Nevanlinna–Pick meromorphic interpolation: The degenerate case and minimal norm solutions

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A B S T R A C T

The Nevanlinna–Pick interpolation problem is studied in the class $\mathcal{S}_\kappa$ of meromorphic functions f with κ poles inside the unit disk $\mathbb{D}$ and with $\|f\|_{L^\infty(\mathbb{T})} \leqslant 1$. In the indeterminate case, the parametrization of all solutions is given in terms of a family of linear fractional transformations with disjoint ranges. A necessary and sufficient condition for the problem being determinate is given in terms of the Pick matrix of the problem. The result is then applied to obtain necessary and sufficient conditions for the existence of a meromorphic function with a given pole multiplicity which satisfies Nevanlinna–Pick interpolation conditions and has the minimal possible L^∞-norm on the unit circle $\mathbb{T}$.

1. Introduction

Let $\mathcal{S}$ stand for the Schur class of analytic functions mapping the unit disk $\mathbb{D}$ into its closure $\overline{\mathbb{D}}$ and let $\mathcal{B}_\kappa$ be the set of finite Blaschke products of degree κ. We denote by $\mathcal{S}_\kappa$ the *generalized Schur class* of meromorphic functions of the form

$$f(z) = \frac{s(z)}{b(z)}, \tag{1.1}$$

where $s \in \mathcal{S}$ and $b \in \mathcal{B}_\kappa$ do not have common zeros. Formula (1.1) is called the Kreĭn–Langer representation of a generalized Schur function f; the entries s and b are defined by f uniquely if we assume that b is normalized to the form

$$b(z) = \prod_{i=1}^{\kappa} \frac{z - a_i}{1 - z\bar{a}_i}, \quad a_i \in \mathbb{D}, \qquad \deg b = \kappa. \tag{1.2}$$

Via nontangential boundary limits, $\mathcal{S}_\kappa$-functions can be identified with the functions from the unit ball of $L^\infty(\mathbb{T})$ which admit meromorphic continuation inside the unit disk with total pole multiplicity equal κ. Also (see [11]) $\mathcal{S}_\kappa$-functions can be characterized as meromorphic functions f on $\mathbb{D}$ with the associated kernel $\frac{1-f(z)f(\zeta)^*}{1-z\zeta}$ having κ negative squares on $\rho(f)$, the domain of analyticity of f. Equivalently, all Schwarz–Pick matrices (which are clearly Hermitian)

$$P_n(f; z_1, \ldots, z_n) := \left[\frac{1 - f(z_i)f(z_j)^*}{1 - z_i\bar{z}_j} \right]_{j,i=1}^{n}, \quad z_1, \ldots, z_n \in \rho(f), \tag{1.3}$$

have at most κ negative eigenvalues (counted with multiplicities), and at least one such matrix has exactly κ negative eigenvalues. In what follows, we denote by $\pi(X)$, $\nu(X)$ and $\delta(X)$ respectively the numbers of positive, negative and zero

eigenvalues, counted with multiplicities, of a Hermitian matrix X. For notational convenience, we will often write f^* rather than $\bar{f}$. We will be mostly concerned about the Nevanlinna–Pick interpolation problem:

NP$_\kappa$: Given n distinct points $z_1, \ldots, z_n \in \mathbb{D}$, complex numbers $f_1, \ldots, f_n$ and an integer $\kappa \geqslant 0$, find all functions $f \in \mathcal{S}_\kappa$ (if exist) which are analytic at $z_1, \ldots, z_n$ and satisfy

$$f(z_i) = f_i \quad \text{for } i = 1, \ldots, n. \tag{1.4}$$

As in the classical case $\kappa = 0$, necessary and sufficient conditions for the problem **NP$_\kappa$** to have a solution can be given in terms of the *Pick matrix*

$$P = \left[\frac{1 - f_i f_j^*}{1 - z_i \bar{z}_j} \right]_{i,j=1}^n \tag{1.5}$$

of the problem. These conditions are contained in Theorem 1.2 below.

Definition 1.1. A Hermitian matrix P of rank d is said to be *saturated* if every $d \times d$ principal submatrix of P is invertible.

Theorem 1.2. *Let P be the Pick matrix of the problem* **NP$_\kappa$**. *Then*:

(1) *The problem has infinitely many solutions if and only if $\kappa \geqslant \nu(P) + \delta(P)$.*
(2) *The problem has a unique solution if and only if $\kappa = \nu(P)$, $\delta(P) > 0$, and P is saturated.*
(3) *Otherwise, the problem has no solutions.*

The absence of solutions in case $\kappa < \nu(P)$ is immediate: if the problem **NP$_\kappa$** admits a solution $f \in \mathcal{S}_\kappa$, then $P_n(f; z_1, \ldots, z_n) = P$ and consequently,

$$\nu(P) = \nu\big(P_n(f; z_1, \ldots, z_n)\big) \leqslant \kappa. \tag{1.6}$$

Statement (1) and the absence of solutions in case $\nu(P) < \kappa < \nu(P) + \delta(P)$ (which is part of statement (3)) were established in [13] in the context of a related to $\mathcal{S}_\kappa$ class $\mathcal{N}_\kappa$ of generalized Nevanlinna functions. These parts will be recovered in Corollaries 2.6 and 2.11 below as consequences of a Schur-type reduction. A new point here is a description of the solution set **S(NP$_\kappa$)** of the problem **NP$_\kappa$** for each fixed $\kappa \geqslant \nu(P) + \delta(P)$. It is known that the set **S(NP$_\kappa$)** can be parametrized by a single linear fractional formula if $\delta(P) = 0$ (see e.g., [2–4,9]) or if P is singular and saturated [10]. Theorem 2.10 below shows that in the general singular case, the set **S(NP$_\kappa$)** can be parametrized by a family of linear fractional transformations with disjoint ranges.

The uniqueness criterion for the problem **NP$_\kappa$** can be established as follows: it turns out that if $\kappa = \nu(P)$ and $\delta(P) > 0$, then the problem **NP$_\kappa$** has at most one solution and the rational function f^0 defined explicitly in terms of interpolation data by formula (2.21) below, is the only candidate. Upon making use of formula (2.21) one can rewrite conditions $f^0(z_i) = f_i$ entirely in terms of interpolation data. This gives a uniqueness criterion which actually reads: *The problem* **NP$_\kappa$** *has a unique solution if and only if the only candidate f^0 is indeed a solution to* **NP$_\kappa$**. The uniqueness criterion given in statement (2) of Theorem 1.2 provides yet additional evidence that the features of the Nevanlinna–Pick problem depend on spectral and structural properties of the associated Pick matrix rather than on individual values of z_i and f_i. The present criterion looks quite satisfactory from computational point of view; the next theorem gives a simple test to verify whether or not a matrix P of the form (1.4) is saturated.

