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A higher order analogue of the Carathéodory–Julia theorem

Vladimir Bolotnikov [a], Alexander Kheifets [b,∗]

[a] Department of Mathematics, The College of William and Mary, Williamsburg, VA 23187-8795, USA
[b] Department of Mathematics, University of Massachusetts Lowell, Lowell, MA 01854, USA

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Abstract

A higher order analogue of the classical Carathéodory–Julia theorem on boundary derivatives is proved.

Keywords: Carathéodory–Julia theorem; Boundary derivatives; De Branges–Rovnyak spaces

1. Introduction

We denote by $\mathcal{S}$ the Schur class of analytic functions mapping the open unit disk $\mathbb{D}$ into its closure. We will write $z \widehat{\to} t_0$ if a point z approaches a boundary point $t_0 \in \mathbb{T}$ nontangentially and we will write $z \to t_0$ if z approaches t_0 unrestrictedly in $\mathbb{D}$. We start with the classical Carathéodory–Julia theorem ([4,5] and also [9, Chapter 4] and [8, Chapter 6]).

Theorem 1.1. *For $w \in \mathcal{S}$ and $t_0 \in \mathbb{T}$, the following are equivalent*:

(1) $d_1 := \liminf_{z \to t_0} \frac{1-|w(z)|^2}{1-|z|^2} < \infty$.
(2) $d_2 := \lim_{z \widehat{\to} t_0} \frac{1-|w(z)|^2}{1-|z|^2} < \infty$.

∗ Corresponding author.
E-mail addresses: vladi@math.wm.edu (V. Bolotnikov), alexander_kheifets@uml.edu (A. Kheifets).

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(3) *The limits*

$$w_0 := \lim_{z \widehat{\to} t_0} w(z) \quad \textit{and} \quad d_3 := \lim_{z \widehat{\to} t_0} \frac{1 - w(z)\bar{w}_0}{1 - z\bar{t}_0}$$

exist and satisfy $|w_0| = 1$ *and* $d_3 \geqslant 0$.

(4) *The limits* $w_0 := \lim_{z \widehat{\to} t_0} w(z)$ *and* $w_1 := \lim_{z \widehat{\to} t_0} w'(z)$ *exist and satisfy* $|w_0| = 1$ *and* $t_0 w_1 \bar{w}_0 \geqslant 0$.

Moreover, when these conditions hold, $d_1 = d_2 = d_3 = t_0 w_1 \bar{w}_0$.

We refer to [7–9] for more details concerning the Carathéodory–Julia theorem.

The purpose of this paper is to establish a higher order analogue of the Carathéodory–Julia theorem. First we introduce some needed notation.

A well-known property of Schur functions w is that the matrix

$$\mathbf{P}_n^w(z) := \left[\frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \frac{1 - |w(z)|^2}{1 - |z|^2} \right]_{i,j=0}^n \tag{1.1}$$

which will be referred to as to a *Schwarz–Pick matrix*, is positive semidefinite for every $n \geqslant 0$ and $z \in \mathbb{D}$.

We extend this notion to boundary points as follows: *given a point* $t_0 \in \mathbb{T}$, *the boundary Schwarz–Pick matrix is*

$$\mathbf{P}_n^w(t_0) = \lim_{z \widehat{\to} t_0} \mathbf{P}_n^w(z) \quad (n \geqslant 0), \tag{1.2}$$

provided the limit in (1.2) *exists*. It is clear that once the boundary Schwarz–Pick matrix $\mathbf{P}_n^w(t_0)$ exists for $w \in \mathcal{S}$, it is positive semidefinite.

Now let us assume that $w \in \mathcal{S}$ has nontangential boundary limits

$$w_j(t_0) := \lim_{z \widehat{\to} t_0} \frac{w^{(j)}(z)}{j!} \quad \text{for } j = 0, \dots, 2n+1 \tag{1.3}$$

and let

$$\mathbb{P}_n^w(t_0) := \begin{bmatrix} w_1(t_0) & \cdots & w_{n+1}(t_0) \\ \vdots & & \vdots \\ w_{n+1}(t_0) & \cdots & w_{2n+1}(t_0) \end{bmatrix} \boldsymbol{\Psi}_n(t_0) \begin{bmatrix} \overline{w_0(t_0)} & \dots & \overline{w_n(t_0)} \\ & \ddots & \vdots \\ 0 & & \overline{w_0(t_0)} \end{bmatrix}, \tag{1.4}$$

where the first factor is a Hankel matrix, the third factor is an upper triangular Toeplitz matrix and where $\boldsymbol{\Psi}_n(t_0) = [\Psi_{j\ell}]_{j,\ell=0}^n$ is the upper triangular matrix with entries

$$\Psi_{j\ell} = \begin{cases} 0, & \text{if } j > \ell, \\ (-1)^\ell \binom{\ell}{j} t_0^{\ell+j+1}, & \text{if } j \leqslant \ell. \end{cases} \tag{1.5}$$

Note that the matrix (1.4) appeared first in [6] in the context of boundary interpolation for Schur class functions.

