{\mathbf{U_0}}\mathbf{P}_{[X]}\mathbf{U}_0\begin{bmatrix} y' \\ e \\ \delta_* \end{bmatrix}\right)(z) + \mathbf{S}(z)\begin{bmatrix} e \\ \delta_* \end{bmatrix}. \tag{5.19}$$

Since the vector v is arbitrary and $I - z\mathbf{P}_{[X]}\mathbf{U}_0$ is invertible, it follows by (5.17), that v' can be chosen arbitrarily in (5.19) . Fix a vector $x \in X$ and take

$$v' = \begin{bmatrix} y' \\ e \\ \delta_* \end{bmatrix} = \begin{bmatrix} P^{\frac{1}{2}}x \\ Mx \\ 0 \end{bmatrix}. \tag{5.20}$$

Then, by definition (5.7) of $\mathbf{U}_0$ and definition (5.1) of $\mathbf{V}$,

$$\mathbf{U_0}\begin{bmatrix} P^{\frac{1}{2}}x \\ Mx \\ 0 \end{bmatrix} = \begin{bmatrix} P^{\frac{1}{2}}Tx \\ Ex \\ 0 \end{bmatrix} \tag{5.21}$$

and thus,

$$\mathbf{P}_{[X]}\mathbf{U}_0\begin{bmatrix} P^{\frac{1}{2}}x \\ Mx \\ 0 \end{bmatrix} = P^{\frac{1}{2}}Tx \quad \text{and} \quad \mathbf{P}_{\mathcal{E}_*\oplus\widetilde{\Delta}}\mathbf{U}_0\begin{bmatrix} P^{\frac{1}{2}}x \\ Mx \\ 0 \end{bmatrix} = \begin{bmatrix} Ex \\ 0 \end{bmatrix}.$$

Plugging the two last relations and (5.20) into (5.19) we get

$$\begin{bmatrix} Ex \\ 0 \end{bmatrix} = \left(\mathcal{F}^+_{\mathbf{U_0}}P^{\frac{1}{2}}x\right)(z) - z\left(\mathcal{F}^+_{\mathbf{U_0}}P^{\frac{1}{2}}Tx\right)(z) + \mathbf{S}(z)\begin{bmatrix} Mx \\ 0 \end{bmatrix}.$$

By linearity of $\mathcal{F}^+_{\mathbf{U_0}}$, we have

$$\begin{bmatrix} Ex \\ 0 \end{bmatrix} = \left(\mathcal{F}^+_{\mathbf{U_0}}P^{\frac{1}{2}}(I - zT)x\right)(z) + \mathbf{S}(z)\begin{bmatrix} Mx \\ 0 \end{bmatrix}$$

and, upon replacing x by $(I - zT)x$, we rewrite the last relation as

$$\begin{bmatrix} E \\ 0 \end{bmatrix}(I - zT)^{-1}x = \left(\mathcal{F}^+_{\mathbf{U_0}}P^{\frac{1}{2}}x\right)(z) + \mathbf{S}(z)\begin{bmatrix} M \\ 0 \end{bmatrix}(I - zT)^{-1}x,$$

which is equivalent to (5.14). The proof of (5.15) is quite similar: we start with an arbitrary vector

$$v = \begin{bmatrix} y \\ e_* \\ \delta \end{bmatrix} \in \begin{bmatrix} [X] \\ \mathcal{E}_* \\ \widetilde{\Delta} \end{bmatrix}$$

and note that by definitions (5.9) and (5.12),

$$\bar{z}\mathbf{P}_{\mathcal{E}\oplus\widetilde{\Delta}_*}\mathbf{U}_0^*\left(I - \bar{z}\mathbf{P}_{[X]}\mathbf{U}_0^*\right)^{-1}\begin{bmatrix} y \\ e_* \\ \delta \end{bmatrix} = \left(\mathcal{F}^-_{\mathbf{U_0}}y\right)(z) + \bar{z}\mathbf{S}(z)^*\begin{bmatrix} e_* \\ \delta \end{bmatrix}. \tag{5.22}$$

Then we introduce the vector

$$v' := \begin{bmatrix} y' \\ e'_* \\ \delta' \end{bmatrix} = \left(I - \bar{z}\mathbf{P}_{[X]}\mathbf{U}_0^*\right)^{-1} \begin{bmatrix} y \\ e_* \\ \delta \end{bmatrix} \tag{5.23}$$

and check that

$$e'_* = e_*, \quad \delta' = \delta, \quad y = y' - \bar{z}\mathbf{P}_{[X]}\mathbf{U}_0^* \begin{bmatrix} y' \\ e_* \\ \delta \end{bmatrix}, \tag{5.24}$$

which allows us to rewrite (5.22) as

$$\bar{z}\mathbf{P}_{\mathcal{E}\oplus\widetilde{\Delta}_*}\mathbf{U}_0^* \begin{bmatrix} y' \\ e_* \\ \delta \end{bmatrix} = \left(\mathcal{F}^-_{\mathbf{U_0}} y'\right)(z) - \bar{z}\left(\mathcal{F}^-_{\mathbf{U_0}}\mathbf{P}_{[X]}\mathbf{U}_0^* \begin{bmatrix} y' \\ e_* \\ \delta \end{bmatrix}\right)(z) + \bar{z}\mathbf{S}(z)^* \begin{bmatrix} e_* \\ \delta \end{bmatrix}. \tag{5.25}$$

By the same arguments as above, v' can be chosen arbitrarily in $[X] \oplus \mathcal{E}_* \oplus \widetilde{\Delta}$ and we let

$$v' = \begin{bmatrix} y' \\ e \\ \delta_* \end{bmatrix} = \begin{bmatrix} P^{\frac{1}{2}}Tx \\ Ex \\ 0 \end{bmatrix}, \quad x \in X. \tag{5.26}$$

Since $\mathbf{U_0}$ is unitary, it follows from (5.21) that

$$\mathbf{U_0}^* \begin{bmatrix} P^{\frac{1}{2}}Tx \\ Ex \\ 0 \end{bmatrix} = \begin{bmatrix} P^{\frac{1}{2}}x \\ Mx \\ 0 \end{bmatrix}$$

and thus,

$$\mathbf{P}_{[X]}\mathbf{U}_0^* \begin{bmatrix} P^{\frac{1}{2}}Tx \\ Ex \\ 0 \end{bmatrix} = P^{\frac{1}{2}}x \quad \text{and} \quad \mathbf{P}_{\mathcal{E}\oplus\widetilde{\Delta}_*}\mathbf{U}_0^* \begin{bmatrix} P^{\frac{1}{2}}Tx \\ Ex \\ 0 \end{bmatrix} = \begin{bmatrix} Mx \\ 0 \end{bmatrix}.$$

Plugging the two last relations and (5.26) into (5.25) we get

$$\bar{z}\begin{bmatrix} Mx \\ 0 \end{bmatrix} = \left(\mathcal{F}^-_{\mathbf{U_0}} P^{\frac{1}{2}}Tx\right)(z) - \bar{z}\left(\mathcal{F}^-_{\mathbf{U_0}} P^{\frac{1}{2}}x\right)(z) + \bar{z}\mathbf{S}(z)^* \begin{bmatrix} Ex \\ 0 \end{bmatrix}.$$