Theorem 1.3. *Let $P \in \mathbb{C}^{n \times n}$ be of the form* (1.5) *and let $d := \operatorname{rank} P < n$. Then P is saturated if and only if at least one $(d+1) \times (d+1)$ principal submatrix $\widetilde{P}$ of P with $\operatorname{rank} \widetilde{P} = d$ is saturated.*

Thus, to verify that P is saturated, it suffices to pick up any $(d+1) \times (d+1)$ principal submatrix $\widetilde{P}$ of P with $\operatorname{rank} \widetilde{P} = \operatorname{rank} P = d$ and to verify invertibility of its two $d \times d$ principal submatrices.

Remark 1.4. An earlier appearance of saturated matrices in Nevanlinna–Pick interpolation theory occured in [12]. D. Sarason showed that the boundary Nevanlinna–Pick interpolation problem has a unique Schur-class solution if and only if the corresponding Pick matrix is singular, positive semidefinite and saturated (we refer to [12] for the precise formulation of the problem and definition of the boundary Pick matrix). For the classical (interior) Schur-class Nevanlinna–Pick problem (in our notation, **NP$_0$**) saturated matrices do not appear explicitly in the uniqueness criterion for a simple reason—if a matrix P of the form (1.5) is positive semidefinite (i.e., $\nu(P) = 0$), then it is automatically saturated.

The proofs of Theorems 1.2 and 1.3 are given in Section 3. In Section 4 we discuss the existence of solutions of the problem **NP$_\kappa$** of the minimal possible L^∞-norm. Our result in this direction is Theorem 1.5 below.

Let H_k^∞ be the set of all functions f of the form (1.1) where $s \in H^\infty$ and $b \in \mathcal{B}_k$ may have common zeros. From this definition it follows that $\mathcal{S}_k = (H_k^\infty \setminus H_{k-1}^\infty) \cap \mathcal{B}L^\infty$ where $\mathcal{B}L^\infty$ denotes the unit ball of $L^\infty(\mathbb{T})$. Let

$$\mathbf{S} := \left\{ g\colon\ g(z_i) = f_i \text{ for } i = 1, \dots, n \right\} \tag{1.7}$$

be the set of all functions g satisfying interpolation conditions (1.4), let

$$P^{(\lambda)} := \left[\frac{\lambda^2 - f_i f_j^*}{1 - z_i \bar{z}_j} \right]_{i,j=1}^n \tag{1.8}$$

(so that the Pick matrix P defined in (1.5) equals $P^{(1)}$) and let $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_m > 0$ be all positive solutions of the equation $\det P^{(\lambda)} = 0$. Then

$$\mu_k := \inf_{g \in \mathbf{S} \cap H_k^\infty} \|g\|_\infty = \inf_{g \in \mathbf{S} \cap (H_k^\infty \setminus H_{k-1}^\infty)} \|g\|_\infty = \begin{cases} \lambda_k & \text{if } k \leqslant m, \\ 0 & \text{if } k > m \end{cases} \tag{1.9}$$

(see e.g., [1]). Our contribution here is a reasonably simple criterion for the existence of a function $g_{k,\min} \in \mathbf{S} \cap H_k^\infty$ for $k \leqslant m$ such that $\|g_{k,\min}\|_\infty = \mu_k$. The case $k > m$ is simple: the function $g \equiv 0$ (the only function in H_k^∞ with $\|g\| = \mu_k = 0$) belongs to $\mathbf{S}$ if and only if all interpolation conditions are homogeneous: $f_1 = \cdots = f_n = 0$ (in which case the equation $\det P^{(\lambda)} = 0$ has no positive solutions).

Theorem 1.5. *Let $0 \leqslant k \leqslant m$. There exists a (unique) function $g \in \mathbf{S} \cap H_k^\infty$ with $\|g\|_\infty = \mu_k = \lambda_k$ if and only if the matrix $P^{(\lambda_k)}$ defined via formula* (1.8) *is saturated. This extremal function belongs to $H_k^\infty \setminus H_{k-1}^\infty$ if and only if either $k = 0$ or $\lambda_k < \lambda_{k-1}$.*

2. Preliminaries

In case $\delta(P) = 0$ and $\kappa \geqslant \nu(P)$, all solutions of the (nondegenerate) problem $\mathbf{NP}_\kappa$ can be parametrized by a linear fractional formula which will be recalled in Theorem 2.2 below. In the degenerate case, we will first parametrize the solution set of a maximal nondegenerate subproblem and then will match the remaining interpolation conditions by an appropriate choice of parameters in the parametrization formula. Without loss of generality we can (and will) assume that the $d \times d$ leading submatrix P_d of P is invertible where $d = \operatorname{rank} P$:

$$P_d := \left[\frac{1 - f_i f_j^*}{1 - z_i \bar{z}_j} \right]_{i,j=1}^d, \qquad \operatorname{rank} P_d = \operatorname{rank} P = d < n. \tag{2.1}$$

We next introduce the 2×2 matrix function $\Theta = [\Theta_{ij}]_{i,j=1}^2$ by

$$\Theta(z) = I + (z-1) \begin{bmatrix} E_d^* \\ C_d^* \end{bmatrix} \left(I - z T_d^* \right)^{-1} P_d^{-1} (I - T_d)^{-1} [\, E_d \quad -C_d \,], \tag{2.2}$$

where

$$T_d = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_d \end{bmatrix}, \qquad E_d = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad C_d = \begin{bmatrix} f_1 \\ \vdots \\ f_d \end{bmatrix}. \tag{2.3}$$

Remark 2.1. The matrix $\Theta(z)$ is invertible at every point $z \in \mathbb{C} \setminus \{z_1, \dots, z_d\}$, as it follows from the equality (see e.g., [6, Lemma 2.2] for the proof)

$$\det \Theta(z) = \prod_{i=1}^d \frac{(z - z_i)(1 - \bar{z}_i)}{(1 - z\bar{z}_i)(1 - z_i)}. \tag{2.4}$$

In (2.2) and in what follows, the symbol I stands for the identity matrix whose dimension will be always clear from the context. We will write $\mathcal{Z}(f)$ for the zero set of a meromorphic function f and $N\{f\}$ for the total number of zeroes of f that fall inside $\mathbb{D}$. We will denote by $\mathcal{B}_k^m$ the set of all rational functions unimodular on the unit circle $\mathbb{T}$ and with m zeroes and k poles inside $\mathbb{D}$, that is, the set of all coprime quotients of finite Blaschke products of degrees m and k. The two next theorems can be found in [5] (Theorems 1.1 and 2.3 there).

