BOUNDARY RIGIDITY FOR SOME CLASSES OF MEROMORPHIC FUNCTIONS

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Abstract. Let f be a function meromorphic on the open unit disk $\mathbb{D}$, with angular boundary limits bounded by one in modulus almost everywhere on the unit circle. We give sufficient conditions in terms of boundary asymptotics at finitely many points on the unit circle $\mathbb{T}$ for f to be a ratio of two finite Blaschke products. A necessary condition is that f has finitely many poles in $\mathbb{D}$, i.e., that f is a generalized Schur function. Similar rigidity statements are presented for generalized Carathéodory and generalized Nevanlinna functions.

1. Introduction

Let H^∞ be the space of bounded analytic functions on the open unit disk $\mathbb{D}$ and let $\mathcal{S}$ be its unit ball (called sometimes the Schur class):

$$\mathcal{S} := \mathcal{B}H^\infty = \{f \in H^\infty : \ \ \|f\|_{H^\infty} := \sup_{z\in\mathbb{D}} |f(z)| \le 1\}. \tag{1.1}$$

The following boundary rigidity result is presented in [15].

Theorem 1.1. *Let $f \in \mathcal{S}$ and let $f(z) = z + O(|z-1|^4)$ as $z \to 1$. Then $f(z) \equiv z$.*

As mentioned in [15], the term $O(|z-1|^4)$ can be replaced by $o(|z-1|^3)$ if $z \in \mathbb{D}$ tends to 1 nontangentially. Furthermore, it was shown in [19] that the conclusion $f(z) \equiv z$ follows from a weaker assumption $\liminf_{r\to 1^-} \frac{\Re f(r) - r}{(1-r)^3} = 0$. The same conclusion follows from the assumption that $\lim_{n\to\infty} \frac{f(z_n) - z_n}{(1-z_n)^3} = 0$ for some sequence $\{z_n\} \subset \mathbb{D}$ converging to 1 nontangentially (not necessarily radially); see [8]. Recently, Theorem 1.1 has been extended in several directions. We refer to [20] for a continuous version of the theorem, to [14, 21] for conditions in terms of boundary behavior of commuting $f, g \in \mathcal{S}$ (that is, $f\circ g = g \circ f$) near their common Denjoy-Wolff point which are sufficient for $f \equiv g$, and to [22] for rigidity under conditions on boundary Schwarzian derivatives. All mentioned results establish the rigidity properties subject to functions' behavior near one boundary point. The multi-point case was considered in [2, 10, 16]. The next theorem appears in [10].

Theorem 1.2. *Let $f \in \mathcal{S}$ and let g be a finite Blaschke product of degree d. Let $t_1, \ldots, t_n$ be points on $\mathbb{T}$ and let*

$$f(z) = g(z) + o(|z-t_i|^{m_i}) \quad \textit{for} \quad i = 1, \ldots, n \tag{1.2}$$

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as z tends to t_i nontangentially and where $m_1,\dots,m_n$ are nonnegative integers. If

$$\left[\frac{m_1+1}{2}\right]+\ldots+\left[\frac{m_n+1}{2}\right]>d=\deg\, g, \tag{1.3}$$

then $f(z)\equiv g(z)$. Otherwise, the uniqueness fails.

In (1.3), $[x]$ denotes the largest integer that does not exceed a real number x. The last statement in Theorem 1.2 means: if condition (1.3) fails for a finite Blaschke product g and nonnegative integers $m_1,\dots,m_n$, then for every choice of n points $t_1,\dots,t_n\in\mathbb{T}$, there are infinitely many functions $f\in\mathcal{S}$ subject to (1.2). Thus, conditions (1.2) are minimal. Theorem 1.2 can be supplemented by the following result which is a fairly straightforward consequence of [11, Theorem 2.1].

Proposition 1.3. *Let us assume that a rational function $g\in\mathcal{S}$ is not a finite Blaschke product. Then for every choice of integers n, $m_1,\dots,m_n$, there are infinitely many functions $f\in\mathcal{S}$ satisfying conditions* (1.2).

Observe that upon identifying H^∞-functions with their boundary functions obtained via nontangential boundary limits we may rewrite definition (1.1) as

$$\mathcal{S}:=H^\infty\cap\mathcal{B}L^\infty \tag{1.4}$$

where $\mathcal{B}L^\infty$ denotes the closed unit ball of $L^\infty(\mathbb{T})$. The second component in (1.4) imposing the metric constraint $\|\cdot\|_{L^\infty}\le 1$ is crucial for deducing rigidity conclusions in Theorems 1.1 and 1.2. This constraint makes sense for any meromorphic function having non-tangential boundary limits almost everywhere on $\mathbb{T}$. The objective of this note is to verify for which classes Ω of such functions, the L^∞-norm constraint provides the rigidity of the following sort:

(R): *If g is a rational function in $\Omega\cap\mathcal{B}L^\infty$ (that is, $g\in\Omega$ and $|g|\le 1$ on $\mathbb{T}$) and if $f\in\Omega\cap\mathcal{B}L^\infty$ satisfies conditions* (1.2) *for some choice of integers n and $m_1,\dots,m_n$ (perhaps depending on g), then $f\equiv g$.*

Notation: For the rest of the paper we fix the following notation.

1. $\mathcal{B}_\kappa$ – the set of all Blaschke products of degree κ.
2. $\mathcal{B}_p/\mathcal{B}_q$ – the set of all coprime quotients $g=b/\theta$ with $b\in\mathcal{B}_p$ and $\theta\in\mathcal{B}_q$, i.e., the set of all rational functions g unimodular on $\mathbb{T}$ and with p zeros and q poles in $\mathbb{D}$ (counted with multiplicities).
3. $\mathcal{S}_\kappa$ – the generalized Schur class (introduced in [18]) consisting of all coprime quotients of the form $f=s/b$ where $s\in\mathcal{S}$ and $b\in\mathcal{B}_\kappa$.
4. $\mathcal{S}_{\le\kappa}:=\bigcup_{r\le\kappa}\mathcal{S}_r$, the set of quotients as in (3), but not necessarily coprime.
5. $Z(f)$ – the zero set of a function f.

The paper is organized as follows. In Section 2 we show that rigidity of type (**R**) cannot occur for meromorphic function beyond generalized Schur classes and we establish such rigidity property for the class $\mathcal{S}_{\le\kappa}$ (Theorem 2.1). As we will see, this meromorphic result follows directly from its particular case covered by Theorem 1.2. Being specialized to the single-point case, Theorem 2.1 gives a rigidity condition in terms of a single asymptotic expansion (Corollary 2.4). We then compare it with another single-point rigidity result recently established in [2]. In Section 4 we prove minimality of conditions in Theorem 2.1 using some results on boundary interpolation which are collected in Section 3. In Section 5 we formulate the analogs of Theorem 2.1 for generalized Carathéodory and generalized Nevanlinna functions where again the proofs will follow directly from Theorem 1.2. We will conclude this note with several open questions which are listed in Section 6.

2. Rigidity for generalized Schur functions

The assumption that $g \in \mathcal{B}L^\infty$ is rational together with conditions (1.2) guarantee that the boundary limits of $f^{(j)}(z)$ exist for $j = 0, \ldots, m_i$ as z tends to t_i nontangentially and moreover,

$$\lim_{z \to t_i} f^{(j)}(z) = g^{(j)}(t_i) \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad j = 0, \ldots, m_i. \tag{2.1}$$

As for the choice of Ω, it looks quite natural to start with $N(\mathbb{D})$, the class of meromorphic functions of bounded type. These functions have nontangential boundary limits almost everywhere on $\mathbb{T}$ and $N(\mathbb{D}) \cap \mathcal{B}L^\infty$ consists of quotients $f = s/b$ where $s \in \mathcal{S}$ and b is a Blaschke product. Let us show that statement (**R**) cannot hold in this setting. Indeed, if $g \in \mathcal{B}L^\infty$ is rational, then it is of the form $g = s_g/b_g$ where s_g is a rational function from $\mathcal{S}$ and b_g is a finite Blaschke product. If θ is a finite Blachke product of degree exceeding the total number of interpolation conditions in (2.1), then there exist infinitely many functions $s \in \mathcal{S}$ such that

$$\lim_{z \to t_i} s^{(j)}(z) = (s_g \cdot \theta)^{(j)}(t_i) \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad j = 0, \ldots, m_i, \tag{2.2}$$

and for every such s, the function $f := \dfrac{s}{b_g \theta}$ satisfies conditions (2.1) and clearly belongs to $N(\mathbb{D}) \cap \mathcal{B}L^\infty$. The existence of infinitely many functions s satisfying (2.2) follows from Theorem 1.2 in case s_g is a finite Blaschke product and from Proposition 1.3 otherwise; in the latter case, θ is not needed.