By linearity of $\mathcal{F}^-_{\mathbf{U_0}}$, we have

$$\bar{z}\begin{bmatrix} Mx \\ 0 \end{bmatrix} = \left(\mathcal{F}^-_{\mathbf{U_0}} P^{\frac{1}{2}}(T - \bar{z}I)x\right)(z) + \bar{z}\mathbf{S}(z)^* \begin{bmatrix} Ex \\ 0 \end{bmatrix}$$

and, upon replacing x by $(\bar{z}I - T)^{-1}x$, we rewrite the latter relation as

$$\bar{z}\begin{bmatrix} M \\ 0 \end{bmatrix}(\bar{z}I - T)^{-1} = -\left(\mathcal{F}^-_{\mathbf{U_0}} P^{\frac{1}{2}}x\right)(z) + \bar{z}\mathbf{S}(z)^* \begin{bmatrix} E \\ 0 \end{bmatrix}(\bar{z}I - T)^{-1},$$

which is equivalent to (5.15). □

6. Description of all solutions of Problem 1.5

Since Problem 1.5 is equivalent to the **AIP** with a specific choice of the data (4.25), Theorem 5.1 gives, in fact, a parametrization of all solutions of Problem 1.5. However, the fact that in the context of Problem 1.5, $X = \mathbb{C}^N$ and $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$, and that the matrices P, T, E and M are of special structure (3.1)-(3.8), allow us to rewrite the results from the previous section more transparently. We assume that the necessary conditions (4.9) for Problem 1.5 to have a solution are in force. Then P satisfies the Stein identity

$$P + M^*M = T^*PT + E^*E \tag{6.1}$$

(by Theorem 3.1) which in turn, gives raise to the isometry

$$\mathbf{V}: \begin{bmatrix} P^{\frac{1}{2}}x \\ Mx \end{bmatrix} \to \begin{bmatrix} P^{\frac{1}{2}}Tx \\ Ex \end{bmatrix}, \quad x \in \mathbb{C}^N$$

that maps

$$\mathcal{D}_{\mathbf{V}} = \text{Ran} \begin{bmatrix} P^{\frac{1}{2}} \\ M \end{bmatrix} \subseteq \begin{bmatrix} [X] \\ \mathcal{E} \end{bmatrix} \quad \text{onto} \quad \mathcal{R}_{\mathbf{V}} = \text{Ran} \begin{bmatrix} P^{\frac{1}{2}}T \\ E \end{bmatrix} \subseteq \begin{bmatrix} [X] \\ \mathcal{E}_* \end{bmatrix},$$

where $[X] = \text{Ran}\, P^{\frac{1}{2}}$. In the present context, the defect spaces (5.6)

$$\Delta = \begin{bmatrix} [X] \\ \mathcal{E} \end{bmatrix} \ominus \mathcal{D}_{\mathbf{V}} \quad \text{and} \quad \Delta_* = \begin{bmatrix} [X] \\ \mathcal{E}_* \end{bmatrix} \ominus \mathcal{R}_{\mathbf{V}}$$

admit a simple characterization.

Lemma 6.1. *If P is nonsingular, then*

$$\Delta = \text{Span} \begin{bmatrix} -P^{-\frac{1}{2}}M^* \\ 1 \end{bmatrix} \quad \text{and} \quad \Delta_* = \text{Span} \begin{bmatrix} -P^{-\frac{1}{2}}(T^{-1})^*E^* \\ 1 \end{bmatrix}. \tag{6.2}$$

If P is singular then $\Delta = \{0\}$ and $\Delta_* = \{0\}$.

Proof. A vector $\begin{bmatrix} [x] \\ e \end{bmatrix} \in \begin{bmatrix} [X] \\ \mathcal{E} \end{bmatrix}$ belongs to Δ if and only if

$$< [x],\, P^{\frac{1}{2}}y > + < e,\, My >= 0$$

for every $y \in X$, which is equivalent to

$$P^{\frac{1}{2}}[x] + M^*e = 0. \tag{6.3}$$

Equation (6.3) has a nonzero solution $\begin{bmatrix} [x] \\ e \end{bmatrix}$ if and only if the vector-column M^* belongs to $[X]$. If P is nonsingular, then $[X] = X$, therefore $M^* \in [X]$, and (6.3) implies the first relation in (6.2). The second relation is proved quite similarly.

Let now P be singular. Then $M^* \notin [X]$. Indeed assuming that $M^* \in \text{Ran}\, P^{\frac{1}{2}}$ we get that $Mx = 0$ for every $x \in \text{Ker}\, P$, which implies, in view of (6.1), that $Tx \in \text{Ker}\, P$ and $Ex = 0$ for every $x \in \text{Ker}\, P$. In particular, $\text{Ker}\, P$ is T-invariant and therefore, at least one eigenvector x_0 of T belongs to $\text{Ker}\, P$, and this vector must satisfy $Ex_0 = 0$. However, by definitions (3.6, (3.7) $Ex_0 \neq 0$ for every eigenvector x_0 of T. The contradiction means that $M^* \notin [X]$ and, therefore, equation (6.3) has only zero solution, i.e. $\Delta = \{0\}$ in case when P is singular. The result concerning Δ_* is established in much the same way. □

Theorem 6.2. *If P is singular, then Problem* 1.5 *has a unique solution*

$$w_0(z) = E\left(\widetilde{P} - zPT\right)^{-1} M^*, \tag{6.4}$$

(which is a finite Blaschke product of degree equal to rank P*), where*

$$\widetilde{P} := P + M^*M = T^*PT + E^*E. \tag{6.5}$$

The inverse in (6.4) *is well defined as an operator on* $\widetilde{X} = \operatorname{Ran}\widetilde{P}$.

Proof. By Lemma 6.1, if P is singular then $\mathcal{D}_{\mathbf{V}} = [X] \oplus \mathcal{E}$ and $\mathcal{R}_{\mathbf{V}} = [X] \oplus \mathcal{E}_*$ where $[X] = \operatorname{Ran} P^{\frac{1}{2}}$. Therefore, the isometry $\mathbf{V}$ defined by (5.1), is already a unitary operator from $[X] \oplus \mathcal{E}$ onto $[X] \oplus \mathcal{E}_*$. Therefore, the solution is unique and is given by the formula (5.5) with $\mathbf{V}$ and $[X]$ in place of $\mathbf{U}$ and $\mathcal{H}$, respectively:

$$w_0(z) = \mathbf{P}_{\mathcal{E}_*}\mathbf{V}\left(I - z\mathbf{P}_{[X]}\mathbf{V}\right)^{-1}|_{\mathcal{E}} \quad (z \in \mathbb{D}). \tag{6.6}$$

Since $\dim[X] < \infty$, it follows that w_0 is a finite Blaschke product of degree equal to $\dim[X] = \operatorname{rank} P$ (see, e.g., [18]). It remains to derive the realization formula (6.4) from (6.6).

Note that by definition of $\widetilde{X}$, it is $\widetilde{P}$-invariant. Since $\widetilde{P}$ is Hermitian, it is invertible on its range $\widetilde{X}$. In what follows, the symbol $\widetilde{P}^{-1}$ will be understood as an operator on $\widetilde{X}$. We define the mappings

$$A = \begin{bmatrix} P^{\frac{1}{2}} \\ M \end{bmatrix} : \widetilde{X} \to [X] \oplus \mathcal{E} \quad \text{and} \quad B = \begin{bmatrix} P^{\frac{1}{2}}T \\ E \end{bmatrix} : \widetilde{X} \to [X] \oplus \mathcal{E}_*.$$

Since

$$A^*A = B^*B = \widetilde{P} \tag{6.7}$$

and since $\widetilde{P}$ is invertible on $\widetilde{X}$, both A and B are nonsingular on $\widetilde{X}$. Since P is singular, it follows (by the proof of Lemma 6.1) that $M^* \notin [X]$ and thus $\dim \widetilde{X} = \dim[X] + 1$. Therefore, A is a bijection from $\widetilde{X}$ onto $[X] \oplus \mathcal{E}$ and B is a bijection from $\widetilde{X}$ onto $[X] \oplus \mathcal{E}_*$. Using (6.7), one can also write the formulas for the inverses

$$A^{-1} = \widetilde{P}^{-1}A^* : [X] \oplus \mathcal{E} \to \widetilde{X}, \quad B^{-1} = \widetilde{P}^{-1}B^* : [X] \oplus \mathcal{E}_* \to \widetilde{X}.$$

