

Carathéodory-Julia type conditions and symmetries of boundary asymptotics for analytic functions on the unit disk

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It is shown that the following conditions are equivalent for the generalized Schur class functions w at a boundary point $t_0 \in \mathbb{T}$: Carathéodory–Julia type condition of order n ([2], [3]); agreeing of asymptotics from inside and outside of the disk \mathbb{D} up to order $2n + 1$ ([11]); t_0 -isometry of the coefficients of the boundary asymptotics; a certain structured matrix \mathbb{P} constructed from these coefficients being Hermitian ([1]). Some interconnections between these properties are established for more general classes of functions.

1. Introduction

In this paper we discuss some questions arising in the context of the Carathéodory–Julia theorem. For a point t_0 on the unit circle \mathbb{T} , let

$$\mathcal{U}_{t_0, \varepsilon} := \{z \in \mathbb{D} : 0 < |z - t_0| < \varepsilon\},$$

$$\Gamma_{t_0, \alpha, \varepsilon}^{\text{int}} := \{z \in \mathcal{U}_{t_0, \varepsilon} : |\arg(z - t_0)| < \alpha\} \quad \text{for } \alpha \in (0, \frac{\pi}{2}),$$

$$\Gamma_{t_0, \alpha, \varepsilon}^{\text{ext}} := \{z \in \mathbb{C} : \frac{1}{\bar{z}} \in \Gamma_{t_0, \alpha, \varepsilon}^{\text{int}}\}$$

and

$$\Gamma_{t_0, \alpha, \varepsilon} := \Gamma_{t_0, \alpha, \varepsilon}^{\text{int}} \cup \Gamma_{t_0, \alpha, \varepsilon}^{\text{ext}}$$

We consider $\mathcal{U}_{t_0, \varepsilon}$ as a punctured neighborhood and $\Gamma_{t_0, \alpha, \varepsilon}$ as a punctured nontangential neighborhood of t_0 that can be split into interior and exterior parts. The parameter ε will be dropped from notation. We write $z \widehat{\rightarrow} t_0$ if a point z approaches a boundary point $t_0 \in \mathbb{T}$ staying inside $\Gamma_{t_0, \alpha}$ for some $\alpha \in (0, \frac{\pi}{2})$ and we will write $z \rightarrow t_0$ if z approaches t_0 unrestrictedly in \mathbb{D} (i.e., staying inside \mathcal{U}_{t_0}). Also, “the

limit exists" will always mean that the limit is finite. For notational convenience, we use the symbol a^* for the complex conjugate of $a \in \mathbb{C}$. The symbol \mathcal{S} will stand for the Schur class of analytic functions w mapping the unit disk into its closure. We start with the classical Carathéodory–Julia theorem [7, 9].

Theorem 1.1. *Let $w \in \mathcal{S}$, $t_0 \in \mathbb{T}$ and let*

$$d_w(z) := \frac{1 - |w(z)|^2}{1 - |z|^2}. \quad (1.1)$$

The following are equivalent:

1. $\tilde{d} := \liminf_{z \rightarrow t_0} d_w(z) < \infty$.
2. $d := \lim_{z \nearrow t_0} d_w(z) < \infty$.
3. *The nontangential limits*

$$w_0 := \lim_{z \nearrow t_0} w(z) \quad \text{and} \quad w_1 := \lim_{z \nearrow t_0} w'(z) \quad (1.2)$$

exist and satisfy $|w_0| = 1$ and $w_1 t_0 w_0^ \in \mathbb{R}$.*

Moreover, if the latter hold, then $\tilde{d} = d = w_1 t_0 w_0^ \geq 0$.*

Higher order analogues of the above results have been presented in [3]. To recall them we first introduce some needed notations and definitions.

Given a function w analytic at $z \in \mathbb{D}$ (not necessarily in the Schur class) and given $n \in \mathbb{Z}_+$, let us introduce the Hermitian matrix

$$\mathbf{P}_n^w(z) := \left[\frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \frac{1 - |w(z)|^2}{1 - |z|^2} \right]_{i,j=0}^n \quad (1.3)$$

which will be referred to as to a *Schwarz-Pick matrix* (a distinguished property of the Schur class is that for $w \in \mathcal{S}$, the matrix $\mathbf{P}_n^w(z)$ is always positive semidefinite. We extend this notion to boundary points as follows: *given $t_0 \in \mathbb{T}$ and given a function w analytic on a neighborhood \mathcal{U}_{t_0} of t_0 , the boundary Schwarz-Pick matrix is defined by*

$$\mathbf{P}_n^w(t_0) := \lim_{z \nearrow t_0} \mathbf{P}_n^w(z), \quad (1.4)$$

provided the limit in (1.4) exists. It is clear that once the boundary Schwarz-Pick matrix $\mathbf{P}_n^w(t_0)$ exists, it is Hermitian (and it is positive semidefinite, if $w \in \mathcal{S}$). We denote the lower diagonal entry in the Schwarz-Pick matrix $\mathbf{P}_n^w(z)$ by

$$d_{w,n}(z) := \frac{1}{(n!)^2} \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \frac{1 - |w(z)|^2}{1 - |z|^2} \quad (1.5)$$

and remark that if $n = 0$, then both $\mathbf{P}_n^w(z)$ and $d_{w,n}(z)$ reduce to $d_w(z)$ introduced in (1.1).

Throughout the paper, we will use notation $w_j(z) := \frac{w^{(j)}(z)}{j!}$ and we reserve the symbol $w_j(t_0)$ for the nontangential boundary limit

$$w_j(t_0) := \lim_{z \nearrow t_0} w_j(z) = \lim_{z \nearrow t_0} \frac{w^{(j)}(z)}{j!} \quad (1.6)$$

provided the latter limit exists. If w is either analytic at $t_0 \in \mathbb{T}$ or if it possesses the nontangential boundary limits (1.6) for $j = 0, \dots, 2n+1$, we let

$$\mathbb{U}_n^w(t_0) := \begin{bmatrix} w_0(t_0)^* & w_1(t_0)^* & \dots & w_n(t_0)^* \\ 0 & w_0(t_0)^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & w_1(t_0)^* \\ 0 & \dots & 0 & w_0(t_0)^* \end{bmatrix}, \quad (1.7)$$

$$\mathbb{H}_n^w(t_0) := \begin{bmatrix} w_1(t_0) & w_2(t_0) & \dots & w_{n+1}(t_0) \\ w_2(t_0) & w_3(t_0) & \dots & w_{n+2}(t_0) \\ \vdots & \vdots & & \vdots \\ w_{n+1}(t_0) & w_{n+2}(t_0) & \dots & w_{2n+1}(t_0) \end{bmatrix} \quad (1.8)$$

where the first matrix is of the upper triangular Toeplitz and the second of the Hankel structure. We also introduce the structured matrix

$$\mathbb{P}_n^w(t_0) = \mathbb{H}_n^w(t_0) \Psi_n(t_0) \mathbb{U}_n^w(t_0) \quad (1.9)$$

where $\Psi_n(t_0) = [\Psi_{j\ell}]_{j,\ell=0}^n$ is the upper triangular matrix

$$\Psi_n(t_0) = \begin{bmatrix} t_0 & -t_0^2 & t_0^3 & \dots & (-1)^n \binom{n}{0} t_0^{n+1} \\ 0 & -t_0^3 & 2t_0^4 & \dots & (-1)^n \binom{n}{1} t_0^{n+2} \\ \vdots & & t_0^5 & \dots & (-1)^n \binom{n}{2} t_0^{n+3} \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & (-1)^n \binom{n}{n} t_0^{2n+1} \end{bmatrix}, \quad (1.10)$$

with the entries

$$\Psi_{j\ell} = \begin{cases} 0, & \text{if } j > \ell \\ (-1)^\ell \binom{\ell}{j} t_0^{\ell+j+1}, & \text{if } j \leq \ell. \end{cases} \quad (1.11)$$

Also we will make use of similar structured matrices associated with $t_0 \in \mathbb{T}$ and a sequence $\{w_j\}$ of complex numbers:

$$\mathbb{U}(w_0, \dots, w_n) = \begin{bmatrix} w_0^* & \dots & w_n^* \\ \vdots & \ddots & \vdots \\ 0 & \dots & w_0^* \end{bmatrix}, \quad \mathbb{H}(w_1, \dots, w_{2n+1}) = [w_{i+j+1}]_{i,j=0}^n \quad (1.12)$$

and

$$\mathbb{P}(t_0, w_0, \dots, w_{2n+1}) := \mathbb{H}(w_1, \dots, w_{2n+1}) \Psi_n(t_0) \mathbb{U}(w_0, \dots, w_n). \quad (1.13)$$

Definition 1.2. Given a point $t_0 \in \mathbb{T}$, we will say that a sequence $\{w_0, \dots, w_n\}$ of complex numbers is t_0 -isometric if

$$\overline{\mathbb{U}(w_0, \dots, w_n)} \Psi_n(t_0) \mathbb{U}(w_0, \dots, w_n) = \Psi_n(t_0) \quad (1.14)$$

where $\overline{\mathbb{U}}$ is the complex conjugate of \mathbb{U} . We will say that an infinite sequence $\{w_i\}_{i=0}^\infty$ is t_0 -isometric if the equality (1.14) holds for every $n \in \mathbb{Z}_+$.

Remark 1.3. It follows from Definition 1.2 by the upper triangular structure of matrices in (1.14) that if $\{w_0, \dots, w_n\}$ is t_0 -isometric, then the subsequence $\{w_0, \dots, w_k\}$ is t_0 -isometric for every $k < n$. In particular, for $k = 0$ we conclude from (1.14) that $|w_0| = 1$.

The following theorem has been established in [3].

Theorem 1.4. *Let $w \in \mathcal{S}$, $t_0 \in \mathbb{T}$, $n \in \mathbb{Z}_+$ and let $d_{w,n}$ be defined as in (1.5). The following are equivalent.*

1. *The following limit inferior is finite*

$$\tilde{d}_n := \liminf_{z \rightarrow t_0} d_{w,n}(z) < \infty. \quad (1.15)$$

2. *The following nontangential boundary limit exists and is finite:*

$$d_{w,n}(t_0) := \lim_{z \widehat{\rightarrow} t_0} d_{w,n}(z) < \infty. \quad (1.16)$$

3. *The boundary Schwarz-Pick matrix $\mathbf{P}_n^w(t_0)$ defined via the nontangential boundary limit (1.4) exists.*
4. *The nontangential boundary limits $w_j(t_0)$ exist for $j = 0, \dots, 2n+1$ and satisfy*

$$|w_0(t_0)| = 1 \quad \text{and} \quad \mathbb{P}_n^w(t_0) \geq 0,$$

where $\mathbb{P}_n^w(t_0)$ is the matrix defined in (1.9).