The latter argument shows that rigidity of type (**R**) cannot be achieved even if we restrict Ω to the class of meromorphic functions with finitely many poles. Let us further reduce Ω to the set of meromorphic functions of bounded type with a fixed bound (say, κ) on the total pole multiplicity. Thus we arrive at the class H^∞_κ introduced in [1] and consisting of all quotients $f = s/b$ where $s \in H^\infty$ and $b \in \mathcal{B}_\kappa$. It follows from this definition and from definition of $\mathcal{S}_\kappa$ that $\mathcal{S}_\kappa = (H^\infty_\kappa \backslash H^\infty_{\kappa-1}) \cap \mathcal{B}L^\infty$ so that

$$H^\infty_\kappa \cap \mathcal{B}L^\infty = \{f = \frac{s}{b} : \;\; s \in \mathcal{S}, \; b \in \mathcal{B}_\kappa\} = \bigcup_{r \le \kappa} \mathcal{S}_r =: \mathcal{S}_{\le \kappa}. \tag{2.3}$$

If g is a rational function from $H^\infty_\kappa \cap \mathcal{B}L^\infty$, then it is of the form $g = s_g/b_g$ where s_g is a rational function from $\mathcal{S}$ and $b_g \in \mathcal{B}_\kappa$. If s_g is not a finite Blaschke product, we repeat the preceding argument (with $\theta \equiv 1$) and invoke Proposition 1.3 to conclude that for every choice of n, $m_1, \ldots, m_n$, there are infinitely many functions f of the form $f = s/b_g$ with $s \in \mathcal{S}$ satisfying (1.2). Thus, a rigidity property of type (**R**) for $f, g \in H^\infty_\kappa \cap \mathcal{B}L^\infty$ and g being a quotient of two finite Blaschke products is all we may have, and the next theorem shows that we indeed have it.

Theorem 2.1. *Let κ, p, q be nonnegative integers and let $t_1, \ldots, t_n$ be n distinct points on $\mathbb{T}$, let $g \in \mathcal{B}_p/\mathcal{B}_q$ and let us assume that a function $f \in H^\infty_\kappa \cap \mathcal{B}L^\infty$ satisfies conditions*

$$f(z) = g(z) + o(|z - t_i|^{m_i}) \quad \textit{for} \quad i = 1, \ldots, n \tag{2.4}$$

as z tends to t_i nontangentially, for some nonnegative integers $m_1,\ldots,m_n$. If

$$\left[\frac{m_1+1}{2}\right]+\ldots+\left[\frac{m_n+1}{2}\right]>\kappa+p, \tag{2.5}$$

then $f\equiv g$ on $\mathbb{D}$.

Proof: Substituting coprime quotient representations for f and g

$$f(z)=\frac{s_f(z)}{b_f(z)}\quad (s_f\in\mathcal{S},\ \ b_f\in\mathcal{B}_\kappa)\quad\text{and}\quad g(z)=\frac{b(z)}{\theta(z)}\quad (b\in\mathcal{B}_p,\ \ \theta\in\mathcal{B}_q) \tag{2.6}$$

into (2.4) and then multiplying both sides in (2.4) by $b_f\cdot\theta\in\mathcal{B}_{\kappa+q}$ we get

$$s_f(z)\theta(z)=b(z)b_f(z)+o(|z-t_i|^{m_i})\quad\text{for}\quad i=1,\ldots,n. \tag{2.7}$$

Since $s_f\cdot\theta\in\mathcal{S}$, $b\cdot b_f\in\mathcal{B}_{\kappa+p}$ and since by (2.5), $\sum_{i=1}^{n}\left[\frac{m_i+1}{2}\right]>\kappa+p=\deg(b\cdot b_f)$, we conclude from (2.7) by Theorem 1.2 that $s_f\cdot\theta\equiv b\cdot b_f$ which is equivalent, by (2.6), to $f\equiv g$. □

Remark 2.2. Observe that the membership $f\in H^\infty_\kappa\cap\mathcal{B}L^\infty$ means that total pole multiplicity of f does not exceed κ. Although in Theorem 2.1, we allow f and g to have different pole multiplicities, this possibility cannot be realized under conditions (2.5).

Remark 2.3. If we impose the additional restriction that f and g have the same pole multiplicity q, then the integer κ in (2.5) can be replaced by q. If we denote by d the MacMillan degree of g (which clearly is equal to $\deg b+\deg\theta=p+q$), then condition (2.5) looks pretty much the same as (1.3).

Being specialized to the case $n=1$, Theorem 2.1 gives the following.

Corollary 2.4. *Let κ,p,q be nonnegative integers, let $g\in\mathcal{B}_p/\mathcal{B}_q$ and let $f\in H^\infty_\kappa\cap\mathcal{B}L^\infty$ be such that*

$$f(z)=g(z)+o(|z-t_0|^{2\kappa+2p+1}) \tag{2.8}$$

as z tends to $t_0\in\mathbb{T}$ nontangentially. Then $f\equiv g$ on $\mathbb{D}$.

For the proof, it is enough to notice that the least integer m satisfying inequality $\left[\frac{m+1}{2}\right]>\kappa+p$ is $m=2\kappa+2p+1$.

We now recall a recent result from [2] where rigidity for functions in $H^\infty_\kappa\cap\mathcal{B}L^\infty$ was established under a slightly stronger condition than (2.8).

Theorem 2.5. *Let t_0 be a point on $\mathbb{T}$ and let us assume that the numbers $\tau_0\in\mathbb{T}$ and $\tau_k,\tau_{k+1},\ldots,\tau_{2k-1}\in\mathbb{C}$ are such that the matrix $\mathbb{P}=\overline{\tau}_0TB$ is Hermitian, where T is the lower triangular Toeplitz matrix with the bottom row equal $[\tau_{2k-1}\ \tau_{2k-2}\ \ldots\ \tau_{k+1},\ \tau_k]$ and $B=[b_{ij}]_{i,j=1}^k$ is the $k\times k$ right lower triangular matrix with the entries*

$$b_{ij}=\begin{cases}0, & \text{if}\quad 2\le i+j\le k,\\ (-1)^{j-1}\binom{j-1}{j+i-k-1}t_0^{j+k-1}, & \text{if}\quad k+1\le i+j\le 2k.\end{cases}$$

Let $g(z)$ be the function defined by

$$g(z)=\frac{a(z)x+b(z)}{c(z)x+d(z)} \tag{2.9}$$

where $x \in \mathbb{T}\backslash\{\tau_0\}$,

$$\begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix} = I_2 - \frac{(1-z\overline{z}_0)p(z)}{(1-z\overline{t}_0)^k} \begin{bmatrix} 1 & -\tau_0 \\ \overline{\tau}_0 & -1 \end{bmatrix},$$

where $z_0 \neq t_0$ *is an arbitrary point on* $\mathbb{T}$ *and* $p(z)$ *is the polynomial (note that the matrix* $\mathbb{P}$ *is invertible by construction) given by* $p(z) = (1-z\overline{t}_0)^k)R(z)\mathbb{P}^{-1}R(z_0)^*$ *where* $R(z) = \begin{bmatrix} \frac{1}{1-z\overline{t}_0} & \frac{z}{(1-z\overline{t}_0)^2} & \dots & \frac{z^{k-1}}{(1-z\overline{t}_0)^k} \end{bmatrix}$. *Then*

(1) *The function* g *is the quotient of two finite Blaschke products with* r *poles in* $\mathbb{D}$ *(where* r *is the number of negative eigenvalues of the matrix* $\mathbb{P}$*) and with the following Taylor expansion at* t_0*:*

$$g(z) = \tau_0 + \sum_{i=k}^{2k-1} \tau_i (z-t_0)^i + O(|z-t_0|^{2k}). \tag{2.10}$$

(2) *If* f *is a function from* $H^\infty_\kappa \cap \mathcal{B}L^\infty$ *with* r *poles in* $\mathbb{D}$ *and such that*

$$f(z) = g(z) + O(|z-t_0|^{2k+2}), \tag{2.11}$$

then $f \equiv g$.