By definition (5.1), $\mathbf{V}A = B$, which can be rephrased as $\mathbf{V} = BA^{-1} = B\widetilde{P}^{-1}A^*$. Plugging this in (6.6) we get (6.4):

$$\begin{aligned} w_0(z) &= \mathbf{P}_{\mathcal{E}_*} B\widetilde{P}^{-1}A^* \left(I - z\mathbf{P}_{[X]}B\widetilde{P}^{-1}A^*\right)^{-1}|_{\mathcal{E}} \\ &= \mathbf{P}_{\mathcal{E}_*} B\widetilde{P}^{-1} \left(I - zA^*\mathbf{P}_{[X]}B\widetilde{P}^{-1}\right)^{-1} A^*|_{\mathcal{E}} \\ &= \mathbf{P}_{\mathcal{E}_*} B \left(\widetilde{P} - zA^*\mathbf{P}_{[X]}B\right)^{-1} A^*|_{\mathcal{E}} \\ &= \mathbf{P}_{\mathcal{E}_*} B \left(\widetilde{P} - zPT\right)^{-1} A^*|_{\mathcal{E}} \\ &= E\left(\widetilde{P} - zPT\right)^{-1} M^*. \end{aligned}$$

All the inverses in the latter chain of equalities (except the first one) are understood as operators on $\widetilde{X}$. They exist, since the first inverse in this chain does, which, in turn, is in effect since $\mathbf{V}$ is unitary. □

Theorem 6.3. *If P is nonsingular, then the set of all solutions of Problem* 1.5 *is parametrized by the formula*

$$w(z) = s_0(z) + s_2(z)\left(1 - \mathcal{E}(z)s(z)\right)^{-1}\mathcal{E}(z)s_1(z), \tag{6.8}$$

where the free parameter $\mathcal{E}$ runs over the Schur class $\mathcal{S}$,

$$\begin{aligned} s_0(z) &= E(\widetilde{P} - zPT)^{-1}M^*, && (6.9)\\ s_1(z) &= \alpha^{-1}\left(1 - zMT(\widetilde{P} - zPT)^{-1}M^*\right), && (6.10)\\ s_2(z) &= \beta^{-1}\left(1 - zE(\widetilde{P} - zPT)^{-1}(T^{-1})^*E^*\right), && (6.11)\\ s(z) &= z\alpha^{-1}\beta^{-1}MP^{-1}\widetilde{P}(\widetilde{P} - zPT)^{-1}(T^{-1})^*E^*, && (6.12) \end{aligned}$$

the matrix $\widetilde{P}$ is given in (6.5) *and α and β are positive numbers given by*

$$\alpha = \sqrt{1 + MP^{-1}M^*} \quad \text{and} \quad \beta = \sqrt{1 + ET^{-1}P^{-1}(T^{-1})^*E^*}. \tag{6.13}$$

The matrix $(\widetilde{P} - zPT)$ is invertible for every $z \in \mathbb{D}$ in this case.

Proof. By Theorem 5.1, all the solutions of Problem 1.5 are parametrized by the formula (6.8) where the coefficients s_0, s_1, s_2 and s are the entries of the characteristic function $\mathbf{S}$ of the universal unitary colligation $\mathbf{U_0}$. By Lemma 6.1, we have $\dim \Delta = \dim \Delta_* = 1$. Since, by the very construction of the universal colligation, $\widetilde{\Delta}$ and $\widetilde{\Delta}_*$ are isomorphic copies of Δ and Δ_*, respectively, we have also $\dim \widetilde{\Delta} = \dim \widetilde{\Delta}_* = 1$, and we will identify each of these two spaces with $\mathbb{C}$. However, we will keep the notations $\widetilde{\Delta}$ and $\widetilde{\Delta}_*$ for the spaces so that not to mix them up. Thus, in the present context, the characteristic function $\mathbf{S}$ of $\mathbf{U}_0$ is a 2×2 matrix valued function and it remains to establish explicit formulas (6.9)–(6.12) for its entries which are scalar valued functions. First we will write relations (5.7) defining the operator $\mathbf{U}_0 : \ X \oplus \mathcal{E} \oplus \widetilde{\Delta}_* \to X \oplus \mathcal{E}_* \oplus \widetilde{\Delta}$ more explicitly. The first relation in (5.7) can be written, by the definition (5.1) of $\mathbf{V}$, as

$$\mathbf{U}_0 \begin{bmatrix} P^{\frac{1}{2}} \\ M \\ 0 \end{bmatrix} = \begin{bmatrix} P^{\frac{1}{2}}T \\ E \\ 0 \end{bmatrix}. \tag{6.14}$$

By Lemma 6.1, the spaces Δ and Δ_* are spanned by the vectors

$$\delta = \begin{bmatrix} -P^{-\frac{1}{2}}M^* \\ 1 \\ 0 \end{bmatrix} \in \begin{bmatrix} X \\ \mathcal{E} \\ \widetilde{\Delta}_* \end{bmatrix} \quad \text{and} \quad \delta_* = \begin{bmatrix} -P^{-\frac{1}{2}}(T^{-1})^*E^* \\ 1 \\ 0 \end{bmatrix} \in \begin{bmatrix} X \\ \mathcal{E}_* \\ \widetilde{\Delta} \end{bmatrix},$$

respectively. Note that

$$\|\delta\|^2 = 1 + MP^{-1}M^* \quad \text{and} \quad \|\delta_*\|^2 = 1 + ET^{-1}P^{-1}(T^{-1})^*E^*. \tag{6.15}$$

By the second relation in (5.7), the vector $\mathbf{U}_0\delta$ belongs to $\widetilde{\Delta}$ and therefore, it is of the form

$$\mathbf{U}_0\delta = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} \tag{6.16}$$

where $|\alpha| = \|\delta\|$, due to unitarity of $\mathbf{U}_0$. The latter equality and the first equality in (6.15) imply that $\alpha \neq 0$. In fact, we can choose the identification map $i : \ \Delta \to \widetilde{\Delta}$ so that α will be as in (6.13). Equality (6.16) is an explicit form of the second relation in (5.7). Similarly, the second identification map $i_* : \ \Delta_* \to \widetilde{\Delta}_*$ can be chosen so that

$$\mathbf{U}_0 \begin{bmatrix} 0 \\ 0 \\ \beta \end{bmatrix} = \delta_*, \tag{6.17}$$

where β is defined as in (6.13). Summarizing equalities (6.14), (6.16) and (6.17) we conclude that $\mathbf{U}_0$ satisfies (and is uniquely determined by) the equation

$$\mathbf{U}_0 A = B, \tag{6.18}$$

where

$$A = \begin{bmatrix} P^{\frac{1}{2}} & -P^{-\frac{1}{2}}M^* & 0 \\ M & 1 & 0 \\ 0 & 0 & \beta \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} P^{\frac{1}{2}}T & 0 & -P^{-\frac{1}{2}}(T^{-1})^*E^* \\ E & 0 & 1 \\ 0 & \alpha & 0 \end{bmatrix} \tag{6.19}$$

are operators from $X \oplus E \oplus \mathcal{E}_*$ to $X \oplus E \oplus \widetilde{\Delta}_*$ and to $X \oplus E_* \oplus \widetilde{\Delta}$, respectively. Since $\mathbf{U}_0$ is unitary, it follows that $A^*A = B^*B$. We denote this matrix by $\widehat{P}$ and a straightforward calculation shows that

$$\widehat{P} := A^*A = B^*B = \begin{bmatrix} \widetilde{P} & 0 & 0 \\ 0 & |\alpha|^2 & 0 \\ 0 & 0 & |\beta|^2 \end{bmatrix} : \quad \begin{bmatrix} X \\ \mathcal{E} \\ \mathcal{E}_* \end{bmatrix} \to \begin{bmatrix} X \\ \mathcal{E} \\ \mathcal{E}_* \end{bmatrix}. \tag{6.20}$$