Moreover, if this is the case, then $\tilde{d}_n = d_{w,n}(t_0)$ and $\mathbf{P}_n^w(t_0) = \mathbb{P}_n^w(t_0)$.

In case $n = 0$, Theorem 1.4 reduces to Theorem 1.1. Note that the structured matrix $\mathbb{P}_n^w(t_0)$ (which first appeared in [11]) is a higher order analogue of the product $w_1 t_0 w_0^*$. It turns out (and will be shown in this paper) that property (3) in Theorem 1.4 follows from a weaker assumption

$$|w_0(t_0)| = 1 \quad \text{and} \quad \mathbb{P}_n^w(t_0) = \mathbb{P}_n^w(t_0)^*, \quad (1.17)$$

even if w does not belong to the Schur class and is analytic on a neighborhood $\mathcal{U}_{t_0, \varepsilon}$ only. The following theorem establishes this fact and presents a number of equivalent reformulations of conditions (1.17).

Theorem 1.5. *Let w be analytic in a neighborhood \mathcal{U}_{t_0} of $t_0 \in \mathbb{T}$ and let us assume that the nontangential boundary limits $w_j(t_0)$ exist for $j = 0, \dots, 2n+1$. The following are equivalent:*

1. *Conditions (1.17) are satisfied.*
2. *The sequence $\{w_0(t_0), \dots, w_{2n+1}(t_0)\}$ is t_0 -isometric, i.e.,*

$$\overline{\mathbb{U}}(w_0, \dots, w_{2n+1}) \mathbf{\Psi}_n(t_0) \mathbb{U}(w_0, \dots, w_{2n+1}) = \mathbf{\Psi}_{2n+1}(t_0).$$

3. *There exists a rational unimodular on \mathbb{T} function f (a ratio of two finite Blaschke products) so that*

$$w(z) = f(z) + o((z - t_0)^{2n+1}) \quad \text{as } z \widehat{\rightarrow} t_0. \quad (1.18)$$

4. The asymptotic relation

$$w(z) = \sum_{j=0}^{2n+1} w_j(t_0)(z - t_0)^j + o((z - t_0)^{2n+1}) \quad (1.19)$$

holds as z tends to t_0 nontangentially from inside and outside of the unit disk \mathbb{D} , where for $|z| > 1$

$$w(z) := \frac{1}{\overline{w(1/\bar{z})}}. \quad (1.20)$$

Moreover, if this is the case, then the boundary Schwarz-Pick matrix $\mathbf{P}_n^w(t_0)$ exists and is equal to $\mathbb{P}_n^w(t_0)$.

Remark 1.6. Extension (1.20) is called the symmetry continuation and has nothing to do with analytic continuation unless the function w is unimodular on an arc of \mathbb{T} . In general, existence of a nontangential asymptotics (1.19) for $w(z)$ from inside of \mathbb{D} with $w(t_0) \neq 0$ implies existence of an asymptotics of the same order from outside for the symmetry continuation of w . However, in general, the coefficients of the asymptotics from outside may differ from the coefficients of the asymptotics from inside. Thus, agreeing of the two asymptotics is a special property of the function w . Theorem 1.4 tells that this property implies (1.15) and Theorem 1.5 shows that Schur class functions it follows form (1.15). In general (1.15) does not imply this property (see example in Section 4). However, if w is in generalized Schur class then it does. We show in Section 4 that Theorem 1.4 can be literally extended to the class of generalized Schur functions. The agreeing of the two asymptotics was actually taken by I.V. Kovalishina as a basis for her constructions in [11], [11] as opposed to [3] and [6], where all the constructions started from (1.15).

The next theorem is a version of Theorem 1.5 under slightly relaxed assumptions: existence of the nontangential boundary limit $w_{2n+1}(t_0)$ is relaxed to the uniform nontangential boundedness of $w^{(2n+1)}$:

$$\sup_{z \in \Gamma_{t_0, \alpha}} |w^{(2n+1)}(z)| < \infty \quad \text{for some } \alpha \in (0, \frac{\pi}{2}). \quad (1.21)$$

It can be shown (see Lemma 3.1 below) that the latter condition guarantees the existence of the nontangential boundary limits $w_j(t_0) := \lim_{z \widehat{\rightarrow} t_0} \frac{w^{(j)}(z)}{j!}$ for $j = 0, \dots, 2n$.

Theorem 1.7. *Let w be analytic in a neighborhood \mathcal{U}_{t_0} of $t_0 \in \mathbb{T}$. The following are equivalent*

1. *The bound (1.21) is in force and the sequence $\{w_0(t_0), \dots, w_{2n}(t_0)\}$ is t_0 -isometric.*
2. *There exist a rational unimodular function f (a ratio of two finite Blaschke products) so that*

$$w(z) = f(z) + O((z - t_0)^{2n+1}) \quad \text{as } z \widehat{\rightarrow} t_0. \quad (1.22)$$

3. The asymptotic relation

$$w(z) = \sum_{j=0}^{2n} w_j(t_0)(z - t_0)^j + O((z - t_0)^{2n+1}),$$

holds as $z \widehat{\rightarrow} t_0$ from inside and outside of \mathbb{D} .

Moreover, if this is the case, then

$$\sup_{z \in \Gamma_{t_0, \alpha}} d_{w,n}(z) < \infty. \quad (1.23)$$

As a corollary we obtain the following extended version of Theorem 1.4.

Theorem 1.8. *Let $w \in \mathcal{S}$, $t_0 \in \mathbb{T}$, $n \in \mathbb{Z}_+$. The statements 1-4 in Theorem 1.4, the statements 1-4 in Theorem 1.5 and the statements 1-3 in Theorem 1.7 are all equivalent.*

For the proof it is enough to note that every statement in Theorem 1.5 implies (1.16) while condition (1.23) is obviously stronger than (1.15).

2. t_0 -isometric sequences

A t_0 -isometric sequence $\{w_0, \dots, w_n\}$ has been characterized in Definition 1.2 by the matrix equality (1.14); in more detail,

$$\begin{bmatrix} w_0 & \dots & w_n \\ & \ddots & \vdots \\ 0 & & w_0 \end{bmatrix} \Psi_n(t_0) \begin{bmatrix} w_0^* & \dots & w_n^* \\ & \ddots & \vdots \\ 0 & & w_0^* \end{bmatrix} = \Psi_n(t_0). \quad (2.1)$$

Remark 2.1. *Note that the latter matrix equality is equivalent to the following $n+1$ (in general, independent) relations*

$$\sum_{j=0}^k \sum_{\ell=0}^{k-j} (-1)^\ell \binom{k-\ell}{j} t_0^{j-\ell} w_j w_\ell^* = 1 \quad \text{for } k = 0, \dots, n. \quad (2.2)$$

Indeed, multiplying both parts in (2.2) by $(-1)^k t_0^{k+1}$ and making use of numbers (1.11) we get

$$\sum_{j=0}^k \sum_{\ell=0}^{k-j} w_j \Psi_{j,k-\ell} w_\ell^* = \Psi_{0,k} \quad \text{for } k = 0, \dots, n,$$

that can be written in the matrix form as

$$\begin{bmatrix} w_0 & \dots & w_n \end{bmatrix} \Psi_n(t_0) \begin{bmatrix} w_0^* & \dots & w_n^* \\ & \ddots & \vdots \\ 0 & & w_0^* \end{bmatrix} = \begin{bmatrix} \Psi_{00} & \dots & \Psi_{0n} \end{bmatrix} \quad (2.3)$$

and express equality of the top rows in (2.1). It was shown in [2] that (2.3) is equivalent to the “whole” equality (2.1).

Lemma 2.2. *Let $t_0 \in \mathbb{T}$, $w_0, \dots, w_n \in \mathbb{C}$ and let $T \in \mathbb{C}^{(n+1) \times (n+1)}$ and $M, E \in \mathbb{C}^{(n+1) \times 1}$ be given by*

$$T_n = \begin{bmatrix} t_0 & 0 & \dots & 0 \\ 1 & t_0 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & 1 & t_0 \end{bmatrix}, \quad E_n = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad M_n = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{bmatrix}. \quad (2.4)$$

The Stein equation

$$X - T_n X T_n^* = E_n E_n^* - M_n M_n^* \quad (2.5)$$

has a solution X if and only if the sequence $\{w_0, \dots, w_n\}$ is t_0 -isometric. Moreover, in this case every solution X to (2.5) is of the form (1.13) for some choice of $w_{n+1}, \dots, w_{2n+1} \in \mathbb{C}$.

The proof can be found in [2, Section 10] along with some other equivalent reformulations of the equality (2.1).

Theorem 2.3. *Let w be analytic on a neighborhood*

$$D_{r, \varepsilon}(t_0) = \{z : r < |z| < \frac{1}{r}, \arg t_0 - \varepsilon < \arg z < \arg t_0 + \varepsilon\}$$

of $t_0 \in \mathbb{T}$ and unimodular on $D_{r, \varepsilon}(t_0) \cap \mathbb{T}$ and let

$$w(z) = \sum_{j=0}^{\infty} w_j(t_0)(z - t_0)^j \quad (2.6)$$

be its Taylor expansion at t_0 . Then

1. *The boundary Schwartz-Pick matrix $\mathbf{P}_n^w(t_0)$ exists and admits the representation*

$$\mathbf{P}_n^w(t_0) = \mathbb{P}_n^w(t_0) \quad (2.7)$$

for every $n \in \mathbb{Z}_+$, where $\mathbb{P}_n^w(t_0)$ is defined via formula (1.9). Therefore, $\mathbb{P}_n^w(t_0)$ is Hermitian.