To embed Theorem 2.5 into our framework we first recall that for every quotient of two finite Blaschke products with the Taylor expansion (2.10), the matrix $\mathbb{P}$ constructed in the theorem is necessarily Hermitian and $\tau_0 = g(t_0)$ is unimodular (see [13, Section 2]. On the other hand, it follows from general results from [7, Section 21] that formula (2.9) parametrizes *all* functions $g \in \mathcal{B}_{k-r}/\mathcal{B}_r$ of the form (2.10). Therefore, the rigidity part in Theorem 2.5 can be reformulated equivalently in the following more compact form.

Theorem 2.6. *Let* $g \in \mathcal{B}_{k-r}/\mathcal{B}_r$ *admit the Taylor expansion* (2.10) *at* $t_0 \in \mathbb{T}$. *If* $f \in H^\infty_r \cap \mathcal{B}L^\infty$ *satisfies the nontangential asymptotic condition* (2.11)*, then* $f \equiv g$.

The main limitation in Theorem 2.6 is that g has quite special Taylor coefficients at t_0 ($\tau_1 = \tau_2 = \ldots = \tau_{k-1} = 0$) (observe that the original Burns-Krantz theorem is of a different type, since there we have $\tau_1 = 1$ and $\tau_2 = \tau_3 = 0$; however it was shown in [2, Section 4] that Theorem 1.1 can be deduced from Theorem 2.5). Corollary 2.4 shows that rigidity holds for *any* quotient of finite Blaschke products. Besides, Corollary 2.4 shows that the term $O(|z-t_0|^{2k+2})$ in (2.11) can be relaxed to $o(|z-t_0|^{2k+1})$, that the order of approximation can be of any parity (not necessarily even) and that rigidity may hold also in case where only a bound for the pole multiplicity of f is known.

Remark 2.7. Observe that Theorem 2.1 does not discuss the optimality of conditions (2.5) and in this regard, it is not a full extent analog of Theorem 1.2. As we will see below, conditions (2.5) are indeed optimal in the sense that if the integers κ, p, q and m_i's are not as in Theorem 2.1, then the rigidity cannot be guaranteed by conditions (2.5).

Let us observe that if g belongs to $\mathcal{B}_p/\mathcal{B}_q$ and $\kappa \neq q$, then rigidity occurs for no function f in $\mathcal{S}_\kappa$ (for the simple reason that f and g have different pole multiplicities) no matter what conditions are imposed. Thus, it suffices to consider the case where $g \in \mathcal{B}_p/\mathcal{B}_q$ and $f \in \mathcal{S}_q$. This remaining case is covered by the following theorem.

Theorem 2.8. *Let p, q, $m_1, \ldots, m_n$ be nonnegative integers and let $g \in \mathcal{B}_p/\mathcal{B}_q$. If*

$$\left[\frac{m_1+1}{2}\right] + \ldots + \left[\frac{m_n+1}{2}\right] \le p+q, \tag{2.12}$$

then for every choice of $t_1, \ldots, t_n \in \mathbb{T}$, there are infinitely many functions $f \in \mathcal{S}_q$ subject to equalities

$$f(z) = g(z) + O(|z-t_i|^{m_i+1}) \quad for \quad i = 1, \ldots, n, \tag{2.13}$$

where z tends to t_i unrestrictedly in $\mathbb{D}$.

The theorem states that if the total order of contact of f and g on $\mathbb{T}$ is not large enough (this is the meaning of condition (2.12)), then we cannot guarantee that $f \equiv g$ even if conditions (2.5) are replaced by slightly stronger conditions (2.13) with the arbitrary (rather than the nontangential) convergence of z to t_i. The proof of Theorem 2.8 will be given in Section 4.

3. Boundary Schwarz-Pick matrices and related interpolation

In this section we collect some preliminary facts needed for the proof of Theorem 2.8. Let g be a meromorphic function with the domain of holomorphy $\mathrm{Dom}(g)$. For every positive integer n, and every n-tuple $\mathbf{z} = (z_1, \ldots, z_n)$ of points in $\mathbb{D} \cap \mathrm{Dom}(g)$ taken with multiplicities k_i from another n-tuple $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$, we let $|\mathbf{k}| := k_1 + \ldots + k_n$ and introduce the $|\mathbf{k}| \times |\mathbf{k}|$ matrix

$$\mathbb{P}^g_{\mathbf{k}}(\mathbf{z}) = \left[\mathbb{P}^g_{k_i,k_j}(z_i, z_j)\right]_{i,j=1}^n \tag{3.1}$$

with the $k_i \times k_j$ block entries

$$\mathbb{P}^g_{k_i,k_j}(z_i, z_j) = \left[\frac{1}{\ell! r!} \frac{\partial^{\ell+r}}{\partial z^\ell \partial \bar{\zeta}^r} \frac{1 - g(z)\overline{g(\zeta)}}{1 - z\bar{\zeta}}\bigg|_{\substack{z = z_i \\ \zeta = \bar{z}_j}}\right]_{\ell=0,\ldots,k_i-1}^{r=0,\ldots,k_j-1}. \tag{3.2}$$

The matrix $P^w_{\mathbf{k}}(\mathbf{z})$ which will be referred to as to a *Schwarz-Pick matrix*, is Hermitian. Its definition can be extended to the boundary setting as follows: given $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{T}^n$, the *boundary Schwarz-Pick matrix* is defined by

$$\mathbb{P}^g_{\mathbf{k}}(\mathbf{t}) = \left[\mathbb{P}^g_{k_i,k_j}(t_i, t_j)\right]_{i,j=1}^n := \lim_{\mathbf{z} \to \mathbf{t}} \mathbb{P}^g_{\mathbf{k}}(\mathbf{z}) \tag{3.3}$$

as $z_i \in \mathbb{D}$ tends to t_i nontangentially for $i = 1, \ldots, n$, provided the limit in (3.3) exists ("the limit exists" also means that it is finite).

Remark 3.1. Since $\mathbb{P}^g_{\mathbf{k}}(\mathbf{z})$ is Hermitian, the boundary Schwarz-Pick matrix is Hermitian whenever it exists.

It is readily seen from the formula for the bottom diagonal entry in $\mathbb{P}^g_{k_i,k_i}(z_i, z_i)$ that higher order Carathéodory-Julia conditions

$$\liminf_{z \to t_i} \frac{\partial^{2k_i-2}}{\partial z^{k_i-1} \partial \bar{z}^{k_i-1}} \frac{1 - |g(z)|^2}{1 - |z|^2} < \infty \quad \text{for} \quad i = 1, \ldots, n, \tag{3.4}$$

where $z \in \mathbb{D}$ tends to t_i arbitrarily (not necessarily nontangentially) are necessary for the limit (3.3) to exist. These conditions are also sufficient for functions in $\mathcal{S}_{\le \kappa}$ as the following theorem shows.