where $\widetilde{P}$ is given in (6.5). Since P is nonsingular so is $\widetilde{P}$ and since $\alpha \neq 0$ and $\beta \neq 0$, $\widehat{P}$ is nonsingular as well. Therefore, A and B are nonsingular. Now we proceed as in the proof of Theorem 6.2: it follows from (6.18) and (6.20) that $\mathbf{U}_0 = BA^{-1} = B\widehat{P}^{-1}A^*$ which being substituted into (5.9) leads us (recall that since P is nonsingular, $[X] = X = C^N$) to

$$\begin{aligned} \mathbf{S}(z) &= \mathbf{P}_{\mathcal{E}_*\oplus\widetilde{\Delta}} B\widehat{P}^{-1}A^* \left(I - z\mathbf{P}_X B\widehat{P}^{-1}A^*\right)^{-1}|_{\mathcal{E}\oplus\widetilde{\Delta}_*} \\ &= \begin{bmatrix} 0 & I_2 \end{bmatrix} B\widehat{P}^{-1}\left(I - zA^*\mathbf{P}_X B\widehat{P}^{-1}\right)^{-1} A^* \begin{bmatrix} 0 \\ I_2 \end{bmatrix} \\ &= \begin{bmatrix} E & 0 & 1 \\ 0 & \alpha & 0 \end{bmatrix}\left(\widehat{P} - zA^*\begin{bmatrix} I_N & 0 \\ 0 & 0 \end{bmatrix} B\right)^{-1}\begin{bmatrix} M^* & 0 \\ 1 & 0 \\ 0 & \beta \end{bmatrix}, \end{aligned} \tag{6.21}$$

where I_2 and I_N are 2×2 and $N\times N$ unit matrices, respectively. The first inverse in this chain of equalities exists for every $z\in\mathbb{D}$ since $\mathbf{U}_0$ is unitary, all the others exist since the first one does. By (6.19) and (6.20),

$$\widehat{P} - zA^*\begin{bmatrix} I_N & 0 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} \widetilde{P} - zPT & 0 & z(T^{-1})^*E^* \\ zMT & |\alpha|^2 & -zMP^{-1}(T^{-1})^*E^* \\ 0 & 0 & |\beta|^2 \end{bmatrix}.$$

Upon inverting the latter triangular matrix and plugging it into (6.21), we eventually get

$$\begin{aligned} \mathbf{S}(z) &= \begin{bmatrix} s_0(z) & s_2(z) \\ s_1(z) & s(z) \end{bmatrix} \\ &= \begin{bmatrix} ER(z)M^* & \beta^{-1}\left(1 - zER(z)(T^{-1})^*E^*\right) \\ \alpha^{-1}\left(1 - zMTR(z)M^*\right) & z\alpha^{-1}\beta^{-1}MP^{-1}\widetilde{P}R(z)(T^{-1})^*E^* \end{bmatrix}, \end{aligned}$$

where $R(z) = \left(\widetilde{P} - zPT\right)^{-1}$, which is equivalent to (6.9)–(6.12). □

In conclusion we will establish some important properties of the coefficient matrix $\mathbf{S}$ constructed in Theorem 6.3.

Theorem 6.4. *Let* $\mathbf{S} = \begin{bmatrix} s_0 & s_2 \\ s_1 & s \end{bmatrix}$ *be the characteristic function of the universal unitary colligation* $\mathbf{U_0}$ *defined in* (6.19), (6.20). *Then*

1. *The function* s_0 *is a solution of Problem* 1.2.
2. *The function* $\mathbf{S}(z)$ *is a rational inner matrix-function of degree at most* N.
3. *The functions* s_1 *and* s_2 *have zeroes of multiplicity* n_i+1 *at each interpolating point* t_i *and do not have other zeroes.*

Proof: By Theorem 6.3, s_0 is a solution of Problem 1.5 (corresponding to the parameter $\mathcal{E}\equiv 0$ in the parametrization formula (6.8)). Therefore, by Theorem 4.5,

$\mathbf{F}^{s_0}x$ belongs to the space H^{s_0} for every $x \in \mathbb{C}^N = X$, where H^{s_0} is the de Branges–Rovnyak space associated to the Schur function s_0 and where

$$\mathbf{F}^{s_0}(t) := \begin{bmatrix} 1 & s_0(t) \\ s_0(t)^* & 1 \end{bmatrix} \begin{bmatrix} E \\ -M \end{bmatrix} (\mathrm{I} - tT)^{-1}. \tag{6.22}$$

Again, by Theorem 4.5, to show that s_0 is a solution of Problem 1.2, it remains to check that $\|\mathbf{F}^{s_0}x\|_{H^{s_0}} = x^*Px$. Letting for short

$$R_T(t) := (\mathrm{I} - tT)^{-1}, \tag{6.23}$$

we note that by (6.22) and by definition of the norm in the de-Branges-Rovnyak space,

$$\|\mathbf{F}^{s_0}x\|^2_{H^{s_0}} = \left\langle \begin{bmatrix} 1 & s_0 \\ s_0^* & 1 \end{bmatrix} \begin{bmatrix} E \\ -M \end{bmatrix} R_T x, \begin{bmatrix} E \\ -M \end{bmatrix} R_T x \right\rangle_{L^2(\mathbb{C}^2)} \tag{6.24}$$

By Theorems 5.2 and 5.3, the function

$$\begin{aligned} \left(\mathcal{F}_{\mathbf{U_0}} P^{\frac{1}{2}} x\right)(t) &= \begin{bmatrix} \mathrm{I}_2 & \mathbf{S}(t) \\ \mathbf{S}(t)^* & \mathrm{I}_2 \end{bmatrix} \begin{bmatrix} E \\ 0 \\ -M \\ 0 \end{bmatrix} R_T(t)x \qquad (6.25) \\ &= \begin{bmatrix} E - s_0(t)M \\ -s_1(t)M \\ s_0(t)^*E - M \\ s_2(t)^*E \end{bmatrix} (\mathrm{I} - tT)^{-1} x \quad \text{belongs to } H^{\mathbf{S}} \end{aligned}$$

for every vector $x \in \mathbb{C}^N$. Note that $E(\mathrm{I} - tT)^{-1} x \not\equiv 0$, unless $x = 0$. Indeed, letting

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}, \quad \text{where} \quad x_i = \begin{bmatrix} x_{i,1} \\ \vdots \\ x_{i,n_i} \end{bmatrix} \quad (i = 1, \dots, k), \tag{6.26}$$

we get, on account of definitions (3.6) and (3.7) of T and E, that

$$E(\mathrm{I} - tT)^{-1} x = \sum_{i=1}^{k} \sum_{j=0}^{n_i} \frac{t^j}{(1 - t\bar{t}_i)^{j+1}} x_{i,j} \not\equiv 0 \tag{6.27}$$

for every $x \neq 0$, since the functions

$$\frac{t^j}{(1 - t\bar{t}_i)^{j+1}} \quad (i = 1, \dots, k;\ j = 0, \dots, n_i)$$

are linearly independent (recall that all the points $t_1, \dots, t_k$ are distinct).