2. *The sequence $\{w_j(t_0)\}_{j=0}^{\infty}$ is t_0 -isometric.*

Proof. Since w is unimodular on $D_{r, \varepsilon}(t_0) \cap \mathbb{T}$, its symmetry continuation and analytic continuation across the arc agree and

$$w(z)\overline{w(1/\bar{z})} = 1 \quad \text{for every } z \in D_{r, \varepsilon}(t_0). \quad (2.8)$$

Therefore,

$$K_w(z, \zeta) := \frac{1 - w(z)\overline{w(\zeta)}}{1 - z\bar{\zeta}} = \frac{w(z) - w(1/\bar{\zeta})}{z - \frac{1}{\bar{\zeta}}} \cdot \frac{\overline{w(\zeta)}}{\bar{\zeta}}. \quad (2.9)$$

The latter formula makes sense for every $z, \zeta \in D_{r, \varepsilon}(t_0)$ such that $z\bar{\zeta} \neq 1$. We extend it by continuity to all of $D_{r, \varepsilon}(t_0) \times D_{r, \varepsilon}(t_0)$ upon setting

$$K_w(z, 1/\bar{z}) = zw'(z)\overline{w(1/\bar{z})}. \quad (2.10)$$

The kernel K_w defined by the formulas (2.9) and (2.10) is analytic in z and conjugate analytic in ζ on $D_{r, \varepsilon}(t_0) \times D_{r, \varepsilon}(t_0)$. Therefore, the boundary Schwarz-Pick matrix $\mathbf{P}_n^w(t_0)$ can be defined by the formula

$$\mathbf{P}_n^w(t_0) = \left[\frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} K_w(z, z) \Big|_{z=t_0} \right]_{i,j=0}^n, \quad (2.11)$$

rather than as the limit of the interior Schwarz-Pick matrices. To express the ij -th entry \mathbf{P}_{ij} in (2.11), we use (2.9) along with the Leibnitz's rule:

$$\begin{aligned} \mathbf{P}_{ij} &= \frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{\zeta}^j} \left(\frac{w(z) - w(1/\bar{\zeta})}{z - \frac{1}{\bar{\zeta}}} \cdot \frac{\overline{w(\zeta)}}{\bar{\zeta}} \right) \Big|_{z=\zeta=t_0} \\ &= \frac{1}{i!j!} \sum_{\ell=0}^j \binom{j}{\ell} \frac{\partial^{i+\ell}}{\partial z^i \partial \bar{\zeta}^\ell} \left(\frac{w(z) - w(1/\bar{\zeta})}{z - \frac{1}{\bar{\zeta}}} \right) \cdot \left(\frac{\overline{w(\zeta)}}{\bar{\zeta}} \right)^{(j-\ell)} \Big|_{z=\zeta=t_0}. \end{aligned} \quad (2.12)$$

We want to show that \mathbf{P}_{ij} is equal to the ij -th entry in the matrix $\mathbb{P}_n^w(t_0)$, i.e. (see the formula (1.9)), that

$$\begin{aligned} \mathbf{P}_{ij} &= \begin{bmatrix} w_{i+1}(t_0) & w_{i+2}(t_0) & \dots & w_{i+j+1}(t_0) \end{bmatrix} \Psi_n(t_0) \begin{bmatrix} w_j(t_0)^* \\ \vdots \\ w_0(t_0)^* \\ 0 \end{bmatrix} \\ &= \sum_{\ell=0}^j w_{i+\ell+1}(t_0) \sum_{k=0}^{j-\ell} \Psi_{\ell, j-k} w_{j-\ell-k}(t_0)^*. \end{aligned} \quad (2.13)$$

It is easily seen (we refer to [4, Section 6] for details) that

$$\frac{1}{i!k!} \frac{\partial^{i+k}}{\partial z^i \partial \left(\frac{1}{\bar{\zeta}}\right)^k} \left(\frac{w(z) - w(1/\bar{\zeta})}{z - \frac{1}{\bar{\zeta}}} \right) \Big|_{z=\frac{1}{\bar{\zeta}}} = \frac{w^{(i+k+1)}(z)}{(i+k+1)!} = w_{i+k+1}(z)$$

for $i, k \in \mathbb{Z}_+$. On the other hand, we have by the chain rule,

$$\frac{1}{\ell!} \frac{\partial^\ell}{\partial \bar{\zeta}^\ell} = (-1)^\ell \sum_{k=0}^{\ell-1} \frac{\binom{\ell-1}{k}}{\bar{\zeta}^{\ell+k+1} (k+1)!} \cdot \frac{\partial^{k+1}}{\partial \left(\frac{1}{\bar{\zeta}}\right)^{k+1}} \quad (\ell = 1, 2, \dots)$$

and combining the two latter formulas gives

$$\begin{aligned}
& \frac{1}{i! \ell!} \frac{\partial^{i+\ell}}{\partial z^i \partial \bar{\zeta}^\ell} \left(\frac{w(z) - w(1/\bar{\zeta})}{z - \frac{1}{\bar{\zeta}}} \right) \Big|_{z=\zeta=t_0} \\
&= (-1)^\ell \sum_{k=0}^{\ell-1} \frac{\binom{\ell-1}{k}}{\bar{\zeta}^{\ell+k+1} i! (k+1)!} \cdot \frac{\partial^{i+k+1}}{\partial z^i \partial \left(\frac{1}{\bar{\zeta}}\right)^{k+1}} \left(\frac{w(z) - w(1/\bar{\zeta})}{z - \frac{1}{\bar{\zeta}}} \right) \Big|_{z=\zeta=t_0} \\
&= (-1)^\ell \sum_{k=0}^{\ell-1} t_0^{\ell+k+1} \binom{\ell-1}{k} w_{i+k+2}(t_0)
\end{aligned}$$

for $\ell = 1, 2, \dots$, while for $\ell = 0$, we have

$$\frac{1}{i!} \frac{\partial^i}{\partial z^i} \left(\frac{w(z) - w(1/\bar{\zeta})}{z - \frac{1}{\bar{\zeta}}} \right) \Big|_{z=\zeta=t_0} = w_{i+1}(t_0).$$

Finally, by the Leibnitz's rule,

$$\frac{1}{(j-\ell)!} \left(\frac{w(\zeta)}{\zeta} \right)^{(j-\ell)} \Big|_{\zeta=t_0} = \sum_{m=0}^{j-\ell} (-1)^m \bar{t}_0^{m+1} w_{j-\ell-m}(t_0)$$

and we substitute the three latter formulas into (2.12) to get

$$\begin{aligned}
\mathbf{P}_{ij} &= w_{i+1}(t_0) \sum_{m=0}^j (-1)^m \bar{t}_0^{m+1} w_{j-m}(t_0)^* \\
&\quad + \sum_{\ell=1}^j \sum_{k=0}^{\ell-1} \sum_{m=0}^{j-\ell} (-1)^{\ell+m} t_0^{\ell+k+m+2} \binom{\ell-1}{k} w_{i+k+2}(t_0) w_{j-\ell-m}(t_0)^* \\
&= w_{i+1}(t_0) \sum_{m=0}^j (-1)^m \bar{t}_0^{m+1} w_{j-m}(t_0)^* \\
&\quad + \sum_{\ell=1}^j w_{i+\ell+1}(t_0) \sum_{k=0}^{j-\ell} (-1)^{j-k} t_0^{j+\ell+1-k} \binom{j-k}{\ell} w_{j-\ell-k}(t_0)^* \\
&= \sum_{\ell=0}^j w_{i+\ell+1}(t_0) \sum_{k=0}^{j-\ell} (-1)^{j-k} t_0^{j+\ell+1-k} \binom{j-k}{\ell} w_{j-\ell-k}(t_0)^*.
\end{aligned}$$

Making use of the numbers (1.11) allows us to rewrite the last equality in the form

$$\mathbf{P}_{ij} = \sum_{\ell=0}^j w_{i+\ell+1}(t_0) \sum_{k=0}^{j-\ell} \Psi_{\ell, j-k} w_{j-\ell-k}(t_0)^*$$

which coincides with (2.13) and therefore, proves equality (2.7). This equality implies in particular, that the structured matrix $\mathbb{P}_n^w(t_0)$ of the form (1.9) is Hermitian

(since $\mathbf{P}_n^w(t_0)$ is). Then it follows by [6, Theorem 7.1], that $\mathbb{P}_n^w(t_0)$ satisfies the Stein identity

$$\mathbb{P}_n^w(t_0) - T_n \mathbb{P}_n^w(t_0) T_n^* = E_n E_n^* - M^w(t_0)_n M_n^w(t_0)^*$$

where T_n and E_n are the same as in (2.4) and where

$$M_n^w(t_0) = \begin{bmatrix} w_0(t_0) & \dots & w_n(t_0) \end{bmatrix}^\top.$$

Then, the sequence $\{w_0, \dots, w_n\}$ is t_0 -isometric by Lemma 2.2. Since $n \in \mathbb{Z}_+$ is arbitrary, the second assertion of the theorem follows. Note that t_0 -symmetry of the sequence $\{w_0, \dots, w_n\}$ can be derived directly upon differentiating identity (2.8). \square

Lemma 2.4. *Let t_0 and $w_0, \dots, w_{2n+1} \in \mathbb{C}$ be such that $|w_0| = 1$ and the matrix $\mathbb{P}(t_0; w_0, \dots, w_{2n+1})$ defined in (1.13) is Hermitian. Then there exist $f_0, \dots, f_{2n+1} \in \mathbb{C}$ with $|f_0| = 1$ such that the matrices $\mathbb{P}(t_0; f_0, \dots, f_{2n+1})$ and $\mathbb{P}(t_0; s_0, \dots, s_{2n+1})$ are positive definite where s_j 's are the numbers given by*

$$s_j = \sum_{\ell=0}^j f_{j-\ell} w_\ell \quad \text{for } j = 0, \dots, 2n+1. \quad (2.14)$$

Proof. We will use the following notation for the matrices introduced in (1.12) and (1.13):

$$\mathbb{U}_k^w := \mathbb{U}(w_0, \dots, w_k) \quad \text{and} \quad \mathbb{H}_k^w := \mathbb{H}(w_0, \dots, w_{2k+1}),$$

and

$$\mathbb{P}_k^w := \mathbb{P}(t_0; w_0, \dots, w_{2k+1}) = \mathbb{H}_k^w \mathbf{\Psi}_k(t_0) \mathbb{U}_k^w \quad (2.15)$$

and for the similar matrices associated with the sequences $\{f_j\}$ and $\{s_j\}$. Note that if s_j 's are defined by the convolution formulas (2.14), then

$$\mathbb{U}_k^s = \mathbb{U}_k^w \mathbb{U}_k^f \quad \text{and} \quad \mathbb{H}_k^s = (\mathbb{U}_k^f)^* \mathbb{H}_k^w + \mathbb{H}_n^f(t_0) \overline{\mathbb{U}_k^s}. \quad (2.16)$$

The desired sequence $\{f_0, \dots, f_{2n+1}\}$ will be constructed inductively. Note that for every $k < n$, the matrix \mathbb{P}_k^w is Hermitian as a principal submatrix of \mathbb{P}_n^w which is Hermitian by the assumption of the lemma. For $k = 0$, we have $\mathbb{P}_0^w := t_0 w_1 w_0^* \in \mathbb{R}$. Let f_0 and f_1 be such that

$$|f_0| = 1 \quad \text{and} \quad \mathbb{P}_0^f = t_0 f_1 f_0^* > \max\{0, -t_0 w_1 w_0^*\}$$

(such a choice is obviously possible). Then for $s_0 = f_0 w_0$ and $s_1 = f_0 w_1 + f_1 w_0$, we have $|s_0| = 1$ and

$$\mathbb{P}_0^s = t_0 s_1 s_0^* = t_0 (f_0 w_1 + f_1 w_0) f_0^* w_0^* = t_0 w_1 w_0^* + t_0 f_1 f_0^* > 0.$$

Let us assume that for $k < n$ we already have $\{f_0, \dots, f_{2k-1}\}$ such that the matrices \mathbb{P}_{k-1}^f and \mathbb{P}_{k-1}^s are positive definite where s_j 's are defined by the formula (2.14) for $j = 0, \dots, 2k-1$. We will show that for an appropriate choice of f_{2k} and f_{2k+1} , the extended matrices \mathbb{P}_k^f and \mathbb{P}_k^s are positive definite which will complete the proof by induction.