Theorem 2.2. *Let P_d of the form* (2.1) *be invertible and let Θ be defined as in* (2.2). *A function f belongs to $\mathcal{S}_\kappa$ and satisfies interpolation conditions*

$$f(z_i) = f_i \quad \text{for } i = 1, \dots, d \tag{2.5}$$

if and only if it is of the form

$$f = \frac{\Theta_{11}S + \Theta_{12}B}{\Theta_{21}S + \Theta_{22}B}, \tag{2.6}$$

for some $S \in \mathcal{S}$ and $B \in \mathcal{B}_{\kappa - \nu(P_d)}$ with $\mathcal{Z}(S) \cap \mathcal{Z}(B) = \emptyset$ and such that

$$\Theta_{21}(z_i)S(z_i) + \Theta_{22}(z_i)B(z_i) \neq 0 \quad (i = 1, \dots, d). \tag{2.7}$$

The correspondence $f \mapsto \frac{S}{B}$ is one-to-one and f is unimodular on $\mathbb{T}$ if and only if S is.

Theorem 2.3. *Let P_d be invertible, let Θ be defined as in* (2.2) *and let*

$$U_{S,B} := \Theta_{11}S + \Theta_{12}B, \qquad V_{S,B}(z) := \Theta_{21}S + \Theta_{22}B. \tag{2.8}$$

(1) $N\{V_{S,B}\} = \nu(P) + \deg B$. *If in addition, S is a finite Blaschke product, then $N\{U_{S,B}\} = \pi(P) + \deg S$.*
(2) $\mathcal{Z}(U_{S,B}) \cap \mathcal{Z}(V_{S,B}) \subseteq \{z_1, \dots, z_d\}$.
(3) *If $V_{S,B}(z_i) = 0$, then $U_{S,B}$ has simple zero at z_i.*

By Theorem 2.2, a function f of the form (2.6) satisfies the first d interpolation conditions in (1.4). The next theorem shows that the remaining $n - d$ conditions

$$f(z_j) = f_j \quad (j = d+1, \dots, n) \tag{2.9}$$

can be matched upon an appropriate choice of the parameters S and B in (2.6).

Theorem 2.4. *Let conditions* (2.1) *be satisfied, let Θ be defined as in* (2.2) *and let $S \in \mathcal{S}$ and $B \in \mathcal{B}_{\kappa - \nu(P_d)}$ have no common zeros. Then f of the form* (2.6) *satisfies interpolation conditions* (2.9) *if and only if*

$$a_j S(z_j) = -c_j B(z_j) \quad \text{for } j = d+1, \dots, n, \tag{2.10}$$

where the numbers a_j, c_j are given by

$$[a_j \quad c_j] = [1 \quad -f_j]\Theta(z_j) \quad \text{for } j = d+1, \dots, n. \tag{2.11}$$

These numbers are such that

$$|a_j| = |c_j| \neq 0 \quad \text{and} \quad \frac{c_i}{a_i} = \frac{c_j}{a_j} \quad \text{for } i, j = d+1, \dots, n. \tag{2.12}$$

Proof. By Remark 2.1, $\det \Theta(z_j) \neq 0$ for $j = d+1, \dots, n$, and therefore the numbers a_j and c_j defined as in (2.11) cannot be both equal to zero. It was shown in [8, Section 3] that the matrix $A = \left[\frac{a_i a_j^* - c_i c_j^*}{1 - z_i \bar{z}_j}\right]_{i,j=d+1}^n$ is the Schur complement of P_d in P. It follows by the rank condition in (2.1) that $A = 0$. Thus, $a_i a_j^* = c_i c_j^*$ for every $i, j \in \{d+1, \dots, n\}$ and (2.12) follows.

To prove the rest, take f in the form (2.6), i.e., in the form $f = \frac{U_{S,B}}{V_{S,B}}$ where $U_{S,B}$ and $V_{S,B}$ are given in (2.8). If f satisfies conditions (2.9), then

$$V_{S,B}(z_j) \neq 0 \quad (j = d+1, \dots, n). \tag{2.13}$$

Indeed, if $V_{S,B}(z_j) = 0$, then $U_{S,B}(z_j) = 0$, since f is analytic at z_j. Therefore,

$$\begin{bmatrix} cU_{S,B}(z_j) \\ V_{S,B}(z_j) \end{bmatrix} = \Theta(z_j) \begin{bmatrix} S(z_j) \\ B(z_j) \end{bmatrix} = 0$$

which is impossible, since $\begin{bmatrix} S(z_j) \\ B(z_j) \end{bmatrix} \neq 0$ and $\Theta(z_j)$ is invertible. Multiplying the jth condition in (2.9) by $V_{S,B}(z_j) \neq 0$ brings us to the equivalent condition $U_{S,B}(z_j) = f_j V_{S,B}(z_j) \neq 0$, that is (on account of (2.8)), to

$$\Theta_{11}(z_j)S(z_j) + \Theta_{12}(z_j)B(z_j) = f_j\big(\Theta_{21}(z_j)S(z_j) + \Theta_{22}(z_j)B(z_j)\big),$$

which can be rewritten as

$$[1 \quad -f_j]\begin{bmatrix} \Theta_{11}(z_j) \\ \Theta_{21}(z_i) \end{bmatrix} S(z_j) = -[1 \quad -f_j]\begin{bmatrix} \Theta_{12}(z_j) \\ \Theta_{22}(z_j) \end{bmatrix} B(z_j),$$

which in turn, coincides with (2.10), by definition (2.11) of a_j and c_j. Conversely, let us assume that $S \in \mathcal{S}$ and $B \in \mathcal{B}_{\kappa - \nu(P_d)}$ satisfy conditions (2.10). We have already seen that these conditions are equivalent to (2.9) (for the function f of the form (2.6)) provided conditions (2.13) are met. But they indeed are: to show this, we first observe that since $a_j, c_j \neq 0$ and since S and B do not have common zeros, it follows from (2.10) that $B(z_j) \neq 0$ for $j = d + 1, \ldots, n$. The equality

$$V_{S,B}(z_j) = \frac{B(z_j) \det \Theta(z_j)}{a_j} \quad (j = d+1, \ldots, n) \tag{2.14}$$

is verified by a straightforward calculation based on formulas (2.10), (2.11). Since $B(z_j) \neq 0$ and $\det \Theta(z_j) \neq 0$, we get (2.13) and complete the proof. $\quad\square$

Let us introduce the number

$$\gamma = -\frac{\Theta_{12}(z_j) - f_j \Theta_{22}(z_j)}{\Theta_{11}(z_j) - f_j \Theta_{21}(z_j)} = -\frac{c_j}{a_j}, \quad j \in \{d+1, \ldots, n\}, \tag{2.15}$$

which is unimodular and whose definition (2.15) does not depend on j, due to (2.12). Let us mention that conditions (2.1) guarantee that $\nu(P) = \nu(P_d)$. The next proposition follows immediately from Theorems 2.2 and 2.4.