Note also that $s_2 \not\equiv 0$ (since $s_2(0) = \beta \neq 0$, by (6.11) and (6.13)) and therefore,

$$s_2(t)E(\mathrm{I} - tT)^{-1} x \not\equiv 0$$

for every $x \in X, x \neq 0$. It is seen from (6.25) that the latter function is the bottom component of $\mathcal{F}_{\mathbf{U_0}} P^{\frac{1}{2}} x$, which leads us to the conclusion that

$$\mathcal{F}_{\mathbf{U_0}} P^{\frac{1}{2}} x \not\equiv 0 \quad \text{for every} \quad x \neq 0.$$

The latter means that the linear map $\mathcal{F}_{\mathbf{U_0}} : [X] \to H^{\mathbf{S}}$ is a bijection. Since $\mathcal{F}_{\mathbf{U_0}}$ is a partial isometry (by Theorem 5.2), it now follows that this map is unitary, i.e., that

$$\left\| \mathcal{F}_{\mathbf{U_0}} P^{\frac{1}{2}} x \right\|^2_{H^{\mathbf{S}}} = \left\| P^{\frac{1}{2}} x \right\|^2_X = x^* P x. \tag{6.28}$$

Furthermore,

$$\left\| \mathcal{F}_{\mathbf{U_0}} P^{\frac{1}{2}} x \right\|^2_{H^{\mathbf{S}}} = \left\langle \begin{bmatrix} \mathrm{I}_2 & \mathbf{S} \\ \mathbf{S}^* & \mathrm{I}_2 \end{bmatrix} \begin{bmatrix} E \\ 0 \\ -M \\ 0 \end{bmatrix} R_T x, \ \begin{bmatrix} E \\ 0 \\ -M \\ 0 \end{bmatrix} R_T x \right\rangle_{L^2(\mathbb{C}^4)},$$

by (6.25) and virtue of formula (2.1) for the norm in $H^{\mathbf{S}}$. Upon taking advantage of the zero entries in the last formula and the partition of the matrix $\mathbf{S}$, we get

$$\begin{aligned} \left\| \mathcal{F}_{\mathbf{U_0}} P^{\frac{1}{2}} x \right\|^2_{H^{\mathbf{S}}} &= \left\langle \begin{bmatrix} 1 & 0 & s_0 & s_2 \\ s_0^* & s_1^* & 1 & 0 \end{bmatrix} \begin{bmatrix} E \\ 0 \\ -M \\ 0 \end{bmatrix} R_T x, \ \begin{bmatrix} E \\ -M \end{bmatrix} R_T x \right\rangle_{L^2(\mathbb{C}^2)} \\ &= \left\langle \begin{bmatrix} 1 & s_0 \\ s_0^* & 1 \end{bmatrix} \begin{bmatrix} E \\ -M \end{bmatrix} R_T x, \ \begin{bmatrix} E \\ -M \end{bmatrix} R_T x \right\rangle_{L^2(\mathbb{C}^2)} \end{aligned} \tag{6.29}$$

Comparing (6.24) and (6.29) and taking into account (6.28) we come to

$$\| \mathbf{F}^{s_0} x \|_{H^{s_0}} = \left\| \mathcal{F}_{\mathbf{U_0}} P^{\frac{1}{2}} x \right\|_{H^{\mathbf{S}}} = x^* P x,$$

which proves the first assertion of the theorem. The second assertion follows since $\dim X = N = \sum_{i=0}^{k} (n_i + 1) < \infty$ (see, e.g., [18]).

To prove the last assertion, we use (6.25) for x in the form (6.26) with the only nonzero entry $x_{i,n_i} = 1$. For this choice of x we have by definitions (3.6)–(3.8) of T, E and N,

$$E(I - tT)^{-1} x = \frac{t^{n_i}}{(1 - t\bar{t}_i)^{n_i+1}} \quad \text{and} \quad M(I - tT)^{-1} x = \frac{\mathbf{c}_i(t)}{(1 - t\bar{t}_i)^{n_i+1}}$$

where

$$\mathbf{c}_i(t) = \sum_{\ell=0}^{n_i} t^{n_i - \ell} (1 - t\bar{t}_i)^{\ell} c^*_{i,\ell}. \tag{6.30}$$

Now we conclude from (6.25) that

$$\frac{s_1(t)\mathbf{c}_i(t)}{(1 - t\bar{t}_i)^{n_i+1}} \in H_2^+ \quad \text{and} \quad \frac{t^{n_i} s_2(t)^*}{(1 - t\bar{t}_i)^{n_i+1}} = \bar{t} \frac{s_2(t)^*}{(\bar{t} - \bar{t}_i)^{n_i+1}} \in H_2^-. \tag{6.31}$$

By (6.30), $\mathbf{c}_i(t_i) = t_i^{n_i} c_{i,0}^* \neq 0$ and thus, the first condition in (6.31) implies that s_1 has the zero of multiplicity at least $n_i + 1$ at t_i. The second condition in (6.31) is equivalent to

$$\frac{s_2(t)}{(t-t_i)^{n_i+1}} \in H_2^+$$

which implies that s_2 has zero of multiplicity at least $n_i + 1$ at t_i. On the other hand, since s_1 and s_2 are rational functions of degree at most $N = \sum_{i=0}^{k}(n_i+1)$ (the second assertion of this theorem) and since they do not vanish identically (by the proof of the first assertion of this theorem), they can not have more than N zeroes. Therefore, they have zeroes of multiplicities $n_i + 1$ at t_i for $i = 1, \ldots, k$ and they do not have other zeroes. □

Some consequences of Theorem 6.4 needed in the next section are proved in the following lemma.

Lemma 6.5. *Let* $\mathbf{S} = \begin{bmatrix} s_0 & s_2 \\ s_1 & s \end{bmatrix}$ *be as in Theorem* 6.4. *Then* $|s(t_i)| = 1$,

$$\frac{s_1^{(n_i+1)}(t_i)}{(n_i+1)!} = \lim_{z\to t_i} \frac{s_1(z)}{(z-t_i)^{n_i+1}} \neq 0, \quad \frac{s_2^{(n_i+1)}(t_i)}{(n_i+1)!} = \lim_{z\to t_i} \frac{s_2(z)}{(z-t_i)^{n_i+1}} \neq 0, \tag{6.32}$$

and

$$s_2^{(n_i+1)}(t_i)^* = (-1)^{n_i} t_i^{2n_i+2} s(t_i)^* s_1^{(n_i+1)}(t_i) c_{i,0}{}^* \ . \tag{6.33}$$

Proof: By the third assertion of Theorem 6.4, the rational functions s_1 and s_2 have zeros of multiplicity $n_i + 1$ at t_i. This implies (6.32). By the second assertion of Theorem 6.4, the matrix-function $\mathbf{S}$ is inner and rational. In particular, it is unitary at $t_i \in \mathbb{T}$ and therefore, $|s_2(t_i)|^2 + |s(t_i)|^2 = 1$ which implies $|s(t_i)| = 1$, since $s_2(t_i) = 0$. Furthermore, by the reflection principle, $\mathbf{S}(1/\bar{z})^*\mathbf{S}(z) \equiv \mathrm{I}_2$,or in more detail,

$$\begin{bmatrix} s_0(1/\bar{z})^* & s_1(1/\bar{z})^* \\ s_2(1/\bar{z})^* & s(1/\bar{z})^* \end{bmatrix} \begin{bmatrix} s_0(z) & s_2(z) \\ s_1(z) & s(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In particular,

$$s_2(1/\bar{z})^* s_0(z) + s(1/\bar{z})^* s_1(z) = 0 \tag{6.34}$$

To verify (6.33), we note first that by the first relation in (6.32),

$$\lim_{z\to t_i} \frac{s(1/\bar{z})^* s_1(z)}{(z-t_i)^{n_i+1}} = \frac{s(t_i)^* s_1^{(n_i+1)}(t_i)}{(n_i+1)!}. \tag{6.35}$$

Since $|t_i| = 1$, the second relation in (6.32) gives

$$\frac{s_2^{(n_i+1)}(t_i)}{(n_i+1)!} = \lim_{z\to t_i} \frac{s_2(\bar{z}^{-1})}{(\bar{z}^{-1}-t_i)^{n_i+1}},$$

which is equivalent, on account of $\bar{z}^{-1} - t_i = -\frac{\bar{z}-\bar{t}_i}{\bar{z}\bar{t}_i}$, to

$$\frac{s_2^{(n_i+1)}(t_i)}{(n_i+1)!} = \lim_{z\to t_i} \frac{(-\bar{z}\bar{t}_i)^{n_i+1}s_2(\bar{z}^{-1})}{(\bar{z}-\bar{t}_i)^{n_i+1}} = (-1)^{n_i+1}\bar{t}_i^{2n_i+2} \lim_{z\to t_i} \frac{s_2(\bar{z}^{-1})}{(\bar{z}-\bar{t}_i)^{n_i+1}}.$$