Since $|f_0| = 1$ and $\mathbb{P}_{k-1}^f > 0$, it follows by an interpolation result (see e.g., [11, 1, 3]), there exists a finite Blaschke product $b(z)$ such that

$$\frac{b^{(j)}(t_0)}{j!} = f_j \quad \text{for } j = 0, \dots, 2k-1. \quad (2.17)$$

Letting

$$f_{2k} = \frac{b^{(2k)}(t_0)}{(2k)!} \quad \text{and} \quad f_{2k+1} = \frac{b^{(2k+1)}(t_0)}{(2k+1)!},$$

define s_{2k} and s_{2k+1} according to (2.14). The matrix

$$\mathbb{P}_k^f := \mathbb{H}_k^f \Psi_k(t_0) \mathbb{U}_k^f. \quad (2.18)$$

is positive semidefinite, since it is equal (by construction) to the matrix $\mathbb{P}_k^b(t_0)$ associated to the finite Blaschke product b via formula (1.9) and therefore, it is equal (by Theorem 2.3) to the boundary Schwarz-Pick matrix $\mathbf{P}_n^b(t_0)$ which is positive semidefinite since $b \in \mathcal{S}$.

Since the structured matrix \mathbb{P}_k^w is Hermitian, it satisfies (see Theorem 7.1 in [6] for the proof) the Stein identity

$$\mathbb{P}_k^w - T_k \mathbb{P}_k^w T_k^* = E_k E_k^* - M_k M_k^*$$

where T_k , E_k and M_k are defined via formulas (2.4). Then it follows by Lemma 2.2 that the sequence $\{w_0, \dots, w_k\}$ is t_0 -isometric:

$$\overline{\mathbb{U}_k^w} \Psi_k(t_0) \mathbb{U}_k^w = \Psi_n(t_0). \quad (2.19)$$

Consider the matrix

$$\mathbb{P}_k^s := \mathbb{H}_k^s \Psi_k(t_0) \mathbb{U}_k^s.$$

Upon substituting relations (2.16) into the right hand side expression and making use of (2.19) we get

$$\begin{aligned} \mathbb{P}_k^s &= \left((\mathbb{U}_k^f)^* \mathbb{H}_k^w + \mathbb{H}_k^f \overline{\mathbb{U}_k^w} \right) \Psi_k(t_0) \mathbb{U}_k^w \mathbb{U}_k^f \\ &= (\mathbb{U}_k^f)^* \mathbb{H}_k^w \Psi_n(t_0) \mathbb{U}_k^w \mathbb{U}_k^f + \mathbb{H}_k^f \overline{\mathbb{U}_k^w} \Psi_n(t_0) \mathbb{U}_k^w \mathbb{U}_k^f \\ &= (\mathbb{U}_k^f)^* \mathbb{H}_k^w \Psi_n(t_0) \mathbb{U}_k^w \mathbb{U}_k^f + \mathbb{H}_k^f \Psi_n(t_0) \mathbb{U}_k^f \end{aligned}$$

which on account of (2.15) and (2.18) can be written as

$$\mathbb{P}_k^s = (\mathbb{U}_k^f)^* \mathbb{P}_k^w \mathbb{U}_k^f + \mathbb{P}_k^f. \quad (2.20)$$

Since the matrices \mathbb{P}_k^w and \mathbb{P}_k^f are Hermitian, the matrix \mathbb{P}_k^s is also Hermitian. It is readily seen from the structure of the matrices on the right hand side of (2.18)

that the only entry in \mathbb{P}_k^f that depends on f_{2k+1} is the lower diagonal entry

$$\begin{aligned} \left[\mathbb{P}_k^f\right]_{k+1,k+1} &= \sum_{\ell=0}^k \sum_{j=0}^{k-\ell} f_{k+\ell+1} \Psi_{\ell,k-j} f_{k-\ell-j}^* \\ &= (-1)^k t_0^{2k+1} f_{2k+1} f_0^* + \sum_{\ell=0}^{k-1} f_{k+\ell+1} \sum_{j=0}^{k-\ell} \Psi_{\ell,k-j} f_{k-\ell-j}^* \end{aligned}$$

which is nonnegative since $\mathbb{P}_k^f \geq 0$. Since $|f_0| = 1$, we conclude that the replacement

$$f_{2k+1} \rightarrow f_{2k+1} + (-1)^k \bar{t}_0^{2k+1} f_0 x \quad \text{where } x > 0, \quad (2.21)$$

increases the lower diagonal entry in \mathbb{P}_k^f by x and does not affect all the other entries. This replacement modifies s_{2k+1} accordingly to (2.14)

$$s_{2k+1} \rightarrow s_{2k+1} + (-1)^k \bar{t}_0^{2k+1} f_0 w_0 x = s_{2k+1} + (-1)^k \bar{t}_0^{2k+1} s_0 x$$

and does not change s_j for $j \leq 2k$. Since the matrix \mathbb{U}_k^f depends only on f_0, \dots, f_k , it follows from (2.20) that the entries in \mathbb{P}_k^s do not change but the lower diagonal entry $[\mathbb{P}_k^s]_{k+1,k+1}$ which gets the same raise x as $[\mathbb{P}_k^f]_{k+1,k+1}$. Thus, modifying f_{2k+1} as in (2.21), one can increase the lower diagonal entries in \mathbb{P}_k^s and \mathbb{P}_k^f by any preassigned $x > 0$. Since the $k \times k$ principal submatrices \mathbb{P}_{k-1}^s and \mathbb{P}_{k-1}^f in $(k+1) \times (k+1)$ matrices \mathbb{P}_k^s and \mathbb{P}_k^f are positive definite by the induction hypothesis, that latter modification makes \mathbb{P}_k^s and \mathbb{P}_k^f positive definite if x is large enough. This completes the proof of the lemma. \square

Theorem 2.3 shows how one can generate infinite t_0 -isometric sequences using unimodular functions analytic at t_0 . Note that t_0 -isometric sequences $\{w_j\}_{j=0}^\infty$ that arise in this way are characterized by the additional condition

$$\limsup_{j \rightarrow \infty} |w_j|^{\frac{1}{j}} < \infty$$

that guarantees a nonzero radius of convergence for the series in (2.6). The finite case is much simpler: the next theorem shows that every *finite* t_0 -isometric sequence arises as a sequence of Taylor coefficients at t_0 of a ratio of two finite Blaschke products.

Theorem 2.5. *Let $t_0 \in \mathbb{T}$ and $w_0, \dots, w_{2n+1} \in \mathbb{C}$. The following are equivalent:*

1. $|w_0| = 1$ and the matrix $\mathbb{P}(t_0; w_0, \dots, w_{2n+1})$ defined in (1.13) is Hermitian.
2. The sequence $\{w_0, \dots, w_{2n+1}\}$ is t_0 -isometric.
3. There exist finite Blaschke products s and f so that the function $w = \frac{s}{f}$ satisfies interpolation conditions

$$w_j(t_0) := \frac{w^{(j)}(t_0)}{j!} = w_j \quad \text{for } j = 0, \dots, 2n+1. \quad (2.22)$$

Proof. The implication (3) \Rightarrow (2) follows by Statement 2 in Theorem 2.3.

(2) \Rightarrow (1): Let us assume that $\{w_0, \dots, w_{2n+1}\}$ is t_0 -isometric and let the matrices T_{2n+1} , E_{2n+1} and M_{2n+1} be defined via formulas (2.4). By Lemma 2.2, the Stein equation

$$X - T_{2n+1}XT_{2n+1}^* = E_{2n+1}E_{2n+1}^* - M_{2n+1}M_{2n+1}^* \quad (2.23)$$

has a solution $X = [X_{ij}]_{i,j=0}^{2n+1}$. Comparing the corresponding entries in the equality (2.23) we come to the system

$$\begin{aligned} t_0 X_{0i} - \bar{t}_0 X_{i0} &= w_{i+1} w_0^* & (i = 0, \dots, 2n), \\ t_0 X_{i+1,j} + \bar{t}_0 X_{i,j+1} + X_{ij} &= w_{i+1} w_{j+1}^* & (i = 1, \dots, 2n; j = 0, \dots, 2n), \end{aligned}$$

which, upon being solved recursively, leads to

$$X_{k\ell} = \sum_{j=0}^{\ell} \sum_{i=0}^j (-1)^j \binom{j}{i} t_0^{i+j+1} w_{k+i+1} w_{\ell-j}^* \quad (2.24)$$

for

$$0 \leq k + \ell \leq 2n - 1. \quad (2.25)$$

Thus, for every solution X of the Stein equation (2.23), the entries with the indeces from the domain (2.25) are uniquely defined by the right hand side of the equation. Moreover, these entries satisfy the symmetry

$$X_{k\ell} = X_{\ell k}^* \quad (0 \leq k + \ell \leq 2n - 1) \quad (2.26)$$

which follows from the abovementioned uniqueness and the fact that X^* satisfies the Stein equation (2.23) if and only if X does. Making use of numbers Ψ_{ij} given in (1.11), one can rewrite (2.24) as

$$X_{k\ell} = \sum_{j=0}^{\ell} \sum_{i=0}^j w_{k+i+1} \Psi_{ij} w_{\ell-j}^* \quad \text{for } 0 \leq k + \ell \leq 2n - 1. \quad (2.27)$$

The latter means that for $k, \ell \in \{0, \dots, n\}$, $X_{k\ell}$ is equal to the corresponding entry in the matrix $\mathbb{P}(t_0; w_0, \dots, w_{2n+1})$ defined by (1.13). In other words, $\mathbb{P}(t_0; w_0, \dots, w_{2n+1})$ is the $(n+1) \times (n+1)$ leading principal submatrix of X . It now follows from (2.26) that $\mathbb{P}(t_0; w_0, \dots, w_{2n+1})$ is Hermitian. Equality $|w_0| = 1$ is a part of the definition of t_0 -symmetry.