Proposition 2.5. *Let conditions* (2.1) *be satisfied, let Θ and γ be defined as in* (2.2) *and* (2.15). *Then all solutions f of the problem* **NP**$_\kappa$ *are parametrized by formula* (2.6) *where $\{S, B\}$ runs through the set of all pairs such that*

(1) $\quad S \in \mathcal{S}, \quad B \in \mathcal{B}_{\kappa - \nu(P)}, \quad \mathcal{Z}(S) \cap \mathcal{Z}(B) = \emptyset;$ \hfill (2.16)

(2) $\quad \Theta_{21}(z_i)S(z_i) + \Theta_{22}(z_i)B(z_i) \neq 0 \quad \text{for } i = 1, \ldots, d;$ \hfill (2.17)

(3) $\quad S(z_j) = \gamma B(z_j) \quad \text{for } j = d+1, \ldots, n.$ \hfill (2.18)

Corollary 2.6. *If $\nu(P) < \kappa < \nu(P) + \delta(P)$, then* **NP**$_\kappa$ *has no solutions.*

Proof. Let us assume that on the contrary, the problem **NP**$_\kappa$ has a solution f which necessarily is of the form (2.6) for some pair $\{S, B\}$ satisfying conditions (2.16)–(2.18). Since $0 < \kappa - \nu(P) < \delta(P) = n - d$, the function γB is a nonconstant finite Blaschke product of degree $\deg B < n - d$. By (2.18), the Schur-class function S coincides with B at $n - d > \deg B$ points inside $\mathbb{D}$ and therefore S is equal to γB identically. Then $\mathcal{Z}(S) \cap \mathcal{Z}(B) \neq \emptyset$ which gives a contradiction. $\quad\square$

Corollary 2.7. *If $\kappa = \nu(P)$ and $\delta(P) > 0$, then* **NP**$_\kappa$ *has at most one solution.*

Proof. Let us consider the associated interpolation problem (2.16)–(2.18). Since $\kappa = \nu(P)$, the function $B \in \mathcal{B}_0$ is a unimodular constant. Since the independent parameter in (2.6) is in fact the ratio S/B, we can let without loss of generality, $B \equiv \gamma^*$. By the maximum modulus principle, there exists only one Schur class function $S \equiv 1$ satisfying conditions (2.16). Substituting $S \equiv 1$ and $B \equiv \gamma^*$ into (2.6) leads us to the function

$$f^0(z) = \frac{\Theta_{11}(z) - \Theta_{12}(z)\gamma^*}{\Theta_{21}(z) - \Theta_{22}(z)\gamma^*}, \tag{2.19}$$

which by Proposition 2.5, is the only possible solution to the problem **NP**$_\kappa$. $\quad\square$

Remark 2.8. By Theorem 2.4, f^0 satisfies interpolation conditions (2.9). By Proposition 2.5, f^0 satisfies conditions (2.5) if and only if its denominator does not vanish at $z_1, \ldots, z_d$:

$$\Theta_{21}(z_i) - \Theta_{22}(z_i)\gamma^* \neq 0 \quad \text{for } i = 1, \ldots, d. \tag{2.20}$$

Therefore, if $\kappa = \nu(P)$, conditions (2.20) are necessary and sufficient for the problem **NP**$_\kappa$ to have a solution.

Note the explicit formula for f^0 in terms of interpolation data:

$$f^0(z) = \frac{1 - (1 - z\bar{z}_j)E_d^*(I - zT_d^*)^{-1}P_d^{-1}(I - \bar{z}_jT_d)^{-1}(E_d - C_df_j^*)}{f_j^* - (1 - z\bar{z}_j)C_d^*(I - zT_d^*)^{-1}P_d^{-1}(I - \bar{z}_jT_d)^{-1}(E_d - C_df_j^*)}, \tag{2.21}$$

where j can be picked arbitrarily form $\{d+1, \ldots, n\}$. Indeed,

$$\begin{bmatrix} \Theta_{11}(z)a_j^* - \Theta_{12}(z)c_j^* \\ \Theta_{21}(z)a_j^* - \Theta_{22}(z)c_j^* \end{bmatrix} = \Theta(z)\left[\begin{pmatrix} a_j^* \\ -c_j^* \end{pmatrix}\right] = \begin{bmatrix} 1 \\ f_j^* \end{bmatrix} - (1 - z\bar{z}_j)\begin{bmatrix} E_d^* \\ C_d^* \end{bmatrix}(I - zT_d^*)^{-1}$$
$$\times P_d^{-1}(I - \bar{z}_j T_d)^{-1}(E_d - C_d f_j^*)$$

(the first equality is clear while establishing the second requires some simple algebraic manipulations with explicit formula (2.2) for Θ) and dividing the top components by the bottom ones leads us to (2.21).

Remark 2.9. If f^0 is a solution to the problem $\mathbf{NP}_\kappa$, then $f^0 \in \mathcal{B}_{\nu(P)}^{\pi(P)}$.