Upon taking adjoints in the latter equality we get

$$\lim_{z\to t_i} \frac{s_2(\bar{z}^{-1})^*}{(z-t_i)^{n_i+1}} = (-1)^{n_i+1}t_i^{2n_i+2}\frac{s_2^{(n_i+1)}(t_i)^*}{(n_i+1)!}$$

and, since $s_0(t_i) = c_{i,0}$ (recall that s_0 is a solution of Problem 1.2), we have also

$$\lim_{z\to t_i} \frac{s_2(\bar{z}^{-1})^* s_0(z)}{(z-t_i)^{n_i+1}} = (-1)^{n_i+1}t_i^{2n_i+2}\frac{s_2^{(n_i+1)}(t_i)^*}{(n_i+1)!}c_{i,0}. \tag{6.36}$$

Now upon multiplying (6.34) by $\dfrac{(n_i+1)!}{(z-t_i)^{n_i+1}}$ and passing to limits as $z \to t_i$, we come, on account of (6.35) and (6.36) to the equality

$$s(t_i)^* s_1^{(n_i+1)}(t_i) + (-1)^{n_i+1}t_i^{2n_i+2}s_2^{(n_i+1)}(t_i)^*c_{i,0} = 0,$$

which is equivalent to (6.33), since $|c_{i,0}| = |t_i| = 1$. □

7. Boundary interpolation problem with equality

In this section we establish a parametrization of all solutions of Problem 1.2. Recall that all solutions w of Problem 1.5 are parametrized by the linear fractional formula (6.8) with the free Schur class parameter $\mathcal{E}$. Thus, for every function w of the form (6.8), we have

$$\delta_{w,i} := \gamma_i - d_{w,n_i}(t_i) \geq 0 \quad (i = 1,\dots,k).$$

Theorem 7.4 below will present the explicit formula for the gaps $\delta_{w,i}$ in terms of the parameter $\mathcal{E}$ leading to w via formula (6.8) . As a consequence of this formula we will get a characterization of all the parameters $\mathcal{E}$, leading to functions w with zero gaps, i.e., to solutions of Problem 1.2. We start with some needed preliminaries. The proof of the first lemma can be found in [21] for the case when $n = 0$. For the case $n > 0$ the proof was given in [6] using pretty much the same ideas.

Lemma 7.1. *Let w be a function analytic in some nontangential neighborhood of a point $t_0 \in \mathbb{T}$ and let $w_0,\dots,w_{2n+1}$ be complex numbers. Then equality*

$$\lim_{z\to t_0} \frac{w(z) - w_0 - (z-t_0)w_1 - \ldots - (z-t_0)^{2n}w_{2n}}{(z-t_0)^{2n+1}} = w_{2n+1}$$

holds if and only if the nontangential limits $\lim_{z\to t_0} \dfrac{w^{(j)}(z)}{j!}$ exist and equal w_j for $j = 0,\dots,2n+1$.

With every triple (ω, t_0, b) consisting of a Schur function $\omega \in \mathcal{S}$, of a point $t_0 \in \mathbb{T}$ and a number $b \in \mathbb{C}$, we associate the quantity

$$\begin{aligned} D_{\omega,b}(t_0) &:= \int_{\mathbb{T}} \frac{1}{|1-t\overline{t}_0|^2} \begin{bmatrix} 1 & -b \end{bmatrix} \begin{bmatrix} 1 & \omega(t) \\ \omega(t)^* & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -b^* \end{bmatrix} m(dt) \\ &= \int_{\mathbb{T}} \left(\left| \frac{1-\omega(t)\overline{b}}{1-t\overline{t}_0} \right|^2 + |b|^2 \frac{1-|\omega(t)|^2}{|1-t\overline{t}_0|^2} \right) m(dt) \\ &= \int_{\mathbb{T}} \left(\left| \frac{\omega(t)-b}{t-t_0} \right|^2 + \frac{1-|\omega(t)|^2}{|t-t_0|^2} \right) m(dt), \end{aligned} \tag{7.1}$$

where $m(dt)$ is the normalized Lebesgue measure on $\mathbb{T}$. It follows from the very definition that

$$0 \leq D_{\omega,\,b}(t_0) \leq \infty.$$

The next theorem (which is a variation of the classical Julia-Carathéodory Theorem and can be mostly found in [21]) characterizes the cases when $D_{\omega,b}(t_0)$ is zero, positive or infinite.

Theorem 7.2. *Let $\omega \in \mathcal{S}$, $t_0 \in \mathbb{T}$, $b \in \mathbb{C}$ and let $D_{\omega,\,b}(t_0)$ be defined as in* (7.1). *Then*

1. $D_{\omega,\ b}(t_0) < \infty$ *if and only if*
$$\liminf_{z\to t_0} \frac{1-|\omega(z)|^2}{1-|z|^2} < \infty \quad \text{and} \quad \lim_{z\to t_0} \omega(z) = b, \tag{7.2}$$
where the second limit is understood as nontangential. In this case $|b| = 1$.
2. $D_{\omega,\ b}(t_0) = \infty$ *if and only if either*
$$\liminf_{z\to t_0} \frac{1-|\omega(z)|^2}{1-|z|^2} = \infty,$$
or the function ω fails to have a nontangential limit b at t_0.
3. $D_{\omega,\ b}(t_0) = 0$ *if and only if $\omega(z) \equiv b$ and $|b| = 1$.*
4. *If $|b| \leq 1$, then the following equality*
$$\lim_{z\to t_0} \frac{1-\omega(z)b^*}{1-z\overline{t}_0} = D_{\omega,\ b}(t_0) \tag{7.3}$$
holds where the limit is understood as nontangential.

Proof. Let H^ω be the de Branges-Rovnyak space associated to the Schur class function ω and let us consider the function

$$K_{t_0,b}(t) = \begin{bmatrix} 1 & \omega(t) \\ \omega(t)^* & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -b^* \end{bmatrix} \frac{1}{1-t\overline{t}_0} = \begin{bmatrix} K_{t_0,b,+}(t) \\ K_{t_0,b,-}(t) \end{bmatrix} \tag{7.4}$$

where

$$K_{t_0,b,+}(t) = \frac{1-\omega(t)b^*}{1-t\overline{t}_0} \quad \text{and} \quad K_{t_0,b,-}(t) = \overline{t}\, \frac{\overline{\omega(t)} - b^*}{\overline{t}-\overline{t}_0} \tag{7.5}$$

By the formula (2.1),

$$\|K_{t_0,b}\|^2_{H^\omega} = \left\langle \begin{bmatrix} 1 & \omega(t) \\ \omega(t)^* & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -b^* \end{bmatrix} \frac{1}{1-t\bar{t}_0}, \begin{bmatrix} 1 \\ -b^* \end{bmatrix} \frac{1}{1-t\bar{t}_0} \right\rangle_{L^2\oplus L^2}$$

which is equal to the first integral in (7.1). Therefore, $D_{\omega,\,b}(t_0) < \infty$ if and only if $K_{t_0,b}$ belongs to L^ω and in this case,

$$D_{\omega,\,b}(t_0) = \|K_{t_0,\,b}\|^2_{L^\omega}. \tag{7.6}$$

On the other hand, if $D_{\omega,\,b}(t_0) < \infty$, then it follows from the second form of $D_{\omega,\,b}(t_0)$ in (7.1) that

$$\int_{\mathbb{T}} \left| \frac{1-\omega(t)b^*}{1-t\bar{t}_0} \right|^2 m(dt) < \infty, \quad \text{i.e., that} \quad K_{t_0,b,+}(t) = \frac{1-\omega(t)b^*}{1-t\bar{t}_0} \in L_2.$$