Theorem 3.2. *Let us assume that $g \in \mathcal{S}_{\le \kappa}$ meets conditions* (3.4). *Then*

(1) *The following nontangential boundary limits exist*

$$g_j(t_i) := \lim_{z\to t_i} \frac{g^{(j)}(z)}{j!} \quad \text{for } j=0,\dots,2k_i-1;\ i=1,\dots,n. \tag{3.5}$$

(2) *The nontangential boundary limit* (3.3) *exists and can be expressed in terms of the limits* (3.5) *as follows:*

$$\mathbb{P}^g_{k_i,k_j}(t_i,t_j) = \mathbb{H}^g_{k_i,k_j}(t_i,t_j)\boldsymbol{\Psi}_{k_j}(t_j)\mathbb{U}^g_{k_j}(t_j)^* \tag{3.6}$$

where $\boldsymbol{\Psi}_{k_j}(t_j)$ *is the* $k_j\times k_j$ *upper triangular matrix with the entries*

$$\psi_{\ell r} = \begin{cases} 0, & \text{if } \ell > r \\ (-1)^r \begin{pmatrix} r \\ \ell \end{pmatrix} t_0^{\ell+r+1}, & \text{if } \ell \le r \end{cases} \qquad (\ell, r = 0,\dots,k_j-1), \tag{3.7}$$

where $\mathbb{U}^g_{k_j}(t_j)$ *is the lower triangular Toeplitz matrix:*

$$\mathbb{U}^g_{k_j}(t_j) = \begin{bmatrix} g_0(t_j) & 0 & \dots & 0 \\ g_1(t_j) & g_0(t_j) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ g_{k_j-1}(t_j) & \dots & g_1(t_j) & g_0(t_j) \end{bmatrix},$$

and where $\mathbb{H}^g_{k_i,k_j}(t_i,t_j)$ *is defined for* $i=j$ *as the Hankel matrix*

$$\mathbb{H}^g_{k_i,k_i}(t_i,t_i) = [g_{\ell+r-1}(t_i)]_{\ell,r=1}^{k_i} \tag{3.8}$$

and entrywise (if $i\neq j$*) by*

$$\begin{aligned} \left[\mathbb{H}_{k_i,k_j}(t_i,t_j)\right]_{\ell,r} &= \sum_{\alpha=0}^{\ell}(-1)^{\ell-\alpha}\begin{pmatrix} \ell+r-\alpha \\ r \end{pmatrix}\frac{g_\alpha(t_i)}{(t_i-t_j)^{\ell+r-\alpha+1}} \\ &\quad -\sum_{\beta=0}^{r}(-1)^{\ell}\begin{pmatrix} \ell+r-\beta \\ \ell \end{pmatrix}\frac{g_\beta(t_j)}{(t_i-t_j)^{\ell+r-\beta+1}} \end{aligned} \tag{3.9}$$

for $\ell=0,\dots,k_i-1$ *and* $r=0,\dots,k_j-1$.

The case $n=1$ was considered in [13] (see Theorem 4.2 there; we also refer to [12] where conditions (3.4) were first introduced and studied). Applying the single-point version to each point t_i individually we get the first statement in Theorem 3.2 and the existence of the angular limits $\lim_{z_i\to t_i} \mathbb{P}^g_{k_i,k_i}(z_i,z_i)$ for the diagonal blocks in $\mathbb{P}^g_{\mathbf{k}}(\mathbf{z})$. The direct differentiation in (3.2) gives

$$\begin{aligned} \left[\mathbb{P}^g_{k_i,k_j}(z_i,z_j)\right]_{\ell,r} &= \sum_{s=0}^{\min\{\ell,r\}} \frac{(\ell+r-s)!}{(\ell-s)!s!(r-s)!}\frac{z_i^{r-s}\bar{z}_j^{\ell-s}}{(1-z_i\bar{z}_j)^{\ell+r-s+1}} \\ &- \sum_{\alpha=0}^{\ell}\sum_{\beta=0}^{r}\sum_{s=0}^{\min\{\alpha,\beta\}} \frac{(\alpha+\beta-s)!}{(\alpha-s)!s!(\beta-s)!}\frac{z_i^{\beta-s}\bar{z}_j^{\alpha-s}g_{\ell-\alpha}(z_i)g_{r-\beta}(z_j)^*}{(1-z_i\bar{z}_j)^{\alpha+\beta-s+1}}. \end{aligned}$$

For $i \neq j$, we pass to the limit in the latter equality as $z_i \to t_i$ and $z_j \to t_j$ and take into account (3.5):

$$\begin{aligned}\left[\mathbb{P}^g_{k_i,k_j}(t_i,t_j)\right]_{\ell,r} &= \sum_{s=0}^{\min\{\ell,r\}} \frac{(\ell+r-s)!}{(\ell-s)!s!(r-s)!}\frac{t_i^{r-s}\bar{t}_j^{\ell-s}}{(1-t_i\bar{t}_j)^{\ell+r-s+1}} \qquad (3.10)\\ &- \sum_{\alpha=0}^{\ell}\sum_{\beta=0}^{r}\sum_{s=0}^{\min\{\alpha,\beta\}} \frac{(\alpha+\beta-s)!}{(\alpha-s)!s!(\beta-s)!}\frac{t_i^{\beta-s}\bar{t}_j^{\alpha-s}g_{\ell-\alpha}(t_i)g_{r-\beta}(t_j)^*}{(1-t_i\bar{t}_j)^{\alpha+\beta-s+1}}.\end{aligned}$$

Verification of the fact that the product on the right hand side of (3.6) gives the matrix with the same entries as in (3.10), is straightforward and will be omitted. □

Remark 3.3. Combining (3.3) and (3.5) gives the following factorization for the matrix $\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})$ (when the latter exists):

$$\mathbb{P}^g_{\mathbf{k}}(\mathbf{t}) = \mathbb{H}^g_{\mathbf{k}}(\mathbf{t})\boldsymbol{\Psi}_{\mathbf{k}}(\mathbf{t})\mathbb{U}^g_{\mathbf{k}}(\mathbf{t})^*, \qquad (3.11)$$

where we have set

$$\mathbb{U}^g_{\mathbf{k}}(\mathbf{t}) = \begin{bmatrix} \mathbb{U}^g_{k_1}(t_1) & & 0 \\ & \ddots & \\ 0 & & \mathbb{U}^g_{k_n}(t_n)\end{bmatrix}, \quad \boldsymbol{\Psi}_{\mathbf{k}}(\mathbf{t}) = \begin{bmatrix} \boldsymbol{\Psi}_{k_1}(t_1) & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{\Psi}_{k_n}(t_n)\end{bmatrix} \qquad (3.12)$$

to be block diagonal matrices and where

$$\mathbb{H}^g_{\mathbf{k}}(\mathbf{t}) = \left[\mathbb{H}_{k_i,k_j}(t_i,t_j)\right]_{i,j=1}^n. \qquad (3.13)$$

Remark 3.4. If g is analytic at $t_1,\ldots,t_n$, one can define the *generalized Löwner matrix*

$$\mathbb{L}^g_{\mathbf{k}}(\mathbf{t}) = \left[\left[\frac{1}{\ell!r!}\frac{\partial^{\ell+r}}{\partial z^\ell \partial\zeta^r}\frac{g(z)-g(\zeta)}{z-\zeta}\bigg|_{\substack{z=t_i\\ \zeta=t_j}}\right]_{\ell=0,\ldots,k_i-1}^{r=0,\ldots,k_j-1}\right]_{i,j=1}^n \qquad (3.14)$$

commonly known for the central role it plays in minimal rational interpolation (see [3]–[5]), [9]. Upon expressing the entries in (3.14) in terms of the Taylor coefficients $g_j(t_i)$ of g at t_i's and comparing them with formulas (3.8) and (3.9) one can easily conclude that $\mathbb{H}^g_{\mathbf{k}}(\mathbf{t}) = \mathbb{L}^g_{\mathbf{k}}(\mathbf{t})$.

In what follows, we will write $\pi(P)$ and $\nu(P)$ for the numbers of positive and negative eigenvalues, counted with multiplicities, of a Hermitian matrix P.

Lemma 3.5. *Let $g \in \mathcal{B}_p/\mathcal{B}_q$ and let two n-tuples $\mathbf{t} \in \mathbb{T}^n$ and $\mathbf{k} \in \mathbb{N}^n$ be given.*

1. *If $|\mathbf{k}| = p+q$, then the boundary Schwarz-Pick matrix $\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})$ is invertible and moreover,*

$$\pi(\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})) = p \quad \text{and} \quad \nu(\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})) = q. \qquad (3.15)$$

2. *If $|\mathbf{k}| \geq p+q$, then* $\operatorname{rank}(\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})) = p+q$.