Proof. Since $|\gamma| = 1$ and Θ is rational, it follows that f^0 is rational and unimodular on $\mathbb{T}$, that is, f^0 is a ratio of two finite Blaschke products. By Theorem 2.3 (part (1)),

$$N\{\Theta_{11} - \Theta_{12}\gamma^*\} = \pi(P) \quad \text{and} \quad N\{\Theta_{21} - \Theta_{22}\gamma^*\} = \nu(P).$$

By conditions (2.20) and by Theorem 2.3 (part (2)), the numerator and the denominator in (2.19) have no common zeros in $\mathbb{D}$. Therefore, f^0 has $\pi(P)$ zeros and $\nu(P)$ poles in $\mathbb{D}$ and thus, it belongs to $\mathcal{B}_{\nu(P)}^{\pi(P)}$. □

Let us now pass to the case where $\kappa \geq \nu(P) + \delta(P)$ and describe all solutions $\{S, B\}$ to the problem (2.16)–(2.18). In addition to these conditions, we will require B to be normalized to the form (1.2) so that different pairs $\{S, B\}$ will lead via (2.6) to different solutions of $\mathbf{NP}_\kappa$. We first observe that for a finite Blaschke product $B \in \mathcal{B}_{\kappa-\nu(P)}$ to satisfy conditions (2.16) and (2.18), it is necessary that

$$B(z_j) \neq 0 \quad \text{for } j = d+1, \dots, n. \tag{2.22}$$

Let us take *any* $B \in \mathcal{B}_{\kappa-\nu(P)}$ subject to (2.22). The matrix

$$\widetilde{P} = \left[\frac{1 - B(z_i)B(z_j)^*}{1 - z_i\bar{z}_j}\right]_{i,j=d+1}^n \tag{2.23}$$

is positive definite as an $(n-d) \times (n-d)$ Schwarz–Pick matrix of a finite Blaschke product B of degree $\deg B = \kappa - \nu(P) \geq \delta(P) \geq n - d$. For a fixed B, equalities (2.18) can be considered as interpolation conditions for an unknown Schur-class function S. This is the classical Nevanlinna–Pick problem which has infinitely many solutions, since its Pick matrix $\widetilde{P}$ is positive definite. The solution set of this problem is parametrized by the linear fractional formula

$$S = \mathbf{T}_{\Psi_B}[\mathcal{E}] := \frac{\Psi_{B,11}\mathcal{E} + \Psi_{B,12}}{\Psi_{B,21}\mathcal{E} + \Psi_{B,22}} \tag{2.24}$$

with the free Schur-class parameter $\mathcal{E} \in \mathcal{S}$ and the coefficient matrix $\Psi_B(z) = [\Psi_{B,ij}(z)]_{i,j=1}^2$ given by

$$\Psi_B(z) = I + (z-1)\begin{bmatrix} \widetilde{E}^* \\ \widetilde{C}^* \end{bmatrix}(I - z\widetilde{T}^*)^{-1}\widetilde{P}^{-1}(I - \widetilde{T})^{-1}[\widetilde{E} \quad -\widetilde{C}], \tag{2.25}$$

where $\widetilde{P}$ is defined in (2.23) and where

$$\widetilde{T} = \begin{bmatrix} z_{d+1} & & \\ & \ddots & \\ & & z_n \end{bmatrix}, \qquad \widetilde{E} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad \widetilde{C} = \begin{bmatrix} \gamma B(z_{d+1}) \\ \vdots \\ \gamma B(z_n) \end{bmatrix}.$$

To get a description of all solutions of the problem $\mathbf{NP}_\kappa$ it remains to substitute (2.24) into (2.6) and to take care of constraints in (2.16) and (2.17).

Theorem 2.10. *Let us assume that $\kappa \geq \nu(P) + \delta(P)$ and that $\operatorname{rank} P_d = \operatorname{rank} P = d$ for the leading submatrix P_d of P. Let $\mathfrak{A}_B(z) = [\mathfrak{A}_{ij}(z)]_{i,j=1}^2$ be defined by*

$$\mathfrak{A}_B(z) = \Theta(z)\begin{bmatrix} 1 & 0 \\ 0 & B(z) \end{bmatrix}\Psi_B(z), \tag{2.26}$$

where Θ and γ are defined in (2.2) and (2.15), respectively, where $B \in \mathcal{B}_{\kappa-\nu(P)}$ is a Blaschke product subject to conditions (2.22), and where $\Psi_B(z)$ is defined from B by formula (2.25). Let $\mathcal{E}$ be a Schur-class function such that

$$\Psi_{B,11}(\zeta)\mathcal{E}(\zeta) + \Psi_{B,12}(\zeta) \neq 0 \quad \text{for } \zeta \in \mathcal{Z}(B) \tag{2.27}$$

and

$$\mathfrak{A}_{21}(z_i)\mathcal{E}(z_i) + \mathfrak{A}_{22}(z_i) \neq 0, \quad for\ i = 1, \dots, d. \tag{2.28}$$

Then the function

$$f = \mathbf{T}_{\mathfrak{A}_B}[\mathcal{E}] := \frac{\mathfrak{A}_{11}\mathcal{E} + \mathfrak{A}_{12}}{\mathfrak{A}_{21}\mathcal{E} + \mathfrak{A}_{22}} \tag{2.29}$$

solves $\mathbf{NP}_\kappa$. *Conversely, every solution* f *of the problem* $\mathbf{NP}_\kappa$ *admits a representation* (2.29) *for a unique choice of a normalized Blaschke product* $B \in \mathcal{B}_{\kappa - \nu(P)}$ *subject to* (2.22) *and a Schur-class function* $\mathcal{E}$ *satisfying* (2.27) *and* (2.28).

Proof. Formula (2.29) is the result of composition of linear fractional transformations (2.6) and (2.24). By Proposition 2.5, f of the form (2.29) solves $\mathbf{NP}_\kappa$ if and only if B and $S = \mathbf{T}_{\Psi_B}[\mathcal{E}]$ are subject to conditions (2.16) and (2.17). Rewriting these conditions in terms of $\mathcal{E}$ from (2.24) gives (2.27) and (2.28).

The uniqueness part can be justified as follows. If f is a solution of $\mathbf{NP}_\kappa$, then it is of the form (2.6) from which we have

$$\frac{S}{B} = \frac{\Theta_{12} - \Theta_{22}f}{\Theta_{21}f - \Theta_{11}} =: \widetilde{\mathcal{E}} \in \mathcal{S}_{\kappa - \nu(P)}.$$

Thus, the function $\widetilde{\mathcal{E}}$ is uniquely determined from f. Due to (2.16), the ratio S/B is the Kreĭn–Langer representation of $\widetilde{\mathcal{E}}$ and since B is normalized to the form (1.2), S and B are defined from f uniquely. Now we conclude from (2.24) that $\mathcal{E}$ is also determined uniquely by $\mathcal{E} = (\Psi_{12} - \Psi_{22}S)/(\Psi_{21}S - \Psi_{11})$. $\quad\square$