(1) \Rightarrow (3): Let us assume that $|w_0| = 1$ and the matrix $\mathbb{P}(t_0; w_0, \dots, w_{2n+1})$ defined in (1.13) is Hermitian. By Lemma 2.4, there exist $f_0, \dots, f_{2n+1} \in \mathbb{C}$ with $|f_0| = 1$ such that the matrices $\mathbb{P}(t_0; f_0, \dots, f_{2n+1})$ and $\mathbb{P}(t_0; s_0, \dots, s_{2n+1})$ are positive definite where s_j 's are the numbers given by (2.14). Since $s_0 = w_0 f_0$, we have $|s_0| = 1$. By the interpolation result mentioned in the proof of Lemma 2.4, there exist finite Blaschke products $s(z)$ and $f(z)$ satisfying the conditions

$$\frac{s^{(j)}(t_0)}{j!} = s_j \quad \text{and} \quad \frac{f^{(j)}(t_0)}{j!} = f_j \quad \text{for } j = 0, \dots, 2n+1.$$

Now it follows from (2.14) that the ratio $w(z) = \frac{s(z)}{f(z)}$ satisfies the conditions (2.22). \square

We conclude the section with several remarks. The first one (for the proof, we refer to [2, Corollary 7.9]) relates the third statements in Theorems 1.5 and 2.5:

Remark 2.6. *Let $t_0 \in \mathbb{T}$, $k \in \mathbb{Z}_+$, $w_0, \dots, w_k \in \mathbb{C}$ and let w be a function analytic in \mathcal{U}_{t_0} . Then the nontangential boundary limits $w_j(t_0)$ exist and satisfy*

$$w_j(t_0) := \lim_{z \widehat{\rightarrow} t_0} \frac{w^{(j)}(z)}{j!} = w_j \quad \text{for } j = 0, \dots, k$$

if and only if w admits the asymptotic expansion

$$w(z) = w_0 + w_1(z - t_0) + \dots + w_k(z - t_0)^k + o((z - t_0)^k) \quad \text{as } z \widehat{\rightarrow} t_0.$$

Remark 2.7. Note that any t_0 -isometric sequence $\{w_0, \dots, w_n\}$ admits a t_0 -isometric extension.

Indeed, if $\{w_0, \dots, w_n\}$ is t_0 -isometric, then $|w_0| = 1$ (by Remark 1.3) and the Stein equation (2.5) has a solution X (by Lemma 2.2). It is easily seen that then X^* also satisfies (2.5) and therefore, $Y := \frac{1}{2}(X + X^*)$ is a Hermitian solution of (2.5). By the second assertion in Lemma 2.2, there exist w_{n+1}, \dots, w_{2n+1} such that

$$Y = \mathbb{P}(t_0, w_0, \dots, w_{2n+1})$$

and since the latter matrix is Hermitian, it follows by Theorem 2.5 that the sequence $\{w_0, \dots, w_{2n+1}\}$ is t_0 -isometric.

3. Proofs of Theorems 1.5 and 1.7

To present the proofs of Theorems 1.5 and 1.7 we still need some preliminary results.

Lemma 3.1. *Let w be analytic on \mathcal{U}_{t_0} and let $w^{(2n+1)}$ be bounded on an open nontangential neighborhood $\Gamma_{t_0, \alpha}$ of $t_0 \in \mathbb{T}$:*

$$|w_{2n+1}(z)| \leq \gamma \quad (z \in \Gamma_{t_0, \alpha}). \quad (3.1)$$

Then the following nontangential limits

$$w_j(t_0) = \lim_{z \rightarrow t_0} w_j(z), \quad w_j(z) := \frac{w^{(j)}(z)}{j!}$$

exist for $j = 0, \dots, 2n$ and

$$w_j(z) = \sum_{i=0}^{2n-j} \binom{j+i}{i} w_{j+i}(t_0)(z - t_0)^i + O((z - t_0)^{2n-j+1}) \quad (z \widehat{\rightarrow} t_0). \quad (3.2)$$

Proof. We start with the Taylor representation

$$w_j(z) = \sum_{i=0}^{2n-j} \frac{w_j^{(i)}(\omega)}{i!} (z - \omega)^i + \int_{\omega}^z \frac{w_j^{(2n-j+1)}(\zeta)}{(2n-j)!} (z - \zeta)^{2n-j} d\zeta \quad (3.3)$$

of the function $w_j(z)$ at a point $\omega \in \Gamma_{t_0, \alpha}$. Since

$$\frac{w_j^{(i)}(z)}{i!} = \frac{w^{(j+i)}(z)}{i!j!} = \frac{(j+i)!}{i!j!} w_{j+i}(z) = \binom{j+i}{i} w_{j+i}(z),$$

representation (3.3) can be written as

$$\begin{aligned} w_j(z) &= \sum_{i=0}^{2n-j} \binom{j+i}{j} w_{j+i}(\omega) (z - \omega)^i \\ &\quad + (2n+1) \binom{2n}{j} \int_{\omega}^z w_{2n+1}(\zeta) (z - \zeta)^{2n-j} d\zeta. \end{aligned} \quad (3.4)$$

The latter integral does not depend on the path of integration between ω and z . We set for short (we integrate along a rectifiable Jordan curve connecting z and ω inside the unit disk \mathbb{D})

$$G_{\omega, z} = \int_{\omega}^z w_{2n+1}(\zeta) (z - \zeta)^{2n-j} d\zeta. \quad (3.5)$$

Since $w(z)$ is bounded on $\Gamma_{t_0, \alpha}$, and $z - \zeta$ is also bounded on $\Gamma_{t_0, \alpha} \times \Gamma_{t_0, \alpha}$, we may conclude that the following two integrals are well defined and the following limit relations hold true

$$G_{\omega, t_0} = \int_{\omega}^{t_0} w_{2n+1}(\zeta) (t_0 - \zeta)^{2n-j} d\zeta = \lim_{z \xrightarrow{\Gamma_{t_0, \alpha}} t_0} F_{\omega, z},$$

and

$$G_{t_0, z} = \int_{t_0}^z w_{2n+1}(\zeta) (z - \zeta)^{2n-j} d\zeta = \lim_{\omega \xrightarrow{\Gamma_{t_0, \alpha}} t_0} F_{\omega, z}.$$

Sending $z \in \mathcal{U}$ to t_0 in (3.4) (for ω fixed) we conclude that there exists the limit

$$w_j(t_0) = \lim_{z \xrightarrow{\Gamma_{t_0, \alpha}} t_0} w_j(z) = \sum_{i=0}^{2n-j} \binom{j+i}{j} w_{j+i}(\omega) (t_0 - \omega)^i + (2n+1) \binom{2n}{j} G_{\omega, t_0}$$

for $j = 0, \dots, 2n$. This proves the first assertion of the lemma and allows us to pass to limits in (3.4) as $\omega \in \Gamma_{t_0, \alpha}$ tends to t_0 for a fixed z to get

$$w_j(z) = \sum_{i=0}^{2n-j} \binom{j+i}{j} w_{j+i}(t_0) (z - t_0)^i + (2n+1) \binom{2n}{j} G_{t_0, z}.$$

Upon choosing the path of integration for $G_{t_0, z}$ to be the line segment connecting the points z and t_0 and making use of (3.1), we get the estimate

$$|G_{t_0, z}| \leq \gamma |z - t_0|^{2n-j+1}, \quad (3.6)$$

which implies (3.2). \square

In the next lemma,

$$\mathbf{P}_{i,j}^w(z) = \frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \frac{1 - |w(z)|^2}{1 - |z|^2} \quad (3.7)$$

stands for the ij -th entry of the Schwarz-Pick matrix $\mathbf{P}_n^w(z)$.

Lemma 3.2. *Let $w(z)$ and $f(z)$ be two functions analytic on \mathbb{D} such that $w^{(2n+1)}(z)$ and $f^{(2n+1)}(z)$ are bounded in some open nontangential neighborhood $\Gamma_{t_0, \alpha}$ of $t_0 \in \mathbb{T}$ and let*

$$\lim_{z \rightarrow t_0} w_j(z) = \lim_{z \rightarrow t_0} f_j(z) =: w_j(t_0) \quad \text{for } j = 0, \dots, 2n \quad (3.8)$$

(note that the existence of limits in (3.8) is guaranteed by Lemma 3.1). Then

$$\mathbf{P}_{i,j}^w(z) - \mathbf{P}_{i,j}^f(z) = O((z - t_0)^{2n-i-j}) \quad (3.9)$$

for $i, j = 0, \dots, n$ as $z \widehat{\rightarrow} t_0$. In particular, $d_{w,n}(z) - d_{f,n}(z) = O(1)$.

Proof. Straightforward differentiation gives

$$\frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \frac{|w(z)|^2}{1 - |z|^2} = \sum_{k=0}^i \sum_{\ell=0}^j w_{i-k}(z) \frac{u_{k,\ell}(z)}{(1 - |z|^2)^{k+\ell+1}} w_{j-\ell}(z)^*, \quad (3.10)$$

where, as before, $w_j(z)$ stands for $\frac{1}{j!} w^{(j)}(z)$ and where

$$u_{k,\ell}(z) = \sum_{m=0}^{\min(k,\ell)} \frac{(k + \ell - m)!}{(k - m)! (\ell - m)! m!} \bar{z}^{k-m} z^{\ell-m} (1 - |z|^2)^m. \quad (3.11)$$

Substituting (3.7), (3.10) and similar formulas for f into (3.9) we get

$$\begin{aligned} \mathbf{P}_{i,j}^w(z) - \mathbf{P}_{i,j}^f(z) &= \frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \frac{|f(z)|^2 - |w(z)|^2}{1 - |z|^2} \\ &= \sum_{k=0}^i \sum_{\ell=0}^j f_{i-k}(z) \frac{u_{k,\ell}(z)}{(1 - |z|^2)^{k+\ell+1}} f_{j-\ell}(z)^* \\ &\quad - \sum_{k=0}^i \sum_{\ell=0}^j w_{i-k}(z) \frac{u_{k,\ell}(z)}{(1 - |z|^2)^{k+\ell+1}} w_{j-\ell}(z)^* \\ &= \sum_{k=0}^i \sum_{\ell=0}^j (f_{i-k}(z) - w_{i-k}(z)) \frac{u_{k,\ell}(z) f_{j-\ell}(z)^*}{(1 - |z|^2)^{k+\ell+1}} \\ &\quad + \sum_{k=0}^i \sum_{\ell=0}^j \frac{w_{i-k}(z) u_{k,\ell}(z)}{(1 - |z|^2)^{k+\ell+1}} (f_{j-\ell}(z)^* - w_{j-\ell}(z)^*). \end{aligned} \quad (3.12)$$

Applying (3.2) to the functions w_k and f_k and making use of equalities (3.8), we conclude that

$$w_k(z) - f_k(z) = O((z - t_0)^{2n-k+1}) \quad \text{as } z \widehat{\rightarrow} t_0$$

for $k = 0, \dots, 2n$. Since $z - t_0 = O(1 - |z|^2)$ when $z \widehat{\rightarrow} t_0$ nontangentially, we have

$$\frac{f_{i-k}(z) - w_{i-k}(z)}{(1 - |z|^2)^{k+\ell+1}} = O((z - t_0)^{2n-i-\ell}), \quad \leq k \leq i \leq n, \quad (3.13)$$

$$\frac{f_{j-\ell}(z) - w_{j-\ell}(z)}{(1 - |z|^2)^{k+\ell+1}} = O((z - t_0)^{2n-j-k}), \quad 0 \leq \ell \leq j \leq n. \quad (3.14)$$

Now asymptotic relation (3.9) follows from (3.12), (3.13) and (3.14). The last assertion follows from (3.9) upon letting $i = j = n$. \square

Lemma 3.3. *Let w and f be two functions analytic on a neighborhood \mathcal{U}_{t_0} of $t_0 \in \mathbb{T}$ and let us assume that the nontangential boundary limits of their $2n + 2$ first derivatives at t_0 exist and are equal:*

$$w_j(t_0) = f_j(t_0) \quad \text{for } j = 0, \dots, 2n + 1. \quad (3.15)$$

Then

$$\mathbf{P}_n^w(z) - \mathbf{P}_n^f(z) = o(1) \quad \text{as } z \widehat{\rightarrow} t_0 \quad (3.16)$$

where $\mathbf{P}_n^w(z)$ and $\mathbf{P}_n^f(z)$ are the Schwarz-Pick matrices associated with w and f via formula (1.3).