Proof: By the assumptions of the lemma, g is of the form

$$g(z) = \frac{b(z)}{\theta(z)} \quad \text{where} \quad b \in \mathcal{B}_p, \;\; \theta \in \mathcal{B}_q \quad \text{and} \quad Z(b)\cap Z(\theta) = \emptyset, \qquad (3.16)$$

and therefore, it satisfies condition (3.4) for every $t_i \in \mathbb{T}$ and every $k_i \geq 1$. Then the boundary Schwarz-Pick matrix $\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})$ exists by Theorem 3.2 and is of the form (3.11), by Remark 3.3. Observe that the upper triangular matrices $\mathbb{U}^g_{\mathbf{k}}(\mathbf{t})$ and $\boldsymbol{\Psi}_{\mathbf{k}}(\mathbf{t})$ are invertible since all their diagonal entries are respectively of the form $\pm t_i^j$ and $g(t_i)$ and therefore, they are unimodular. Thus,

$$\operatorname{rank}(\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})) = \operatorname{rank}(\mathbb{H}^g_{\mathbf{k}}(\mathbf{t})).$$

Since g is analytic on $\mathbb{T}$, the generalized Löwner matrix $\mathbb{L}^g_{\mathbf{k}}(\mathbf{t})$ exists and equals $\mathbb{H}^g_{\mathbf{k}}(\mathbf{t})$, by Remark 3.4. It follows from (3.16), that the MacMillan degree of g equals $\deg g = p+q$. On the other hand, if $|\mathbf{k}| \geq \deg g$, then $\operatorname{rank}(\mathbb{L}^g_{\mathbf{k}}(\mathbf{t}))) = \deg g$, according to Lemma 2.5 in [3]. Summarizing we conclude that whenever $|\mathbf{k}| \geq \deg g = p+q$, we have

$$\operatorname{rank}(\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})) = \operatorname{rank}\left(\mathbb{H}^g_{\mathbf{k}}(\mathbf{t})\right) = \operatorname{rank}\left(\mathbb{L}^g_{\mathbf{k}}(\mathbf{t})\right) = \deg\, g = p+q, \tag{3.17}$$

which proves the second statement of the lemma. Recall that $\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})$ is a $|\mathbf{k}| \times |\mathbf{k}|$ matrix. Thus, if $|\mathbf{k}| = r+\ell$, then it follows from (3.17) that the matrix $\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})$ is invertible. To prove (3.15) we first check the equality

$$\mathbb{P}^b_{\mathbf{k}}(\mathbf{t}) - \mathbb{P}^\theta_{\mathbf{k}}(\mathbf{t}) = \mathbb{U}^\theta_{\mathbf{k}}(\mathbf{t})\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})\mathbb{U}^\theta_{\mathbf{k}}(\mathbf{t})^* \tag{3.18}$$

where $b \in \mathcal{B}_p$ and $\theta \in \mathcal{B}_q$ are the finite Blaschke products from representation (3.16) of g and the matrices $\mathbb{P}^b_{\mathbf{k}}(\mathbf{t})$, $\mathbb{P}^\theta_{\mathbf{k}}(\mathbf{t})$ and $\mathbb{U}^b_{\mathbf{k}}(\mathbf{t})$ are defined via formulas (3.11)–(3.13). To prove (3.18), we apply $\frac{1}{\ell!r!}\frac{\partial^{\ell+r}}{\partial z^\ell \partial \zeta^r}$ to both parts of the self-evident identity

$$\frac{1-b(z)\overline{b(\zeta)}}{1-z\bar{\zeta}} - \frac{1-\theta(z)\overline{\theta(\zeta)}}{1-z\bar{\zeta}} = \theta(z)\frac{1-g(z)\overline{g(\zeta)}}{1-z\bar{\zeta}}\overline{\theta(\zeta)}$$

and evaluating the resulting identity at $z = z_i$ and $\zeta = t_j$ for all needed values of i, j, ℓ and r, we get equalities between the corresponding entries in the matrix equality (3.18). Since the matrix $\mathbb{U}^\theta_{\mathbf{k}}(\mathbf{t})$ is invertible, it follows from (3.18) that

$$\pi(\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})) = \pi\left(\mathbb{P}^b_{\mathbf{k}}(\mathbf{t}) - \mathbb{P}^\theta_{\mathbf{k}}(\mathbf{t})\right) \quad \text{and} \quad \nu(\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})) = \nu\left(\mathbb{P}^b_{\mathbf{k}}(\mathbf{t}) - \mathbb{P}^\theta_{\mathbf{k}}(\mathbf{t})\right). \tag{3.19}$$

Since b is a finite Blaschke product, it follows that the boundary Schwarz-Pick matrix $\mathbb{P}^b_{\mathbf{k}}(\mathbf{t})$ is positive semidefinite and satisfies

$$\pi(\mathbb{P}^b_{\mathbf{k}}(\mathbf{t})) = \operatorname{rank}(\mathbb{P}^b_{\mathbf{k}}(\mathbf{t})) = \min\{|\mathbf{k}|, \deg b\} = p, \tag{3.20}$$

where the second equality holds by [10, Lemma 2.1] and the others are evident. Similarly,

$$\pi(\mathbb{P}^\theta_{\mathbf{k}}(\mathbf{t})) = \operatorname{rank}(\mathbb{P}^\theta_{\mathbf{k}}(\mathbf{t})) = \min\{|\mathbf{k}|, \deg \theta\} = q. \tag{3.21}$$

Combining (3.19)–(3.21) gives

$$\pi(\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})) \leq \pi(\mathbb{P}^b_{\mathbf{k}}(\mathbf{t})) + \nu(\mathbb{P}^\theta_{\mathbf{k}}(\mathbf{t})) = p, \quad \nu(\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})) \leq \nu(\mathbb{P}^b_{\mathbf{k}}(\mathbf{t})) + \pi(\mathbb{P}^\theta_{\mathbf{k}}(\mathbf{t})) = q, \tag{3.22}$$

and since $\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})$ is invertible, i.e., since

$$\pi(\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})) + \nu(\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})) = |\mathbf{k}| = p+q,$$

inequalities (3.22) imply (3.15). □

We now put boundary Schwarz-Pick matrices in the interpolation context. Given

$$\mathbf{t} = \{t_1, \ldots, t_n\} \in \mathbb{T}^n, \quad \mathbf{k} = \{k_1, \ldots, k_n\} \in \mathbb{N}^n, \quad \{b_{ij}\}_{j=0,\ldots,2k_i-1}^{i=1,\ldots,n} \ (b_{ij} \in \mathbb{C}), \tag{3.23}$$

define the $|\mathbf{k}|\times|\mathbf{k}|$ matrix P by formulas similar to (3.11)–(3.13), but with $g_j(t_i)$ replaced by b_{ij}:

$$P=[P_{ij}]_{i,j=1}^n \quad \text{with} \quad P_{ij}=H_{ij}\cdot\mathbf{\Psi}_{k_j}(t_j)\cdot U_j^*, \tag{3.24}$$

where $\mathbf{\Psi}_{k_j}(t_j)$ is the upper triangular matrix with the entries given in (3.7), where U_j is the lower triangular Toeplitz matrix and H_{ii} is the Hankel matrix defined by

$$U_j=\begin{bmatrix} b_{j,0} & & 0 \\ \vdots & \ddots & \\ b_{j,k_j-1} & \dots & b_{j,0}\end{bmatrix}, \quad H_{ii}=\begin{bmatrix} b_{i,1} & \cdots & b_{i,k_i} \\ \vdots & & \vdots \\ b_{i,k_i} & \cdots & b_{i,2k_i-1}\end{bmatrix} \tag{3.25}$$

for $i=1,\dots,n$ and where the matrices H_{ij} (for $i\neq j$) are defined entrywise by

$$\begin{aligned}[H_{ij}]_{\ell,r} \;&=\; \sum_{\alpha=0}^{\ell}(-1)^{\ell-\alpha}\begin{pmatrix}\ell+r-\alpha\\ r\end{pmatrix}\frac{b_{i,\alpha}}{(t_i-t_j)^{\ell+r-\alpha+1}} \\ &\quad -\sum_{\beta=0}^{r}(-1)^{\ell}\begin{pmatrix}\ell+r-\beta\\ \ell\end{pmatrix}\frac{b_{j,\beta}}{(t_i-t_j)^{\ell+r-\beta+1}}.\end{aligned} \tag{3.26}$$

Theorem 3.6. *Let P be constructed from data* (3.23) *by formulas* (3.24)–(3.26). *For the existence of infinitely many functions $g\in\mathcal{B}_{\pi(P)}/\mathcal{B}_{\nu(P)}$ subject to interpolation conditions*

$$g_j(t_i):=\frac{g^{(j)}(t_i)}{j!}=b_{ij} \quad (j=0,\dots,2k_i-1;\; i=1,\dots,n), \tag{3.27}$$

it is necessary and sufficient that

$$P=P^*, \quad \det P\neq 0 \quad \textit{and} \quad |b_{i,0}|=1 \quad \textit{for} \quad i=1,\dots,n. \tag{3.28}$$

The matrix P constructed in (3.24)–(3.26) is called *the Pick matrix* of the interpolation problem with data (3.23) and interpolation conditions (3.27).