Corollary 2.11. *Let* P *of the form* (1.5) *be the Pick matrix of the problem* $\mathbf{NP}_\kappa$. *If* $\kappa \geqslant \nu(P) + \delta(P)$, *then* $\mathbf{NP}_\kappa$ *has infinitely many solutions.*

Proof. Let us fix any $B \in \mathcal{B}_{\kappa - \nu(P)}$ such that $\mathcal{Z}(B) \cap \{z_1, \dots, z_n\} = \emptyset$ and let Ψ_B and $\mathfrak{A}_B$ be defined as in (2.25) and (2.26). Then

$$\big|\Psi_{B,11}(\zeta)\big| + \big|\Psi_{B,12}(\zeta)\big| > 0 \quad \text{and} \quad \big|\mathfrak{A}_{21}(z_i)\big| + \big|\mathfrak{A}_{22}(z_i)\big| > 0 \tag{2.30}$$

for $\zeta \in \mathcal{Z}(B)$ and $i = 1, \dots, d$. Indeed, if $\zeta \in \mathcal{Z}(B)$, then $\zeta \notin \{z_{d+1}, \dots, z_n\}$. Therefore $\Psi(\zeta)$ is invertible (by virtue of Remark 2.1), and the first relation in (2.30) follows. Assuming that $\mathfrak{A}_{21}(z_i) = \mathfrak{A}_{22}(z_i) = 0$, for some $i \in \{1, \dots, d\}$, we get from (2.26)

$$\big[\, \Theta_{21}(z_i) \quad \Theta_{22}(z_i)B(z_i) \,\big]\Psi_B(z_i) = 0. \tag{2.31}$$

Since $\Psi_B(z_i)$ is invertible and $B(z_i) \neq 0$, we conclude from (2.31) that $\Theta_{21}(z_i) = \Theta_{22}(z_i) = 0$. This is a contradiction since for Θ of the form (2.2), the entries Θ_{21} and Θ_{22} cannot have common zeros (see [5, Lemma 2.1] for the proof). This contradiction completes the proof of the second relation in (2.30). One can easily conclude from inequalities (2.30) that there are infinitely many functions $\mathcal{E} \in \mathcal{S}$ subject to constraints (2.27) and (2.28). By Theorem 2.10, each such function leads via linear fractional formula (2.29) to a solution f of the problem $\mathbf{NP}_\kappa$. $\quad\square$

3. Proof of Theorems 1.2 and 1.3

We first recall a result from [7, Theorem 3.4].

Theorem 3.1. *Let* $f \in \mathcal{B}_\kappa^m$ *be the ratio of two finite Blaschke products. Then the Schwarz–Pick matrix* $P_n(f; \zeta_1, \dots, \zeta_n)$ *constructed via formula* (1.3) *has* m *positive and* κ *negative eigenvalues* (*counted with multiplicities*) *whenever* $n \geqslant m + \kappa$ *and* $\zeta_1, \dots, \zeta_r \in \mathbb{D} \cap \rho(f)$.

Corollary 3.2. *If* $f \in \mathcal{B}_\kappa^m$, *then the Schwarz–Pick matrix* $P_n(f; \zeta_1, \dots, \zeta_n)$ *is saturated for every* $n \geqslant m + \kappa$ *and any* n *points* $\zeta_1, \dots, \zeta_n \in \mathbb{D} \cap \rho(f)$.

Proof. By Theorem 3.1, $d := \operatorname{rank} P_n(f; \zeta_1, \dots, \zeta_n) = m + \kappa$. On the other hand, any $d \times d$ principal submatrix $\widehat{P}$ of $P_n(f; \zeta_1, \dots, \zeta_n)$ is itself a Schwarz–Pick matrix for f based on certain d points from $\{\zeta_1, \dots, \zeta_n\}$. Then again by Theorem 3.1, $\operatorname{rank} \widehat{P} = d$ and thus, $\widehat{P}$ is invertible. $\quad\square$

Proof of Theorem 1.2. Let P be the Pick matrix of the problem $\mathbf{NP}_\kappa$. If $\kappa \geqslant \nu(P) + \delta(P)$, then $\mathbf{NP}_\kappa$ has infinitely many solutions, by Corollary 2.11. If $\nu(P) < \kappa < \nu(P) + \delta(P)$ or if $\kappa < \nu(P)$, then the problem $\mathbf{NP}_\kappa$ has no solutions (by Corollary 2.6 and by (1.6)). The last case not covered yet is where $\kappa = \nu(P) < \nu(P) + \delta(P)$ (i.e., $\kappa = \nu(P)$ and P is singular). In this case, the problem $\mathbf{NP}_\kappa$ has at most one solution (by Corollary 2.7) and the function f^0 defined in (2.19) is the unique candidate.

Since all the cases listed above are disjoint, in order to complete the proof of Theorem 1.2, it suffices to show that f^0 *is a solution to the problem* $\mathbf{NP}_\kappa$ *if and only if P is saturated.*

The "only if" part: if f^0 solves $\mathbf{NP}_\kappa$, then $P_n(f^0; z_1, \ldots, z_n) = P$ and since $f_0 \in \mathcal{B}_{\nu(P)}^{\pi(P)}$ (by Remark 2.9), P is saturated by Corollary 3.2.

The "if" part: Let us assume that P is singular and that f^0 does not solve $\mathbf{NP}_\kappa$. Then some of conditions (2.20) fail. After rearrangements (if necessary) we may assume that only the $\ell \geqslant 1$ first conditions in (2.20) fail, i.e.,

$$\Theta_{21}(z_i) - \Theta_{22}(z_i)\gamma^* = 0 \quad \text{for } i = 1, \ldots, \ell.$$

By Theorem 2.3 (part (3)), the numerator in (2.19) must vanish at the same points

$$\Theta_{11}(z_i) - \Theta_{12}(z_i)\gamma^* = 0 \quad \text{for } i = 1, \ldots, \ell,$$

and after zero cancellations in (2.19), it turns out that $f^0 \in \mathcal{B}_{\nu(P)-\ell}^{\pi(P)-\ell}$. By Remark 2.8, f^0 still satisfies interpolation conditions $f^0(z_i) = f_i$ for $i = \ell+1, \ldots, n$. Therefore, the matrix $\widetilde{P}_{n-\ell} := P_{n-\ell}(f^0; z_{\ell+1}, \ldots, z_n)$ is an $(n-\ell) \times (n-\ell)$ principal submatrix of P. By Theorem 3.1,

$$\text{rank}\, \widetilde{P}_{n-\ell} = \big(\pi(P) - \ell\big) + \nu(P) - \ell = \text{rank}\, P - 2\ell. \tag{3.1}$$

Now we will show that P is not saturated separately for two cases.