Proof. By Remark 2.6, equalities (3.15) are equivalent to the asymptotic relation

$$w(z) - f(z) = o((z - t_0)^{2n+1}) \quad \text{as } z \widehat{\rightarrow} t_0. \quad (3.17)$$

Therefore we have

$$w_k(z) - f_k(z) = o((z - t_0)^{2n-k+1}) \quad \text{for } k = 1, \dots, 2n + 1 \quad \text{as } z \rightarrow t_0.$$

Substituting the latter asymptotics into (3.9) we get

$$\mathbf{P}_{i,j}^w(z) - \mathbf{P}_{i,j}^f(z) = o((z - t_0)^{2n-i-j}) \quad (i, j = 0, \dots, n)$$

which clearly imply (3.16). \square

Proof of Theorem 1.5: The third statement in Theorem 1.5 can be reformulated equivalently: *there exists a rational unimodular function f satisfying conditions (3.15).* Now we apply Theorem 2.5 to the sequence $\{w_0, \dots, w_{2n+1}\}$ where $w_j := w_j(t_0)$ to conclude that the Statements 1-3 in Theorem 1.5 are equivalent.

Assume that the asymptotic equality (1.18) holds (or equivalently, the equalities (3.15) hold) for a rational unimodular function f . Then $\mathbb{P}_n^w(t_0) = \mathbb{P}_n^f(t_0)$, by the definition (1.9), and the relation (3.16) holds by Lemma 3.3. Furthermore, since f is rational and unimodular on \mathbb{T} , the boundary Schwarz-Pick matrix $\mathbf{P}_n^f(t_0)$ exists and is equal to $\mathbb{P}_n^f(t_0)$, by Statement 1 in Theorem 2.3. Now we pass to nontangential limits in (3.16) to get

$$\mathbf{P}_n^w(t_0) = \lim_{z \widehat{\rightarrow} t_0} \mathbf{P}_n^w(z) = \lim_{z \widehat{\rightarrow} t_0} \mathbf{P}_n^f(z) = \mathbf{P}_n^f(t_0) = \mathbb{P}_n^f(t_0) = \mathbb{P}_n^w(t_0).$$

To complete the proof of Theorem 1.5, it suffices to show that (3) \Rightarrow (4) \Rightarrow (2).

Proof of (3) \Rightarrow (4): Assume that the asymptotic equality

$$w(z) = f(z) + o((z - t_0)^{2n+1}) \quad (3.18)$$

holds (or equivalently, the equalities (3.15) hold) for a rational unimodular function f as $z \widehat{\rightarrow} t_0$ from inside of the disk \mathbb{D} . Then we also have for $|z| > 1$,

$$w(z) := \overline{w(1/\bar{z})}^{-1} = \overline{f(1/\bar{z})}^{-1} + o\left(\overline{\left(\frac{1}{\bar{z}} - t_0\right)^{2n+1}}\right) = f(z) + o((z - t_0)^{2n+1}) \quad (3.19)$$

as $z \widehat{\rightarrow} t_0$ from outside of \mathbb{D} , where $f(z)$ stands for both symmetry and analytic continuation, that agree since f is rational and unimodular on \mathbb{T} . Since f is analytic in a neighborhood of t_0 , its asymptotics form inside and outside of the unit disk agree. The asymptotic equality (1.19) follows from the fact that the limits $w_j(t_0)$ exist for $j = 0, \dots, n$ as $z \widehat{\rightarrow} t_0$ from inside and outside of \mathbb{D} (by Remark 2.6) and are equal.

Proof of (4) \Rightarrow (2): Let us assume that the asymptotic relations (1.19) hold from inside and outside of \mathbb{D} . By definition of symmetry continuation we have

$$\overline{w(1/\bar{z})} \equiv w(z)^{-1}, \quad |z| < 1.$$

Since

$$\frac{1}{z} - \bar{t}_0 = \frac{-\bar{t}_0(z - t_0)}{z} = O((z - t_0)) \quad \text{as } z \rightarrow t_0,$$

it follows, by substituting $1/\bar{z}$, $|z| < 1$ in (1.19), that

$$\overline{w(1/\bar{z})} = \sum_{j=0}^k w_j(t_0)^* \left(\frac{1}{z} - \bar{t}_0\right)^j + o((z - t_0)^k)$$

for $k = 0, \dots, 2n + 1$ and therefore, that

$$\begin{aligned} z^k w(z)^{-1} \equiv z^k \overline{w(1/\bar{z})} &= \sum_{j=0}^k w_j(t_0)^* z^{k-j} (1 - z\bar{t}_0)^j + o((z - t_0)^k) \\ &= \sum_{j=0}^k (-\bar{t}_0)^j w_j(t_0)^* z^{k-j} (z - t_0)^j + o((z - t_0)^k). \end{aligned}$$

for $|z| < 1$. Simple rearrangements based on the relation $z^\ell = \sum_{i=0}^{\ell} \binom{\ell}{i} t_0^{\ell-i} (z - t_0)^i$,

lead us to

$$z^k w(z)^{-1} = \sum_{j=0}^k \left(\sum_{\ell=0}^{k-j} (-1)^\ell t_0^{j-\ell} \binom{k-\ell}{j} w_\ell(t_0)^* \right) (z - t_0)^{k-j} + o((z - t_0)^k).$$

Upon multiplying the corresponding parts both in the latter equality and (1.19) (for $|z| < 1$) we get

$$\begin{aligned} & \left(\sum_{j=0}^{2n+1} w_j(t_0)(z-t_0)^j \right) \left(\sum_{j=0}^k \left(\sum_{\ell=0}^{k-j} (-1)^\ell t_0^{j-\ell} \binom{k-\ell}{j} w_\ell(t_0)^* \right) (z-t_0)^{k-j} \right) \\ &= z^k + o((z-t_0)^k) = \sum_{j=0}^k \binom{k}{j} t_0^{k-j} (z-t_0)^j + o((z-t_0)^k) \end{aligned}$$

Comparing the coefficients of $(z-t_0)^k$ in the latter equality we eventually arrive at

$$\sum_{j=0}^k \sum_{\ell=0}^{k-j} (-1)^\ell \binom{k-\ell}{j} t_0^{j-\ell} w_j(t_0) w_\ell(t_0)^* = 1 \quad \text{for } k = 0, \dots, 2n+1.$$

By Remark 2.1, the latter equalities mean that the sequence $\{w_0(t_0), \dots, w_{2n+1}(t_0)\}$ is t_0 -isometric. \square

Proof of Theorem 1.7: Let us assume that $w^{(2n+1)}$ is bounded on $\Gamma_{t_0, \alpha}$ and that the sequence $\{w_0, \dots, w_{2n}\}$ of the nontangential boundary limits $w_j := w_j(t_0)$ is t_0 -isometric (the existence of the latter limits is guaranteed by Lemma 3.1). Let w_{2n+1} be any number such that the extended sequence $\{w_0, \dots, w_{2n}, w_{2n+1}\}$ is t_0 -isometric (such a number exists by Remark 2.7). By Theorem 2.5, there exists a rational unimodular function f such that $f_j(t_0) = w_j$ for $j = 0, \dots, 2n+1$. The functions f and w meets all the conditions in Lemma 3.2. Therefore,

$$d_{w,n}(z) - d_{f,n}(z) = O(1) \tag{3.20}$$

as z tends to t_0 nontangentially. Since f is rational unimodular function the limit $d_{f,n}(t_0)$ exists and in particular, $d_{f,n}(z)$ is bounded on $\Gamma_{t_0, \alpha}$. Then (3.20) implies (1.21). Equivalences (1) \Leftrightarrow (2) \Leftrightarrow (3) in Theorem 1.7 are established in much the same way as those in Theorem 1.5. \square

4. Generalized Schur functions

Condition (1.15) does not make much sense for general analytic functions. For example, the function $w(z) = e^{\frac{1}{1-z}}$ meets condition (1.15) (for $n = 0$) at $t_0 = 1$, since $|w(z)| = 1$ on the radius $[0, 1)$; however, it has no radial limit as z approaches 1. Thus the extension of Theorem 1.4 (at least in its present formulation) to general analytic functions is unlikely. However, this theorem can be extended to some classes of functions that are close in some sense to \mathcal{S} . In this section we will show how it can be done for the so-called *generalized Schur functions*. Recall that a function w belongs to the generalized Schur class \mathcal{S}_κ if it admits a representation of the form

$$w(z) = \frac{s(z)}{b(z)}, \tag{4.1}$$

where s is a Schur class function and b is a finite Blaschke product b of degree κ , having disjoint zeroes in \mathbb{D} . A well known property of \mathcal{S}_κ functions is that the kernel

$$K_w(z, \zeta) := \frac{1 - w(z)\overline{w(\zeta)}}{1 - z\bar{\zeta}}$$

has κ negative squares on $\rho(w)$, the domain of analyticity of w intersected with \mathbb{D} . Therefore, the Schwarz-Pick matrix $\mathbf{P}_n^w(z)$ defined in (1.3) makes sense and has not more than κ negative eigenvalues (counted with multiplicities) for every $z \in \rho(w)$ and every $n \in \mathbb{Z}_+$; in formulas: $\text{sq}_- \mathbf{P}_n^w(z) \leq \kappa$.

Remark 4.1. *It is that once the boundary Schwarz-Pick matrix $\mathbf{P}_n^w(t_0)$ exists, $\text{sq}_- \mathbf{P}_n^w(t_0) \leq \kappa$.*

The next theorem extends Theorem 1.4.