Proof of Theorem 3.6: For the proof of necessity, let us assume that $g\in\mathcal{B}_{\pi(P)}/\mathcal{B}_{\nu(P)}$ satisfies (3.27). Then the boundary Schwarz-Pick matrix $\mathbb{P}_{\mathbf{k}}^g(\mathbf{t})$ exists and equals P. By Remark 3.1, P is Hermitian. Let us assume that P is singular so that

$$|\mathbf{k}|>\pi(P)+\nu(P)=\deg g. \tag{3.29}$$

To get a contradiction we assume that f is another function in $\mathcal{B}_{\pi(P)}/\mathcal{B}_{\nu(P)}$ satisfying conditions (3.27) (by the assumption of the theorem, there are infinitely many such functions). Then we have

$$f(z)-g(z)=o(|z-t_i|^{2k_i-1}) \quad \text{for} \quad i=1,\dots,n.$$

Since f has the same pole multiplicity as g and since by (3.28),

$$\sum_{i=1}^n\left[\frac{(2k_i-1)+1}{2}\right]=\sum_{i=1}^n k_i=|\mathbf{k}|>\deg g=p+q,$$

it follows from Theorem 2.1 that $f\equiv g$ which is the desired contradiction. Thus, P is invertible. The necessity of equalities $|b_{i0}|=1$ is obvious since $|g|=1$ on $\mathbb{T}$. The proof of sufficiency can be found in [7, Chapter 21]. □

In fact much more was done in [7]: assuming that the necessary conditions (3.28) are satisfied, the set of *all* rational functions $f \in \mathcal{S}_{\nu(P)}$ satisfying conditions

$$f_j(t_i) = g_j(t_i) = b_{ij} \quad \text{for } \ j = 0, \ldots, 2k_i - 1 \quad \text{and} \quad i = 1, \ldots, n,$$

was parametrized by the linear fractional formula $f = \dfrac{a\mathcal{E} + b}{c\mathcal{E} + d}$ where the coefficient matrix $\Theta = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is a rational function of MacMillan degree $\deg \Theta = |\mathbf{k}|$ and $\mathcal{E}$ is an arbitrary rational Schur class function such that

$$c(t_i)\mathcal{E}(t_i) + d(t_i) \neq 0 \quad \text{for} \quad i = 1, \ldots, n.$$

It is worth mentioning that this description along with the explicit formula for Θ in terms of data (3.23) were established in [7] in a more general bitangential matrix-valued setting. For the proof of Theorem 2.8 we need one more auxiliary statement.

Proposition 3.7. *Let $\widetilde{P} = [p_{ij}]$ be an $r \times r$ Hermitian matrix and let us assume that its principal submatrix $P = [p_{i_\alpha, i_\beta}]_{\alpha,\beta=1}^{\ell}$ is invertible. Then the $r - \ell$ diagonal entries p_{ii} for $i \notin \{i_1, \ldots, i_\ell\}$ (we will call these entries the diagonal entries of $\widetilde{P}$ complementary to the principal submatrix P) can be modified to produce an invertible matrix $\widetilde{P}'$ such that $\nu(\widetilde{P}') = \nu(P)$.*

Proof: Without loss of generality we can assume that P is the leading principal submatrix of $\widetilde{P}$ so that $\widetilde{P} = \left[\begin{smallmatrix} P & R^* \\ R & D \end{smallmatrix}\right]$. Let us modify the diagonal entries in D as follows

$$\widetilde{P}' = \begin{bmatrix} P & R^* \\ R & D' \end{bmatrix} \quad \text{where } \ D' = D + \rho I_{r-\ell}$$

and let us choose $\rho > 0$ large enough so that the matrix $D' - RP^{-1}R^*$ is positive definite. By the standard Schur complement argument, we then have

$$\nu(\widetilde{P}') = \nu(P) + \nu(D' - RP^{-1}R^*) = \nu(P)$$

and $\det(\widetilde{P}') = \det(P) \cdot \det(D' - RP^{-1}R^*) \neq 0$, which completes the proof. □

4. The proof of Theorem 2.8

Given $g \in \mathcal{B}_p/\mathcal{B}_q$, $t_1, \ldots, t_n \in \mathbb{T}$ and nonnegative integers $m_1, \ldots, m_n$ subject to

$$\left[\frac{m_1 + 1}{2}\right] + \ldots + \left[\frac{m_n + 1}{2}\right] = p + q, \tag{4.1}$$

we will show that there are infinitely many *rational* functions $f \in \mathcal{S}_q$ satisfying asymptotic equalities (2.13) or equivalently, interpolation conditions

$$f_j(t_i) = b_{ij} := g_j(t_i) \quad \text{for} \quad 0 \le j \le m_i; \ \ 1 \le i \le n. \tag{4.2}$$

Observe that for rational f, conditions (2.4), (2.13) and (4.2) are equivalent. Define the integers $k_i := \left[\frac{m_i+1}{2}\right]$ for $i = 1, \ldots, n$ and the tuple $\mathbf{k} = (k_1, \ldots, k_n)$ so that assumption (3.1) takes the form

$$k_1 + \ldots + k_n = |\mathbf{k}| = \kappa + p \tag{4.3}$$

and so that $m_i = 2k_i - 1$ or $m_i = 2k_i$. Reindexing if necessary, we can assume without loss of generality that the first ℓ integers $m_1, \ldots, m_\ell$ are odd while the remaining ones (if any) are even. Now we split conditions (4.2) into two parts:

$$f_j(t_i) = b_{ij} := g_j(t_i) \quad \text{for} \quad 0 \le j \le 2k_i - 1; \ \ i = 1, \ldots, n \tag{4.4}$$

and

$$f_{2k_i}(t_i) = b_{i,2k_i} := g_{2k_i}(t_i) \quad \text{for} \quad i = \ell+1,\dots,n. \tag{4.5}$$

Let us consider the interpolation problem with interpolation conditions (4.4). Its Pick matrix P coincides with the boundary Schwarz-Pick matrix $\mathbb{P}^g_{\mathbf{k}}(\mathbf{t})$. Equality (4.3) allows us to apply Lemma 3.5 and to conclude that P is invertible and satisfies

$$\pi(P) = p, \quad \nu(P) = q. \tag{4.6}$$

If $\ell = n$, so that (4.4) contains all the interpolation conditions we wish to satisfy by f, then we conclude from Theorem 3.6 that there are infinitely many functions $f \in \mathcal{B}_p/\mathcal{B}_q$ satisfying conditions (4.2).

If $\ell < n$ (i.e., if the set of conditions (4.5) is not empty) one more step is needed. In this case we attach interpolation conditions

$$f_{2k_i+1}(t_i) = b_{i,2k_i+1} := g_{i,2k_i+1}(t_i) \quad \text{for} \quad i = \ell+1,\dots,n \tag{4.7}$$

to (4.4) and (4.5) and consider the extended interpolation problem with interpolation conditions (4.4), (4.5) and (4.7). The collection of b_{ij}'s appearing in (4.4) and (4.5) will be called the *original data*, the collection $\{b_{i,2k_i+1}\}$ from (3.10) will be called the *supplementary data* whereas their union will be referred to as to the *extended data*.

For the extended interpolation problem we have an even number of conditions at each interpolating point t_i and its Pick matrix $\widetilde{P}$ equals $\mathbb{P}^g_{\widetilde{\mathbf{k}}}(\mathbf{t})$ where g and $\mathbf{t}$ are the same as above and where

$$\widetilde{\mathbf{k}} = (\widetilde{k}_1,\dots,\widetilde{k}_n) = (k_1,\dots,k_\ell,k_{\ell+1}+1,\dots,k_n+1) \in \mathbb{N}^n.$$

The matrix $\widetilde{P}$ can be written in terms of b_{ij}'s via formulas (3.24):

$$\widetilde{P} = \left[\widetilde{P}_{ij}\right]_{i,j=1}^n \quad \text{where} \quad \widetilde{P}_{ij} = \widetilde{H}_{ij}\cdot \mathbf{\Psi}_{\widetilde{k}_j}(t_j)\cdot \widetilde{U}_j^* \tag{4.8}$$

and where $\widetilde{U}_j$ and $\widetilde{H}_{ij}$ are defined by formulas (3.25), (3.26) with k_i replaced by $\widetilde{k}_i$. It is clear that all the entries in $\widetilde{P}$ are completely determined by the extended data. However, it turns out that all its entries but ℓ diagonal ones are uniquely determined from the *original data*. Indeed, if $i \neq j$, then $\widetilde{H}_{ij}$ and $\widetilde{U}_j$ (and therefore, $\widetilde{P}_{ij}$) are expressed via formulas (3.25), (3.26) in terms of the numbers $b_{i,0},\dots,b_{i,\widetilde{k}_i-1}$ and $b_{j,0},\dots,b_{j,\widetilde{k}_j-1}$ all of which are contained in the original data, since $\widetilde{k}_i - 1 \le k_i \le 2k_i - 1$.