Case 1. If $n - \ell \geqslant d := \text{rank}\, P$, then by Theorem 3.1, every $d \times d$ principal submatrix of $\widetilde{P}_{n-\ell}$ has the same rank as $\widetilde{P}_{n-\ell}$ (that is, $d - 2\ell$) and therefore, is singular. These submatrices are also principal submatrices of P, and therefore, P is not saturated.

Case 2. If $n - \ell < d := \text{rank}\, P$, then $\widetilde{P}_{n-\ell}$ is a principal submatrix of $P' = \big[\frac{1 - f_i f_j^*}{1 - z_i \bar{z}_j}\big]_{i,j=n-d+1}^n$, the $d \times d$ bottom principal submatrix of P. Since the dimension of P' is greater than the dimension of $\widetilde{P}_{n-\ell}$ by $d - n + \ell$, we have

$$\text{rank}\, P' \leqslant \text{rank}\, \widetilde{P}_{n-\ell} + 2(d - n + \ell).$$

Now we substitute (3.1) into the last inequality to get

$$\text{rank}\, P' \leqslant d - 2\ell + 2(d - n + \ell) = 3d - 2n < d.$$

Thus, P' is singular and therefore, P is not saturated. This completes the proof of Theorem 1.2. $\quad\square$

Proof of Theorem 1.3. The "only if" direction is trivial. To prove the "if" part, let $\widetilde{P}$ be a $(d+1) \times (d+1)$ saturated principal submatrix of P with rank $\widetilde{P} = \text{rank}\, P = d$. Without loss of generality we can assume that $\widetilde{P} = P_{d+1}$, the leading submatrix of P. Since P_{d+1} is the Pick matrix of the subproblem of $\mathbf{NP}_\kappa$ with interpolation conditions $f(z_i) = f_i$ ($i = 1, \ldots, d+1$), it follows by Theorem 1.2, that this subproblem admits a unique solution f^0 in the class $\mathcal{S}_{\kappa - \nu(P)}$ and this solution is given by formula (2.21). By Remark 2.8, f^0 also satisfies conditions (2.9) and therefore it is a (unique) solution of the "whole" problem $\mathbf{NP}_\kappa$ with $\kappa = \nu(P)$. Therefore P is saturated by Theorem 1.2 (part (2)). $\quad\square$

4. Extremal functions

In this section we prove Theorem 1.5. Although the statements from the next proposition are known (see [1]), we include a short proof.

Proposition 4.1. *Let $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_m > 0$ be all positive solutions of the equation* $\det P^{(\lambda)} = 0$, *where $P^{(\lambda)}$ is defined in* (1.8). *Let the set $\mathbf{S}$ be defined as in* (1.7) *and let $\widetilde{\lambda} \in \mathbb{R}_+ \setminus \{\lambda_0, \lambda_1, \ldots, \lambda_m\}$. Then:*

(1) *If $\widetilde{\lambda} > \lambda_k$, then there exists $g \in \mathbf{S} \cap (H_k^\infty \setminus H_{k-1}^\infty)$ with $\|g\|_\infty \leqslant \widetilde{\lambda}$.*
(2) *If $\widetilde{\lambda} < \lambda_k$, then $\|h\|_\infty > \widetilde{\lambda}$ for every $h \in \mathbf{S} \cap H_k^\infty$.*

Proof. The scaled matrix $\widetilde{\lambda}^{-2} \cdot P^{(\widetilde{\lambda})}$ is the Pick matrix of the Nevanlinna–Pick problem with interpolation conditions

$$f(z_i) = \widetilde{\lambda}^{-1} f_i \quad \text{for } i = 1, \ldots, n, \tag{4.1}$$

and this matrix is invertible, since $\widetilde{\lambda}$ is not equal to any of λ_i's. If $\widetilde{\lambda} > \lambda_k$, then $\nu(P^{(\widetilde{\lambda})}) \leqslant k$ and then by Theorem 2.2, there is a function $f \in \mathcal{S}_k$ satisfying conditions (4.1). Then the function $g = \widetilde{\lambda} f$ satisfies conditions in (1.7) and belongs to $H_k^\infty \setminus H_{k-1}^\infty$ with $\|g\|_\infty \leqslant \widetilde{\lambda}$. This proves part (1). Now let $\widetilde{\lambda} < \lambda_k$ so that

$$\nu\big(P^{(\widetilde{\lambda})}\big) \geqslant k+1. \tag{4.2}$$

Assuming that there exists a $h \in \mathbf{S} \cap H_k^\infty$ with $\|h\|_\infty \leqslant \widetilde{\lambda}$, we conclude that the function $f = \widetilde{\lambda}^{-1} h$ satisfies conditions (4.1) and belongs $\mathcal{S}_\kappa$ for some $\kappa \leqslant k$. But then the Pick matrix $\widetilde{\lambda}^{-2} \cdot P^{(\widetilde{\lambda})}$ of the problem (4.1) has at most κ negative eigenvalues. This contradicts to (4.2) and completes the proof. Note that for $k \leqslant m$,

$$\lambda_k \leqslant \inf_{g \in \mathbf{S} \cap H_k^\infty} \|g\|_\infty \leqslant \inf_{g \in \mathbf{S} \cap (H_k^\infty \setminus H_{k-1}^\infty)} \|g\|_\infty \leqslant \lambda_k, \tag{4.3}$$

where the second inequality is obvious while the first and the third follow by parts (2) and (1) respectively. Equalities (1.9) (for $k \leqslant m$) follow from (4.3). The case where $k > m$ can be handled similarly. $\quad\square$

Proof of Theorem 1.5. We seek a function $g \in \mathbf{S} \cap H_k^\infty$ such that $\|g\|_\infty = \lambda_k$ or equivalently, such that $\|g\|_\infty \leqslant \lambda_k$—this equivalence follows by Proposition 4.1 (part (2)) according to which $\|g\|_\infty \geqslant \lambda_k$ for every $g \in \mathbf{S} \cap H_k^\infty$. It is convenient to seek g in the form $g = \lambda_k f$ where

$$f \in H_k^\infty \cap \mathcal{B}L^\infty \quad \text{and} \quad f(z_i) = \lambda_k^{-1} f_i \quad \text{for } i = 1, \dots, n. \tag{4.4}$$

Thus, the extremal function $g_{k,\min}$ with $\|g_{k,\min}\|_\infty = \lambda_k$ exists if and only if the Nevanlinna–Pick problem (4.4) has a solution in $\mathcal{S}_\kappa$ for some $\kappa \leqslant k$. The extremal function belongs to $\mathbf{S} \cap H_k^\infty \setminus H_{k-1}^\infty$ if and only if the problem (4.4) has a solution in $\mathcal{S}_k$. The Pick matrix of this problem equals $\lambda_k^{-2} \cdot P^{(\lambda_k)}$ and is singular by the definition of λ_i's. We will consider separately two cases.