Theorem 4.2. *Let $w \in \mathcal{S}_\kappa$, $t_0 \in \mathbb{T}$, $n \in \mathbb{Z}_+$ and let $d_{w,n}$ be defined as in (1.5). The following are equivalent.*

1. $\tilde{d} := \liminf_{z \rightarrow t_0} d_{w,n}(z) < \infty$.
2. $d_{w,n}(t_0) := \lim_{z \rightarrow t_0} d_{w,n}(z) < \infty$.
3. *The boundary Schwarz-Pick matrix $\mathbf{P}_n^w(t_0)$ exists.*
4. *The nontangential boundary limits $w_j(t_0)$ exist for $j = 0, \dots, 2n+1$ and satisfy*

$$|w_0(t_0)| = 1, \quad \mathbb{P}_n^w(t_0) = \mathbb{P}_n^w(t_0)^* \quad \text{and} \quad \text{sq}_- \mathbb{P}_n^w(t_0) \leq \kappa,$$

where $\mathbb{P}_n^w(t_0)$ is the matrix defined in (1.9).

Moreover, when these conditions hold, then

$$\tilde{d} = d_{w,n}(t_0) \quad \text{and} \quad \mathbf{P}_n^w(t_0) = \mathbb{P}_n^w(t_0). \quad (4.2)$$

For w of the form (4.1) we have

$$\frac{1 - |w(z)|^2}{1 - |z|^2} = \frac{1 - |s(z)|^2}{|b(z)|^2(1 - |z|^2)} - \frac{1 - |b(z)|^2}{|b(z)|^2(1 - |z|^2)}.$$

Upon applying $\frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j}$ to both parts of the latter equality and arranging the obtained $(n+1)^2$ equalities in the matrix form, we get, by the definitions (1.3),

$$\mathbf{P}_n^w(z) = L^{s,b}(z) - L^{b,b}(z), \quad (4.3)$$

where we have set

$$L^{s,b}(z) := \left[\frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \frac{1}{b(z)} \frac{1 - |s(z)|^2}{1 - |z|^2} \frac{1}{\overline{b(z)}} \right]_{i,j=0}^n, \quad (4.4)$$

$$L^{b,b}(z) := \left[\frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \frac{1}{b(z)} \frac{1 - |b(z)|^2}{1 - |z|^2} \frac{1}{\overline{b(z)}} \right]_{i,j=0}^n. \quad (4.5)$$

Note that if K is a sesqui-analytic kernel and f is analytic at z , then the equality

$$\left[\frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} f(z) K(z, z) \overline{f(z)} \right]_{i,j=0}^n = \mathbb{U}_n^f(z)^* \cdot \left[\frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} K(z, z) \right]_{i,j=0}^n \cdot \mathbb{U}_n^f(z)$$

holds by the Leibnitz rule. Upon applying this equality to

$$K(z, \zeta) = \frac{1 - s(z)\overline{s(\zeta)}}{1 - z\bar{\zeta}} \quad \text{and} \quad f(z) = \frac{1}{b(z)},$$

and taking into account that $\mathbb{U}_n^{1/b}(z) = \mathbb{U}_n^b(z)^{-1}$ whenever $b(z) \neq 0$, we get

$$L^{s,b}(z) := \mathbb{U}_n^b(z)^{-*} \mathbf{P}_n^s(z) \mathbb{U}_n^b(z)^{-1} \quad (4.6)$$

where $\mathbf{P}_n^s(z)$ is the Schwarz-Pick matrix defined via formula (1.3). Letting $s = b$ in the latter formula, we get

$$L^{b,b}(z) = \mathbb{U}_n^b(z)^{-*} \mathbf{P}_n^b(z) \mathbb{U}_n^b(z)^{-1}. \quad (4.7)$$

Remark 4.3. Let s be a Schur function, let b be a finite Blaschke product and let $L^{s,b}(z)$ and $L^{b,b}(z)$ be defined as in (4.4) and (4.5). Then

1. The nontangential limit

$$L^{s,b}(t_0) := \lim_{z \widehat{\rightarrow} t_0} L^{s,b}(z) \quad (4.8)$$

exists if and only if the boundary Schwarz-Pick matrix $\mathbf{P}_n^s(t_0)$ exists and in this case

$$L^{s,b}(t_0) = \mathbb{U}_n^b(t_0)^{-*} \mathbf{P}_n^s(t_0) \mathbb{U}_n^b(t_0)^{-1}. \quad (4.9)$$

2. The limit

$$L^{b,b}(t_0) = \lim_{z \rightarrow t_0} L^{b,b}(z) = \mathbb{U}_n^b(t_0)^{-*} \mathbf{P}_n^b(t_0) \mathbb{U}_n^b(t_0)^{-1} \quad (4.10)$$

exists for every $n \in \mathbb{Z}_+$, when z tends to $t_0 \in \mathbb{T}$ unrestrictedly in \mathbb{C} .

Proof. Since b is analytic at $t_0 \in \mathbb{T}$ and $b(t_0) \neq 0$, the limit matrix $\mathbb{U}_n^b(t_0) := \lim_{z \rightarrow t_0} \mathbb{U}_n^b(z)$ exists and is invertible. Now the first assertion follows from (4.6) upon passing to limits as $z \widehat{\rightarrow} t_0$. The second assertion follows since $\mathbf{P}_n^b(t_0)$ exists, by Theorem 2.3. \square

Corollary 4.4. Let $w \in \mathcal{S}_\kappa$ be of the form (4.1) with a Schur function s and a finite Blaschke product b . Then the boundary Schwarz-Pick matrix $\mathbf{P}_n^w(t_0)$ exists if and only if $\mathbf{P}_n^s(t_0)$ exists and in this case,

$$\mathbf{P}_n^w(t_0) = \mathbb{U}_n^b(t_0)^{-*} (\mathbf{P}_n^s(t_0) - \mathbf{P}_n^b(t_0)) \mathbb{U}_n^b(t_0)^{-1}. \quad (4.11)$$

Proof. Since the limit (4.10) exists no matter how z tends to t_0 , it follows from (4.3) that $\mathbf{P}_n^w(t_0)$ exists if and only if $L^{s,b}(t_0)$ exists; the latter is equivalent to the existence of $\mathbf{P}_n^s(t_0)$, by Remark 4.3. Passing to the limits in (4.3) as $z \widehat{\rightarrow} t_0$ and making use of (4.9) and (4.10) we arrive at (4.11). \square

Lemma 4.5. *Let $s \in \mathcal{S}_0$ be a Schur function, let f be a function analytic at $t_0 \in \mathbb{T}$ and let $n \in \mathbb{Z}_+$. Then*

$$\liminf_{z \rightarrow t_0} \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \left(f(z) \frac{1 - |s(z)|^2}{1 - |z|^2} \overline{f(z)} \right) < \infty$$

if and only if (the “if” part is obvious) the following limits exist and are finite:

$$\lim_{z \xrightarrow{t_0}} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \left(f(z) \frac{1 - |s(z)|^2}{1 - |z|^2} \overline{f(z)} \right) < \infty \quad (i, j = 0, \dots, n). \quad (4.12)$$

Furthermore,

$$\liminf_{z \rightarrow t_0} \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \left(f(z) \frac{1 - |s(z)|^2}{1 - |z|^2} \overline{f(z)} \right) = \lim_{z \xrightarrow{t_0}} \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \left(f(z) \frac{1 - |s(z)|^2}{1 - |z|^2} \overline{f(z)} \right). \quad (4.13)$$

The above result has been proved in [5] in the context of operator-valued Schur functions s and vector valued functions f . The result is useful even in the scalar-valued setting due to the following

Corollary 4.6. *Let s be a Schur function, let b be a finite Blaschke product and let*

$$L_{nn}^{s,b}(z) = \frac{1}{(n!)^2} \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \frac{1 - |s(z)|^2}{|b(z)|^2(1 - |z|^2)}, \quad (4.14)$$

the lower diagonal element of the matrix $L^{s,b}(z)$ given in (4.4) be subject to

$$\liminf_{z \rightarrow t_0} L_{nn}^{s,b}(z) < \infty \quad (t_0 \in \mathbb{T}). \quad (4.15)$$

Then the nontangential boundary limit (4.8) exists and

$$\lim_{z \xrightarrow{t_0}} L_{nn}^{s,b}(z) = \liminf_{z \rightarrow t_0} L_{nn}^{s,b}(z). \quad (4.16)$$

Furthermore, the boundary Schwarz-Pick matrix $\mathbf{P}_n^s(t_0)$ exists.

Proof. Condition means that we can apply Lemma 4.5 for $f(z) = \frac{1}{b(z)}$. Then relations (4.12) mean that all the entries in the matrix $L^{s,b}(z)$ possess nontangential boundary limits and thus, the limit (4.8) exists. Then $\mathbf{P}_n^s(t_0)$ exists by Remark 4.3. Finally, equality (4.13) with $f = \frac{1}{b}$ gives (4.16). \square

Proof of Theorem 4.2: Let $w \in \mathcal{S}_\kappa$ be a generalized Schur function with the Krein–Langer representation (4.1) and let us assume that

$$\tilde{d} := \liminf_{z \rightarrow t_0} d_{w,n}(z) < \infty$$

Equating the lower diagonal entries in the matrix identity (4.3) gives

$$d_{w,n}(z) = L_{nn}^{s,b}(z) - L_{nn}^b(z) \quad (4.17)$$

where $L_{nn}^{s,b}(z)$ is given in (4.14) and where, according to (4.5)

$$L_{nn}^b(z) = \frac{1}{(n!)^2} \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \frac{1 - |b(z)|^2}{|b(z)|^2(1 - |z|^2)}.$$

By Remark 4.3, the limit

$$L_{nn}^b(t_0) = \lim_{z \rightarrow t_0} L_{nn}^b(z)$$

exists. Then it follows from (4.17) that

$$\liminf_{z \rightarrow t_0} d_{w,n}(z) = \liminf_{z \rightarrow t_0} L_{nn}^{s,b}(z) - L_{nn}^b(t_0) \quad (4.18)$$

and since the limit inferior on the left is finite, we conclude that condition (4.15) is satisfied. Then the boundary Schwarz-Pick matrix $\mathbf{P}_n^s(t_0)$ exists by Corollary 4.6. By Corollary 4.4, the boundary Schwarz-Pick matrix $\mathbf{P}_n^w(t_0)$ also exists. Now we can pass to the limits in (4.17) as $z \widehat{\rightarrow} t_0$:

$$d_{w,n}(t_0) = \lim_{z \widehat{\rightarrow} t_0} d_{w,n}(z) = \lim_{z \widehat{\rightarrow} t_0} L_{nn}^{s,b}(z) - L_{nn}^b(t_0)$$

which implies, on account of (4.18) and of equality (4.16) (that holds, by Corollary 4.6)

$$d_{w,n}(t_0) = \liminf_{z \rightarrow t_0} d_{w,n}(z).$$

This completes the proof of implications (1) \Rightarrow (3) \Rightarrow (2) in Theorem 4.2 and also the first equality in (4.2).