Now we examine the diagonal blocks $\widetilde{P}_{ii}$ for $i > \ell$ (if $i \le \ell$, then $\widetilde{P}_{ii} = P_{ii}$ is completely determined by the original data). By (4.8) and (3.25),

$$\widetilde{P}_{ii} = \begin{bmatrix} b_{i,1} & b_{i,2} & \cdots & b_{i,k_i} \\ b_{i,2} & b_{i,3} & \cdots & b_{i,k_i+1} \\ \vdots & \vdots & & \vdots \\ b_{i,k_i} & b_{i,k_i+1} & \dots & b_{i,2k_i-1} \end{bmatrix} \mathbf{\Psi}_{k_i}(t_i) \begin{bmatrix} \overline{b}_{i,0} & \dots & \overline{b}_{i,k_i-1} \\ & \ddots & \vdots \\ 0 & & \overline{b}_{i,0} \end{bmatrix}. \tag{4.9}$$

It is readily seen from (4.9) that the only entry in $\widetilde{P}_{ii}$ that depends on the supplementary data is the the bottom diagonal entry

$$\gamma_i := \left[\widetilde{P}_{ii}\right]_{k_i,k_i} = \begin{bmatrix} b_{i,k_i} & \cdots & b_{i,2k_i-1} \end{bmatrix} \mathbf{\Psi}_{k_i}(t_i) \begin{bmatrix} b_{i,k_i-1} & \cdots & b_{i,0} \end{bmatrix}^* \tag{4.10}$$

which, on account of (3.7), can be written as

$$\gamma_i = (-1)^{k_i-1} t_i^{2k_i-1} b_{i,2k_i-1}\overline{b}_{i,0} + \Phi(t_i, b_{i,0}, \ldots, b_{i,2k_i-2}) \tag{4.11}$$

where the second term on the right does not depend on $b_{i,2k_i-1}$. Since $\widetilde{P} = \mathbb{P}^g_{\widetilde{\mathbf{k}}}(\mathbf{t})$, it follows that $\widetilde{P}$ is Hermitian. Furthermore, $P = \mathbb{P}^g_{\mathbf{k}}(\mathbf{t})$ is an (invertible) principal submatrix of $\widetilde{P}$ and the diagonal entries in $\widetilde{P}$ complementary to P are exactly γ_i's from (4.11), the bottom diagonal entries in the blocks $\widetilde{P}_{ii}$ of $\widetilde{P}$ for $i = \ell+1, \ldots, n$. By Proposition 3.7, upon replacing γ_i in $\widetilde{P}$ by appropriately chosen (sufficiently large) positive numbers γ_i' (for $i = \ell+1, \ldots, n$) and keeping all the other entries the same, one gets an invertible matrix $\widetilde{P}'$ with $\nu(\widetilde{P}') = \nu(P) = q$. Furthermore, for each chosen γ_i', there exists (the unique) $b'_{i,2k_i-1}$ such that

$$\gamma_i' = (-1)^{k_i-1} t_i^{2k_i-1} b'_{i,2k_i-1}\overline{b}_{i,0} + \Phi(t_i, b_{i,0}, \ldots, b_{i,2k_i-2})$$

where the second term on the right is the same as in (4.11) (since $b_{i,0} \neq 0$, the latter equality can be solved for $b'_{i,2k_i-1}$). Now we replace the supplementary interpolation conditions (4.7) by

$$f_{2k_i+1}(t_i) = b'_{i,2k_i+1} \quad \text{for } \ i = \ell+1, \ldots, n \tag{4.12}$$

where the numbers on the right have nothing to do with the function g anymore. The Pick matrix of the modified extended interpolation problem with interpolation conditions (4.4), (4.5) and (4.12) is $\widetilde{P}'$. Since it is invertible, and since $\nu(\widetilde{P}') = \nu(P) = q$ and $\pi(\widetilde{P}') = p + \ell$, it follows from Theorem 3.6 that there are infinitely many functions $f \in \mathcal{B}_{p+\ell}/\mathcal{B}_q$ satisfying conditions (4.4), (4.5) and (4.12). Thus, we have shown that under assumption (4.1), there exist infinitely many rational functions $f \in \mathcal{S}_q$ satisfying conditions (4.2) or equivalently, asymptotic equalities (2.13). It is clear that one comes up with the same conclusion if the equality assumption (4.1) is replaced by inequality (2.4). This completes the proof of Theorem 2.8.

5. Rigidity for generalized Carathéodory and generalized Nevanlinna functions

The generalized Schur class $\mathcal{S}_\kappa$ can be alternatively characterized as the class of all functions f meromorphic on $\mathbb{D}$ and such that the kernel $S_f(z,\zeta) = \dfrac{1 - f(z)\overline{f(\zeta)}}{1 - z\overline{\zeta}}$ has κ negative squares on $\mathbb{D}\cap\mathrm{Dom}(f)$. A related to $\mathcal{S}\kappa$ is the class $\mathcal{C}_\kappa$ of generalized Carathéodory functions h which by definition, are meromorphic on $\mathbb{D}$ and such that the associated kernel $C_h(z,\zeta) = \dfrac{h(z) + \overline{h(\zeta)}}{1 - z\overline{\zeta}}$ has κ negative squares on $\mathbb{D}\cap\mathrm{Dom}(h)$. It is convenient to include the function $h \equiv \infty$ into $\mathcal{C}_0$. Then the Caley transform

$$f \mapsto h = \frac{1+f}{1-f} \tag{5.1}$$

establishes a one-to-one correspondence between $\mathcal{S}_\kappa$ and $\mathcal{C}_\kappa$ and therefore, between $\mathcal{S}_{\le\kappa}$ and $\mathcal{C}_{\le\kappa} := \bigcup\limits_{r\le\kappa} \mathcal{C}_r$. The representation $f = s/b$ for an $f \in \mathcal{S}_\kappa$ combined with (5.1) implies that h belongs to $\mathcal{C}_{\le\kappa}$ if and only if it is of the form

$$h = \frac{b+s}{b-s} \quad \text{where} \quad b \in \mathcal{B}_\kappa, \ s \in \mathcal{S} \quad \text{and} \quad Z(s) \cap Z(b) = \emptyset \tag{5.2}$$

Theorem 2.1 in the present setting looks as follows.

Theorem 5.1. *Let κ, p, q be nonnegative integers and let g be of the form*

$$g = \frac{b_2 + b_1}{b_2 - b_1} \quad \text{where} \quad b_1 \in \mathcal{B}_p, \quad b_2 \in \mathcal{B}_q \quad \text{and} \quad Z(b_1) \cap Z(b_2) = \emptyset. \tag{5.3}$$

Let us assume that a function $h \in \mathcal{C}_{\le\kappa}$ satisfies asymptotic equations

$$h(z) = g(z) + o(|z - t_i|^{m_i}) \quad \text{for} \quad i = 1, \ldots, n \tag{5.4}$$

at some points $t_1, \ldots, t_n \in \mathbb{T}$ and some nonnegative integers $m_1, \ldots, m_n$ which in turn, are subject to (2.5). *Then $h \equiv g$.*

Proof: Substituting (5.2) and (5.3) into (5.4) and then multiplying both sides in (5.4) by $(b_2 - b_1)(b - s)$ we eventually get

$$s(z)b_2(z) = b(z)b_1(z) + o(|z - t_i|^{m_i}) \quad \text{for} \quad i = 1, \ldots, n. \tag{5.5}$$

Since $s \cdot b_2 \in \mathcal{S}$ and $b \cdot b_1 \in \mathcal{B}_{\kappa+p}$, we invoke Theorem 1.2 (as in the proof of Theorem 2.1) to conclude from (5.5) that $s \cdot b_2 \equiv b \cdot b_1$ which implies that $h \equiv g$, thanks to (5.2) and (5.3). □

Remark 5.2. Note that analyticity of g at t_i is not required in Theorem 5.1.