Since $1-t\bar{t}_0$ is an outer function, it follows, by Smirnov maximum principle [22], that $K_{t_0,b,+} \in H_2^+$. Similarly, it follows from the third representation of $D_{\omega,\,b}(t_0)$ in (7.1) that $K_{t_0,b,-} \in H_2^-$. Therefore, $K_{t_0,b}$ belongs to H^ω by Definition 2.1. Thus, we have shown that

$$D_{\omega,b}(t_0) < \infty \iff K_{t_0,b} \in L^\omega \iff K_{t_0,b} \in H^\omega.$$

Now the first assertion of the lemma follows from Theorem 2.3 (the case when $n = 0$): the function $K_{t_0,b}$ of the form (7.4) belongs to H^ω if and only if conditions in (7.2) are satisfied. In this case $|b| = 1$, since

$$1-|b|^2 = \lim_{z\to t_0}(1-|\omega(z)|^2) = \lim_{z\to t_0} \frac{1-|\omega(z)|^2}{1-|z|^2}(1-|z|^2) = 0.$$

The second assertion is simply the formal negation of the first one. To prove the third assertion, we observe that $D_{\omega,\,b}(t_0) = 0$ if and only if

$$\begin{bmatrix} 1 & -b \end{bmatrix} \begin{bmatrix} 1 & \omega(t) \\ \omega(t)^* & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -b^* \end{bmatrix} = 0$$

almost everywhere on $\mathbb{T}$, which occurs if and only if

$$\begin{bmatrix} 1 & -b \end{bmatrix} \begin{bmatrix} 1 & \omega(t) \\ \omega(t)^* & 1 \end{bmatrix} = 0$$

almost everywhere on $\mathbb{T}$. The latter equality collapses to $\omega(t) - b = 1 - b\omega(t)^* = 0$ which implies the requisite.

The proof of the fourth assertion splits up into three cases.

Case 1: Let $D_{\omega,\,b}(t_0) < \infty$. Then by the first statement, conditions (7.2) are satisfied. Then by Theorem 2.2, the kernels

$$K_z(t) = \begin{bmatrix} 1 & \omega(t) \\ \omega(t)^* & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\omega(z)^* \end{bmatrix} \frac{1}{1-t\bar{z}}$$

converge to $K_{t_0,\,b}$ in norm of H^ω:

$$K_z \xrightarrow{H^w} K_{t_0,\,b}, \tag{7.7}$$

as $z \to t_0$ nontangentially. By the reproducing property (2.3) (for $j=0$),

$$\langle f,\ K_z\rangle_{H^\omega} = f_+(z) \quad \text{for every} \quad f = \begin{bmatrix} f_+ \\ f_- \end{bmatrix} \in H^\omega. \tag{7.8}$$

Then, upon making subsequent use of (7.6), (7.7), (7.8) and of the explicit formula (7.5) for $K_{t_0,b,+}$, we get (7.3):

$$\begin{aligned} D_{\omega,\,b}(t_0) = \|K_{t_0,\,b}\|^2_{H^\omega} &= \lim_{z\to t_0} \langle K_{t_0,\,b}, K_z\rangle_{H^\omega} \\ &= \lim_{z\to t_0} K_{t_0,b,+}(z) = \lim_{z\to t_0} \frac{1-\omega(z)b^*}{1-z\bar{t}_0}\ . \end{aligned} \tag{7.9}$$

Case 2: Let $D_{\omega,\,b}(t_0)=\infty$ and $\liminf\limits_{z\to t_0} \dfrac{1-|\omega(z)|^2}{1-|z|^2} < \infty$.

The second assumption guarantees (by Theorem 2.3), that there exists the nontangential limit $\omega(t_0) = \lim\limits_{z\to t_0}\omega(z)$ and that the function $K_{t_0,\omega(t_0)}$ defined via (7.4), belongs to H^ω. Then by virtue of (7.9), we have

$$\lim_{z\to t_0} \frac{1-\omega(z)\omega(t_0)^*}{1-z\bar{t}_0} = D_{\omega,\omega(t_0)}(t_0) = \|K_{t_0,\omega(t_0)}\|^2_{H^\omega} < \infty.$$

Since $D_{\omega,b}(t_0)=\infty$, it follows that $b\neq\omega(t_0)$. It remains to note that (7.3) again holds since

$$\lim_{z\to t_0} \frac{1-\omega(z)b^*}{1-z\bar{t}_0} = \lim_{z\to t_0} \frac{1-\omega(z)\omega(t_0)^* + \omega(z)(b^*-\omega(t_0)^*)}{1-z\bar{t}_0} = \infty.$$

Case 3: Let $D_{\omega,\,b}(t_0)=\infty$ and $\liminf\limits_{z\to t_0} \dfrac{1-|\omega(z)|^2}{1-|z|^2} = \infty$. Since

$$\begin{aligned} 2\Re(1-\omega(z)b^* &= (1-\omega(z)b^*) + (1-b\omega(z)^*) \\ &= |1-\omega(z)b^*|^2 + 1 - |b|^2|\omega(z)|^2 \geq 1-|b|^2|\omega(z)|^2, \end{aligned}$$

it follows that if $|b|\leq 1$, then

$$|1-\omega(z)b^*| \geq \Re(1-\omega(z)b^*) \geq \frac{1}{2}(1-|\omega(z)|^2). \tag{7.10}$$

Furthermore, for every z in the following nontangential neighborhood

$$\Gamma_a(t_0) = \{z\in\mathbb{D}:\ |t_0-z| < a(1-|z|)\}, \quad a>1,$$

of t_0, we have

$$\frac{1-|z|^2}{|1-z\bar{t}_0|} \geq \frac{1-|z|}{|1-z\bar{t}_0|} > \frac{1}{a}$$

which together with (7.10) leads us to

$$\left|\frac{1-\omega(z)b^*}{1-z\bar{t}_0}\right| \geq \frac{1}{2}\,\frac{1-|\omega(z)|^2}{|1-z\bar{t}_0|} = \frac{1}{2}\,\frac{1-|\omega(z)|^2}{1-|z|^2}\cdot\frac{1-|z|^2}{|1-z\bar{t}_0|} > \frac{1}{2a}\,\frac{1-|\omega(z)|^2}{1-|z|^2}.$$

Therefore,

$$\lim_{z\to t_0}\frac{1-\omega(z)b^*}{1-z\bar{t}_0} = \infty = D_{\omega,\ b}(t_0),$$

which completes the proof of the theorem. □

Corollary 7.3. *If a Schur function ω is analytic in a neighborhood of $t_0 \in \mathbb{T}$ and $|\omega(t_0)| = 1$, then $D_{\omega,\,\omega(t_0)}(t_0) < \infty$. In particular, $D_{\omega,\,\omega(t_0)}(t_0) < \infty$ for every rational $\omega \in \mathcal{S}$ with $|\omega(t_0)| = 1$.*

Proof. If w meets the assumed properties, then the limit

$$\lim_{z\to t_0} \frac{1-\omega(z)\overline{\omega(t_0)}}{1-z\bar{t}_0} = \lim_{z\to t_0}\left(\frac{\omega(t_0)-\omega(z)}{t_0-z}\right)\frac{\overline{\omega(t_0)}}{\bar{t}_0} = \omega'(t_0)\frac{\overline{\omega(t_0)}}{\bar{t}_0}$$

is finite, then, by the fourth assertion in Lemma 7.2, $D_{\omega,\,\omega(t_0)}(t_0) < \infty$. □

The next theorem presents an explicit formula for the gap $\gamma_i - d_{w,n_i}(t_i)$ for any solution w of Problem 1.5. Recall that by Theorem 6.3, all solutions of Problem 1.5 are parametrized by formula (6.8).