Furthermore, since $s \in \mathcal{S}$, the existence of $\mathbf{P}_n^s(t_0)$ guarantees (by Theorem 1.4) the existence of the nontangential boundary limits

$$s_j(t_0) := \lim_{z \rightarrow t_0} \frac{s^{(j)}(z)}{j!} \quad \text{for } j = 0, \dots, 2n+1 \quad (4.19)$$

such that $|s_0(t_0)| = 1$ and the matrix $\mathbb{P}_n^s(t_0)$ defined via formula (1.9) is Hermitian. Then the sequence $\{s_0(t_0), \dots, s_{2n+1}(t_0)\}$ is t_0 -isometric, by Theorem 2.5:

$$\overline{\mathbb{U}}_{2n+1}^s(t_0) \mathbf{\Psi}_{2n+1}(t_0) \mathbb{U}_{2n+1}^s(t_0) = \mathbf{\Psi}_{2n+1}(t_0). \quad (4.20)$$

The sequence $\{b_0(t_0), \dots, b_{2n+1}(t_0)\}$ is t_0 -isometric, by the second assertion in Theorem 2.3: Thus, we have

$$\overline{\mathbb{U}}_{2n+1}^b(t_0) \mathbf{\Psi}_{2n+1}(t_0) \mathbb{U}_{2n+1}^b(t_0) = \mathbf{\Psi}_{2n+1}(t_0). \quad (4.21)$$

Since b is analytic at t_0 and since $b(t_0) \neq 0$, the existence of the boundary limits

$$w_j(t_0) := \lim_{z \rightarrow t_0} \frac{w^{(j)}(z)}{j!} \quad (j = 0, \dots, 2n+1) \quad (4.22)$$

follows from (4.1) and (4.21). Since $s = wb$, we have $\mathbb{U}_{2n+1}^s(t_0) = \mathbb{U}_{2n+1}^w(t_0) \mathbb{U}_{2n+1}^b(t_0)$ so that

$$\mathbb{U}_{2n+1}^w(t_0) = \mathbb{U}_{2n+1}^s(t_0) \mathbb{U}_{2n+1}^b(t_0)^{-1}.$$

The latter equality together with (4.20) and (4.21) implies

$$\overline{\mathbb{U}}_{2n+1}^w(t_0) \mathbf{\Psi}_{2n+1}(t_0) \mathbb{U}_{2n+1}^w(t_0) = \mathbf{\Psi}_{2n+1}(t_0)$$

which means that the sequence $\{w_0(t_0), \dots, w_{2n+1}(t_0)\}$ is t_0 -isometric. By Theorem 1.5, conditions (1.17) are satisfied and the matrix $\mathbb{P}_n^w(t_0)$ constructed from the limits (4.22) via formula (1.9), is equal to the boundary Schwarz-Pick matrix $\mathbf{P}_n^w(t_0)$. This proves the second equality in (4.2). Furthermore, this equality implies that $\mathbb{P}_n^w(t_0)$ is Hermitian and (by Remark 4.1) that $\text{sq}_- \mathbf{P}_n^w(t_0) \leq \kappa$. This completes the proof of Theorem 4.2. \square

The next theorem partly generalizes Theorem 4.2.

Theorem 4.7. *Let $w = \frac{s}{f}$ be the ratio of two Schur class functions satisfying conditions*

$$\liminf_{z \rightarrow t_0} d_{s,n}(z) < \infty \quad \text{and} \quad \liminf_{z \rightarrow t_0} d_{f,n}(z) < \infty \quad (4.23)$$

where $t_0 \in \mathbb{T}$ and $n \in \mathbb{Z}_+$. Then

1. *The nontangential boundary limits $w_j(t_0)$ exist for $j = 0, \dots, 2n+1$ and $|w_0(t_0)| = 1$.*
2. *The boundary Schwarz-Pick matrix $\mathbf{P}_n^w(t_0)$ exists and is equal to the matrix $\mathbb{P}_n^w(t_0)$ defined in (1.9). In particular, $\mathbb{P}_n^w(t_0)$ is Hermitian.*

By Theorem 1.4, conditions (4.23) guarantee the existence of the the boundary Schwarz-Pick matrices $\mathbf{P}_n^s(t_0)$ and $\mathbf{P}_n^f(t_0)$. The rest is the same as in the proof of Theorem 4.2.

Note that in case when $w = \frac{s}{f}$ is the ratio of two Schur class functions, the condition

$$\liminf_{z \rightarrow t_0} d_{w,n}(z) < \infty \quad (4.24)$$

follows from (4.23) but does not imply (4.23) in general. Thus, the conclusions in Theorem 4.7 are obtained under apparently too strong assumptions. Conditions (4.23) in Theorem 4.7 cannot be relaxed to (4.24) and we do not know what conditions in terms of $d_{w,n}(z)$ may replace (4.23).

5. Infinite t_0 -isometric sequences and related results

Theorem 2.5 shows that any finite t_0 -isometric sequence $\{w_0, \dots, w_n\}$ arises as a sequence of the first $n+1$ Taylor coefficients of a rational unimodular function at t_0 (moreover, one can use rational unimodular functions with the only pole at the origin to get all finite t_0 -isometric sequences. In this section we characterize infinite t_0 -isometric sequences in similar terms.

Definition 5.1. We will say that a sequence $\{w_j\}_{j=0}^\infty$ of complex numbers is t_0 -positive if $|w_0| = 1$ and the matrix $\mathbb{P}(t_0, w_0, \dots, w_{2n+1})$ defined as in (1.13) is positive semidefinite for all $n \geq 0$.

Remark 5.2. By Theorem 2.5, a sequence t_0 -isometric if and only if $|w_0| = 1$ and the matrix $\mathbb{P}(t_0, w_0, \dots, w_{2n+1})$ is Hermitian for every $n \geq 0$. In particular, any t_0 -positive sequence is t_0 -isometric.

Definition 5.3. Let $t_0 \in \mathbb{T}$. We will say that a Schur function w belongs to the class \mathcal{S}_{t_0} if

$$\liminf_{z \rightarrow t_0} d_{w,n}(z) < \infty \quad \text{for all } n \geq 0 \quad (5.1)$$

or equivalently, if

$$\lim_{z \rightarrow t_0} d_{w,n}(z) < \infty \quad \text{for all } n \geq 0, \quad (5.2)$$

where $d_{w,n}(z)$ is defined as in (1.5). We denote by \mathcal{QS}_{t_0} the class of functions w of the form $w = \frac{s}{f}$ where $s, f \in \mathcal{S}_{t_0}$.

Theorem 5.4. Let $t_0 \in \mathbb{T}$ and let $w \in \mathcal{S}_{t_0}$. Then the nontangential boundary limits $w_j(t_0)$ exist for every $j \geq 0$ and the sequence $\{w_j(t_0)\}_{j=0}^\infty$ is t_0 -positive. Furthermore, any t_0 -positive sequence arises in this way.

Proof. The first assertion follows from Theorem 1.4 and Definition (5.1) of t_0 -positivity. In regard to the second assertion recall a known interpolation result [10] (analogous to the classical Hamburger's solvability criteria for the power moment problem [8]) asserting that if a sequence $\{w_j\}_{j=0}^\infty$ is t_0 -positive, then (and only then) there exists a Schur class function w such that

$$w_j(t_0) := \lim_{z \rightarrow t_0} \frac{w^{(j)}(z)}{j!} = w_j \quad \text{for } j \geq 0.$$

For this function w we have $|w_0| = 1$ and $\mathbb{P}_n^w(t_0) \geq 0$ for all $n \geq 0$ and then it follows by Theorem 1.4 (implication (4) \Rightarrow (2)) that condition (1.16) holds for all $n \geq 0$, i.e., that $w \in \mathcal{S}_{t_0}$. \square

Theorem 5.5. Let $t_0 \in \mathbb{T}$ and let $w \in \mathcal{QS}_{t_0}$. Then the nontangential boundary limits $w_j(t_0)$ exist for every $j \geq 0$ and the sequence $\{w_j(t_0)\}_{j=0}^\infty$ is t_0 -isometric. Furthermore, any t_0 -isometric sequence arises in this way.

Proof. The first assertion follows from Theorem 4.7. To prove the second assertion, let us assume that $\{w_j(t_0)\}_{j=0}^\infty$ is a t_0 -isometric sequence. Then by Remark 5.2, $|w_0| = 1$ and $\mathbb{P}(t_0, w_0, \dots, w_{2n+1})$ is Hermitian for every $n \geq 0$. Then we use the inductive construction from the proof of Lemma 2.4 to get a t_0 -positive sequence $\{f_j\}_{j=0}^\infty$ such that the sequence $\{s_j\}_{j=0}^\infty$ defined by

$$s_j = \sum_{\ell=0}^j f_{j-\ell} w_\ell \quad \text{for } j = 0, 1, \dots \quad (5.3)$$

is t_0 -positive. By Theorem 5.4, there exist functions $s(z)$ and $f(z)$ in \mathcal{S}_{t_0} satisfying the conditions

$$\frac{s^{(j)}(t_0)}{j!} = s_j \quad \text{and} \quad \frac{f^{(j)}(t_0)}{j!} = f_j \quad \text{for } j = 0, 1, \dots \quad (5.4)$$

Now it follows from (5.3) that the quotient $w(z) = \frac{s}{f}$ (which belongs to \mathcal{QS}_{t_0}) satisfies the conditions (5.2). \square

For general analytic functions we have the following result which is an infinite analogue of Theorem 1.5.

Theorem 5.6. *Let w be analytic in a neighborhood \mathcal{U}_{t_0} of $t_0 \in \mathbb{T}$. The following are equivalent:*

1. *The nontangential boundary limits $w_j(t_0)$ exist for every $j \in \mathbb{Z}_+$ and the sequence $\{w_j(t_0)\}_{j=0}^\infty$ is t_0 -isometric.*
2. *There exist a function $f \in \mathcal{QS}_{t_0}$ such that*

$$w(z) = f(z) + o((z - t_0)^n) \quad \text{for all } n \geq 0 \quad \text{as } z \widehat{\rightarrow} t_0. \quad (5.5)$$

3. *w admits the analytic continuation by symmetry $w(z) = \frac{1}{\overline{w(1/\bar{z})}}$ across t_0 into an external nontangential neighborhood*

$$\widetilde{\Gamma}_{t_0, \alpha, \varepsilon} := \{z \notin \mathbb{D} : |z - t_0| < \varepsilon, |\arg(z - t_0)| < \alpha\}$$

of t_0 for every $\alpha \in (0, \frac{\pi}{2})$ and $\varepsilon = \varepsilon(\alpha) > 0$.

Proof. For the proof of (1) \Leftrightarrow (3) it suffices to note that the nontangential asymptotic equality (1.19) holding for every $n \in \mathbb{Z}_+$ from inside and outside of the unit disk, is equivalent to Statement 3 in Theorem 5.6. Equivalence (1) \Leftrightarrow (2) follows from Theorem 5.5: since the limits $f_j(t_0)$ exist for $j \geq 0$, the relation (5.5) is equivalent to existence of the limits $w_j(t_0) = f_j(t_0)$ for $j \geq 0$. \square

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