Another popular class related to $\mathcal{S}_\kappa$ is the class $\mathcal{N}_\kappa$ of generalized Nevanlinna functions, that is, the functions h meromorphic on the open upper half-plane $\mathbb{C}^+$ and such that the associated kernel $N_h(z,\zeta) = \dfrac{h(z) - \overline{h(\zeta)}}{z - \bar{\zeta}}$ has κ negative squares on $\mathbb{C}^+ \cap \mathrm{Dom}(h)$. The function $h \equiv \infty$ is assumed to be in $\mathcal{N}_0$. The classes $\mathcal{N}_\kappa$ and $\mathcal{S}_\kappa$ are related by

$$h(\zeta) = i \cdot \frac{1 + f(\gamma(\zeta))}{1 - f(\gamma(\zeta))}, \quad \gamma(\zeta) = \frac{\zeta - i}{\zeta + i} \tag{5.6}$$

which allows us to characterize $\mathcal{N}_\kappa$-functions by the fractional representation

$$h = i \cdot \frac{b + s}{b - s} \tag{5.7}$$

where s (analytic and bounded by one in modulus in $\mathbb{C}^+$) and $b \in \mathcal{B}_\kappa$ do not have common zeroes. For the rest of the paper we denote by $\mathcal{B}_k(\mathbb{C}^+)$ the set of finite Blaschke products of the form

$$b(\zeta) = \prod_{i=1}^{k} \frac{\zeta - a_i}{\zeta - \bar{a}_i} \qquad (\zeta,\, a_i \in \mathbb{C}^+).$$

Here is Theorem 2.1 for generalized Nevanlinna functions.

Theorem 5.3. *Let κ, p, q be two nonnegative integers, let g be of the form*

$$g = i \cdot \frac{b_2 + b_1}{b_2 - b_1} \quad \text{where} \quad b_1 \in \mathcal{B}_p(\mathbb{C}^+), \;\; b_2 \in \mathcal{B}_q(\mathbb{C}^+), \;\; Z(b_1) \cap Z(b_2) = \emptyset. \tag{5.8}$$

Let $\lambda_1, \ldots, \lambda_n$ be real points, let $m_1, \ldots, m_n$ be nonnegative integers and let us assume that a function $h \in \mathcal{N}_{\le\kappa}$ satisfies the asymptotic equations

$$h(\zeta) = g(\zeta) + o(|\zeta - \lambda_i|^{m_i}) \quad \text{for} \quad i = 2, \ldots, n \tag{5.9}$$

as $\zeta \in \mathbb{C}^+$ tends to λ_i nontangentially and the asymptotic equation

$$h(\zeta) = g(\zeta) + o(|\zeta|^{-m_1}) \tag{5.10}$$

as z tends to infinity staying inside the angle $\{z:\ \epsilon < \arg z < \pi - \epsilon\}$. If the numbers $m_1, \dots, m_n$ are subject to (2.5)*, then $h \equiv g$.*

Proof: Let $z := \gamma(\zeta)$ where γ is given in (5.6). Then $t_1 := \gamma(\infty) = 1 \in \mathbb{T}$ and since $\lambda_i \in \mathbb{T}$, we have $t_i := \gamma(\lambda_i) \in \mathbb{T}$ for $i = 2, \dots, n$. Observe that

$$|z - t_j| = |\gamma(\zeta) - \gamma(\lambda_j)| = \frac{2|\zeta - \lambda_j|}{|(\zeta + i)(\lambda_j + i)|} = O(|\zeta - \lambda_j|)$$

for $j = 2, \dots, n$ and $|z - t_1| = |z - 1| = |\gamma(\zeta) - 1| = \frac{2}{|\zeta + i|} = O(|\zeta|^{-1})$. Therefore, and since γ maps $\mathbb{C}^+$ onto $\mathbb{D}$ conformally, we can write (5.9) and (5.10) as

$$h(\gamma^{-1}(z)) = g(\gamma^{-1}(z)) + o(|z - t_i|^{m_i}) \quad \text{for} \quad i = 1, \dots, n \tag{5.11}$$

It remains to note that the functions $-ih \circ \gamma^{-1}$ and $-ih \circ \gamma^{-1}$ are generalized Carathéodory functions satisfying the assumptions of Theorem 5.1. Therefore, they are equal identically and thus, $h \equiv g$. □

6. Concluding remarks and open questions

The results presented above indicate some additional potential contained in Theorem 1.2 which has not been pointed out in [10]. However, we see at least three settings the extensions of Theorem 2.1 to which would be of considerable interest.

1. Rigidity forced by countably many boundary interpolation conditions. Let $g \in N(\mathbb{D}) \cap \mathcal{B}L^\infty$ and let the nontangenial boundary limits $\lim_{z \to t_i} g^{(j)}(z)$ exist for $0 \le j \le m_j \le \infty$ and $1 \le i \le n \le \infty$ where now n and $m_1, \dots, m_n$ can be infinite.

Question 1. For what g as above and what n and m_i the following statement holds true: *whenever $f \in N(\mathbb{D}) \cap \mathcal{B}L^\infty$ satisfies asymptotic equalities* (2.4)*, $f \equiv g$?*

The class of functions g for which the above rigidity holds might be much larger than $\mathcal{B}_\ell/\mathcal{B}_r$ as well as infinitely many interpolation conditions may guarantee rigidity for functions beyond $\mathcal{S}_{\le\infty} := \bigcup_{r \ge 0} \mathcal{S}_r$.

2. Rigidity for matrix valued functions. A $p \times q$ matrix valued function F is said to belong to the class $N_{p\times q}(\mathbb{D})$ if all its entries are the functions from $N(\mathbb{D})$ and it belongs to $\mathcal{B}L^\infty_{p\times q}$ if $\|F(t)\|_{op} \le 1$ almost everywhere on $\mathbb{T}$. As in the scalar case we define $N_{p\times q}(\mathbb{D}) \cap \mathcal{B}L^\infty_{p\times q}$ and we modify conditions (2.4) as follows:

$$\|F(z)c_i - G(z)c_i\| = o(|z - t_i|^{m_i}) \quad \text{for} \quad i = 1, \dots, n, \tag{6.1}$$

where now c_i are vectors from $\mathbb{C}^q$. In other words we assume that sufficient conditions for rigidity may be split not only between different points on $\mathbb{T}$ but also between different directions in $\mathbb{C}^q$. The question is the same as in the scalar case:

Question 2. For what $G \in N_{p\times q}(\mathbb{D}) \cap \mathcal{B}L^\infty_{p\times q}$ having nontangenial boundary limits $\lim\limits_{z \to t_i} G^{(j)}(z)$ (or just $\lim\limits_{z \to t_i} G^{(j)}(z)c_i$) for $0 \le j \le m_j \le \infty$ and $1 \le i \le n \le \infty$ and what $c_i \in \mathbb{C}^q$, the following statement holds true: *whenever $F \in N(\mathbb{D}) \cap \mathcal{B}L^\infty$ satisfies asymptotic equalities* (6.1)*, $F \equiv G$?*

3. Rigidity without boundary interpolation conditions. In all previous cases we assumed that g itself admits nontangential boundary expansions of the requisite orders which allowed us to rewrite conditions (2.4) in interpolation form (2.1). Then the question concerning rigidity reduces to the question whether or not certain boundary interpolation problem has a unique solution. We expect that rigidity may occur in a quite different situation. We formulate the question in the holomorphic setting of Theorem 1.2 and expect that the meromorphic counterpart is not much harder.

Question 3. Does there exist $g \in \mathcal{S}$ which does not have nontangential boundary limits at $t_1, \ldots, t_n \in \mathbb{T}$ and such that for every $f \in \mathcal{S}$ satisfying conditions (2.4) for some integers $m_1, \ldots, m_n$, it holds that $f \equiv g$?

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