Theorem 7.4. *Let w be a solution of Problem* 1.5*, i.e. a function of the form* (6.8)

$$w(z) = s_0(z) + s_2(z)\left(1-\mathcal{E}(z)s(z)\right)^{-1}\mathcal{E}(z)s_1(z) \tag{7.11}$$

with a parameter $\mathcal{E} \in \mathcal{S}$. Then for $i = 1,\ldots,k$,

$$\gamma_i - d_{w,n_i}(t_i) = \frac{1}{((n_i+1)!)^2}\cdot\frac{|s_2^{(n_i+1)}(t_i)|^2}{D_{\mathcal{E},s(t_i)^*}(t_i)+D_{s,s(t_i)}(t_i)} \tag{7.12}$$

for $i = 1,\ldots,k$, where $D_{\mathcal{E},s(t_i)^*}(t_i)$ and $D_{s,s(t_i)}(t_i)$ are defined according to (7.1).

Proof: Since w is a solution of Problem 1.5 and therefore satisfies conditions (1.26)–(1.28), it follows by Lemma 7.1 that

$$w_{2n_i+1}(t_i) = \lim_{z\to t_i}\frac{w(z)-c_{i,0}-(z-t_i)c_{i,1}-\ldots-(z-t_i)^{2n}c_{i,2n_i}}{(z-t_i)^{2n_i+1}}$$

for $i = 1,\ldots,k$. Since s_0 is a solution of Problem 1.2 (by the first statement in Theorem 6.4), we have (again by Lemma 7.1)

$$c_{i,2n_i+1} = \lim_{z\to t_i}\frac{s_0(z)-c_{i,0}-(z-t_i)c_{i,1}-\ldots-(z-t_i)^{2n}c_{i,2n_i}}{(z-t_i)^{2n_i+1}}.$$

Now it follows from the two latter equalities that

$$c_{i,2n_i+1} - w_{2n_i+1}(t_i) = \lim_{z\to t_i}\frac{s_0(z)-w(z)}{(z-t_i)^{2n_i+1}},$$

which being substituted into (1.24), leads us to to

$$\gamma_i - d_{w,n_i}(t_i) = (-1)^{n_i}t_i^{2n_i+1}\lim_{z\to t_i}\frac{s_0(z)-w(z)}{(z-t_i)^{2n_i+1}}\,{c_{i,0}}^*.$$

Substituting (7.11) into the latter equality gives

$$\gamma_i - d_{w,n_i}(t_i) = -(-1)^{n_i}t_i^{2n_i+1}\lim_{z\to t_i}\frac{s_2(z)(1-\mathcal{E}(z)s(z))^{-1}\mathcal{E}(z)s_1(z)}{(z-t_i)^{2n_i+1}}\,c_{i,0}^*. \tag{7.13}$$

Taking into account relations (6.32) (i.e., the fact that t_i is a zero of multiplicity n_i of s_1 and s_2), we rephrase (7.13) as

$$\gamma_i - d_{w,n_i}(t_i) = \frac{(-1)^{n_i} t_i^{2n_i+2}}{((n_i+1)!)^2} s_2^{(n_i+1)}(t_i) \lim_{z\to t_i} \frac{(1-z\bar{t}_i)\mathcal{E}(z)}{1-\mathcal{E}(z)s(z)} s_1^{(n_i+1)}(t_i) c_{i,0}^*. \quad (7.14)$$

Due to (6.33), the latter equality simplifies to

$$\gamma_i - d_{w,n_i}(t_i) = \frac{|s_2^{(n_i+1)}(t_i)|^2}{((n_i+1)!)^2} \lim_{z\to t_i} \frac{\mathcal{E}(z)s(t_i)(1-z\bar{t}_i)}{1-\mathcal{E}(z)s(z)}. \quad (7.15)$$

Since $|s(t_i)| = 1$ (by Lemma 6.5), we have

$$\frac{1-\mathcal{E}(z)s(z)}{1-z\bar{t}_i} = \frac{1-s(z)s(t_i)^*}{1-z\bar{t}_i} + s(z)\frac{1-\mathcal{E}(z)s(t_i)}{1-z\bar{t}_i} s(t_i)^*. \quad (7.16)$$

By the fourth assertion of Lemma 7.2,

$$\lim_{z\to t_i} \frac{1-s(z)s(t_i)^*}{1-z\bar{t}_i} = D_{s,s(t_i)}(t_i), \quad \lim_{z\to t_i} \frac{1-\mathcal{E}(z)s(t_i)}{1-z\bar{t}_i} = D_{\mathcal{E},s(t_i)^*}(t_i). \quad (7.17)$$

Taking advantage of (7.17) we pass to limits in (7.16) as $z \to t_i$ to get

$$\lim_{z\to t_i} \frac{1-\mathcal{E}(z)s(z)}{1-z\bar{t}_i} = D_{s,s(t_i)}(t_i) + D_{\mathcal{E},s(t_i)^*}(t_i). \quad (7.18)$$

Since s is rational and $|s(t_i)| = 1$, it follows by Corollary 7.3 that $D_{s,s(t_i)}(t_i)$ is finite. Since, by Theorem 6.4, $\mathbf{S}(t)$ is unitary for $t \in \mathbb{T}$ and $s_1(z), s_2(z)$ are not identical zeros, then $s(z)$ is not a unimodular constant. Therefore, $D_{s,s(t_i)}(t_i) \neq 0$, by the third assertion in Lemma 7.2. Thus,

$$0 < D_{s,s(t_i)}(t_i) < \infty.$$

If the second limit in (7.17) is also finite, then

$$\lim_{z\to t_i} \mathcal{E}(z) = s(t_i)^*,$$

by the first assertion in Lemma 7.2. Therefore, (7.15) turns into (7.12) in this case. If $D_{\mathcal{E},s(t_i)^*}(t_i)$ is infinite, then, in view of (7.18), the denominator in (7.15) tends to ∞. Since the numerator $\mathcal{E}(z)s(t_i)$ is bounded, the limit in (7.15) is 0. Thus, (7.12) holds in this case also. Theorem follows. □

Proof of Statement 2 in Theorem 1.6**:** As it was already pointed out, w is a solution of Problem 1.2 if and only if it is of the form (7.11) with some (uniquely determined) parameter $\mathcal{E} \in \mathcal{S}$ and satisfies

$$\delta_{w,i} := \gamma_i - d_{w,n_i}(t_i) = 0 \quad (i = 1, \ldots, k).$$

The formula for $\delta_{w,i}$ is given in (7.12) and it is easily seen that $\delta_{w,i} = 0$ if and only if

$$D_{\mathcal{E},\, s(t_i)^*}(t_i) + D_{s,\, s(t_i)}(t_i) = \infty.$$

Since $D_{s,\,s(t_i)}(t_i) < \infty$ (by Corollary 7.3), the latter is equivalent to $D_{\mathcal{E},\,s(t_i)^*}(t_i) = \infty$ which happens, by the second assertion in Lemma 7.2, if and only if either

$$\liminf_{z\to t_i} \frac{1-|\mathcal{E}(z)|^2}{1-|z|^2} = \infty,$$

or the function $\mathcal{E}$ fails to have the nontangential limit $s(t_i)^*$ at t_i. □

Note that vanishing of the gap at the point t_i depends on the local behavior of the parameter $\mathcal{E}$ at this point only. The number $s(t_i)^*$ absorbs all the interpolation data, though. Note also that the maximum value of the gap $\delta_{w,i}$ is assumed when $D_{\mathcal{E},\ s(t_i)^*}(t_i) = 0$, which happens if and only if $\mathcal{E}(z) \equiv s(t_i)^*$.

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MSC: 30E05, 446E22, 47A57

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