Case 1. Let $\lambda_0 = \cdots = \lambda_k$. Then $\nu(P^{(\lambda_k)}) = 0$ and $\delta(P^{(\lambda_k)}) \geqslant k+1$. By Theorem 1.3, the problem (4.4) has a unique solution $f \in \mathcal{S}_0$ and does not have solutions in $\mathcal{S}_\kappa$ for $\kappa = 1, \dots, k$. Thus, the problem has a solution in $\mathcal{S}_k$ if and only if $k = 0$.

Case 2. Let $\lambda_{\ell-1} > \lambda_\ell = \cdots = \lambda_k$. Then $P^{(\lambda_k)} = P^{(\lambda_\ell)}$,

$$\nu\big(P^{(\lambda_k)}\big) = \ell \quad \text{and} \quad \delta\big(P^{(\lambda_k)}\big) \geqslant k - \ell + 1.$$

By Theorem 1.3, the problem (4.4) has a (unique) solution in $\mathcal{S}_\ell$ if and only if the matrix $P^{(\lambda_k)}$ is saturated and it does not have solutions in $\mathcal{S}_\kappa$ for $\kappa = \ell+1, \dots, k$. Thus, the problem has a solution in $\mathcal{S}_k$ if and only if $k = \ell$ in which case we have $\lambda_{k-1} > \lambda_k$.

For both cases, the unique solution of the problem (4.4) (again by Theorem 1.3) is of the form $f = s/b$ where $s \in \mathcal{B}_{\pi(P^{(\lambda_k)})}$ and $b \in \mathcal{B}_{\nu(P^{(\lambda_k)})}$. Now it follows from representation $g = \lambda_k f$ that the extremal function $g_{k,\min}$ (if exists) is of the form

$$g_{k,\min} = \lambda_k \cdot \frac{s}{b}, \quad \text{where } s \in \mathcal{B}_{\pi(P^{(\lambda_k)})} \text{ and } b \in \mathcal{B}_{\nu(P^{(\lambda_k)})}. \quad\square$$

Remark 4.2. A realization formula for $g_{k,\min}$ in terms of interpolation data can be obtained as follows. For a given k, let us rearrange the interpolation nodes so that the $d \times d$ leading submatrix $P_d^{(\lambda_k)} = \big[\frac{\lambda_k^2 - f_i f_j^*}{1 - z_i z_j^*}\big]_{i,j=1}^d$ of $P^{(\lambda_k)}$ satisfies

$$\operatorname{rank} P_d^{(\lambda_k)} = \operatorname{rank} P^{(\lambda_k)} = d < n \quad \text{and} \quad \nu\big(P_d^{(\lambda_k)}\big) = \nu\big(P^{(\lambda_k)}\big).$$

Let T_d, E_d and C_d be the same as in (2.3). Applying formula (2.21) to the degenerate Nevanlinna–Pick problem (4.4), we arrive at

$$g_{k,\min}(z) = \lambda_k f_{k,\min} = \lambda_k^2 \cdot \frac{1 - (1 - z\bar{z}_j) E_d^* D(z)^{-1} (\lambda_k^2 E_d - C_d f_j^*)}{f_j^* - (1 - z\bar{z}_j) C_d^* D(z)^{-1} (\lambda_k^2 E_d - C_d f_j^*)},$$

where $D(z) = (I - \bar{z}_j T_d) P_d^{(\lambda_k)} (I - z T_d^*)$.

Acknowledgments

References

[1] V.M. Adamjan, D.Z. Arov, M.G. Kreĭn, Analytic properties of the Schmidt pairs of a Hankel operator and the generalized Schur–Takagi problem, Mat. Sb. 86 (128) (1971) 34–75.

[2] J.A. Ball, I. Gohberg, L. Rodman, Interpolation of Rational Matrix Functions, Oper. Theory Adv. Appl., vol. 45, Birkhäuser-Verlag, 1990.

[3] J.A. Ball, J.W. Helton, A Beurling–Lax theorem for the Lie group $U(m,n)$ which contains most classical interpolation theory, J. Operator Theory 9 (1983) 107–142.

[4] J.A. Ball, J.W. Helton, Interpolation problems of Pick–Nevanlinna and Loewner type for meromorphic matrix functions: Parametrizations of the set of all solutions, Integral Equations Operator Theory 9 (1986) 155–203.

[5] V. Bolotnikov, On a multi-point interpolation problem for generalized Schur functions, preprint, http://lanl.arxiv.org/pdf/0812.4564.

[6] V. Bolotnikov, On Carathéodory–Fejér problem for generalized Schur functions, Integral Equations Operator Theory 50 (1) (2004) 9–41.

[7] V. Bolotnikov, A. Kheifets, On negative inertia of Pick matrices associated with generalized Schur functions, Integral Equations Operator Theory 56 (3) (2006) 323–355.

[8] V. Bolotnikov, A. Kheifets, L. Rodman, Nevanlinna–Pick interpolation: Pick matrices have bounded number of negative eigenvalues, Proc. Amer. Math. Soc. 132 (2004) 769–780.

[9] L.B. Golinskii, A generalization of the matrix Nevanlinna–Pick problem, Izv. Akad. Nauk Armyan. SSR Ser. Mat. 18 (1983) 187–205.

[10] M. Kaltenbäck, H. Woracek, The Krein formula for generalized resolvents in degenerated inner product spaces, Monatsh. Math. 127 (2) (1999) 119–140.

[11] M.G. Kreĭn, H. Langer, Über die verallgemeinerten Resolventen und die charakteristische Funktion eines isometrischen Operators im Raume Π_κ, Coll. Math. Soc. János Bolyai 5 (1972) 353–399.

[12] D. Sarason, Nevanlinna–Pick interpolation with boundary data, Integral Equations Operator Theory 30 (1998) 231–250.

[13] H. Woracek, Nevanlinna–Pick interpolation: The degenerated case, Linear Algebra Appl. 252 (1997) 141–158.