We denote the lower right corner in the Schwarz–Pick matrix $\mathbf{P}_n^w(z)$ by

$$d_{w,n}(z) := \frac{1}{(n!)^2} \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \frac{1-|w(z)|^2}{1-|z|^2} \tag{1.6}$$

and formulate a higher order analogue of Theorem 1.1.

Theorem 1.2. *For $w \in \mathcal{S}$, $t_0 \in \mathbb{T}$ and $n \in \mathbb{Z}_+$, the following are equivalent*:

(1)
$$\liminf_{z \to t_0} d_{w,n}(z) < \infty. \tag{1.7}$$

(2)
$$\lim_{z \widehat{\to} t_0} d_{w,n}(z) < \infty. \tag{1.8}$$

(3) *The boundary Schwarz–Pick matrix $\mathbf{P}_n^w(t_0)$ exists.*

(4) *The nontangential boundary limits* (1.3) *exist and satisfy*

$$\left|w_0(t_0)\right| = 1 \quad \text{and} \quad \mathbb{P}_n^w(t_0) \geqslant 0, \tag{1.9}$$

where $\mathbb{P}_n^w(t_0)$ is the matrix defined in (1.4).

Moreover, when these conditions hold, the limits in (1.7) *and* (1.8) *are equal and furthermore,*

$$\mathbf{P}_n^w(t_0) = \mathbb{P}_n^w(t_0). \tag{1.10}$$

Note that equality (1.10) was established in [6] under assumptions of the nature different from the one of Carathédory–Julia. Equality (1.10) enables one to compute boundary Schwarz–Pick matrices in terms of boundary values of w and of its derivatives, which in some cases (e.g., if w is rational) is much easier to do than to use the original definition (1.2) of $\mathbf{P}_n^w(t_0)$. On the other hand, (1.9) imposes certain restriction on the boundary limits (1.3).

When $n = 0$, Theorem 1.2 reduces to Theorem 1.1 with statement (3) excluded. A higher order analogue of this statement has been studied in [2, Section 9] and will be recalled in Section 5.

The paper is organized as follows. In Section 2 we discuss the de Branges–Rovnyak spaces of analytic functions and their reproducing kernels. Section 3 deals with boundary analogues of these reproducing kernels that (as it will be shown) make sense only if condition (1.7) is satisfied. The proof of Theorem 1.2 is presented in Section 4. Some further results related to Theorem 1.2 are briefly reviewed in Section 5.

2. De Branges–Rovnyak spaces and their reproducing kernels

In this section we recall definitions of Hilbert spaces L^w and H^w associated to a Schur function w and discuss their properties that we will need in what follows. We use the standard notation L_2 for the Lebesgue space of square integrable functions on the unit circle $\mathbb{T}$; the symbols H_2^+ and H_2^- stand for the Hardy spaces of functions with vanishing negative (respectively, nonnegative) Fourier coefficients. The elements in H_2^+ and H_2^- will be identified with their unique analytic (respectively, conjugate-analytic) continuations inside the unit disk and consequently H_2^+ and H_2^- will be identified with the Hardy spaces of the unit disk.

Let w be a Schur function and let

$$W(t) := \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix}.$$

The space L^w is the range space $W^{1/2}(L_2 \oplus L_2)$ endowed with the range norm. In more detail: for every element f in L^w, there exists a unique $g_f \in L_2 \oplus L_2$ which is orthogonal to $\operatorname{Ker} W(t)$ for almost all $t \in \mathbb{T}$ and such that $f = W^{1/2} g_f$. This unique g_f will be denoted by $g_f := W^{[-1/2]} f$ and the L^w-norm is defined by

$$\|f\|_{L^w}^2 := \|g_f\|_{L_2 \oplus L_2}^2 = \int_{\mathbb{T}} \left\| \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix}^{[-1/2]} f(t) \right\|_{\mathbb{C}^2}^2 m(dt),$$

where $m(dt)$ stands for the normalized arc length Lebesgue measure on $\mathbb{T}$. Since

$$\operatorname{Ran} W(t) = \operatorname{Ran} W(t)^{1/2}$$

almost everywhere on $\mathbb{T}$, then we have also

$$\langle f, h \rangle_{L^w} = \int_{\mathbb{T}} \left\langle \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix}^{[-1]} f(t),\ h(t) \right\rangle_{\mathbb{C}^2} m(dt). \tag{2.1}$$

Here the inverse means that we choose an arbitrary vector function $g(t)$ satisfying $f(t) = W(t)g(t)$. This g does not necessarily have to be in $L_2(\mathbb{C}^2)$. However, the integrand in (2.1) does not depend on the choice of such $g(t)$ if $h \in L^w$ and the integral is finite.

Definition 2.1. A function $f = \left[\begin{smallmatrix} f_+ \\ f_- \end{smallmatrix}\right]$ is said to belong to the de Branges–Rovnyak space H^w if it belongs to L^w and if $f_+ \in H_2^+$ and $f_- \in H_2^-$.

The space H^w is a closed subspace of L^w; in what follows, P_{H^w} denotes the orthogonal projection of L^w onto H^w.

Recall that H_2^+ and H_2^- are reproducing kernel Hilbert spaces with reproducing kernels

$$k_z(t) = \frac{1}{1 - t\bar{z}} \quad \text{and} \quad \tilde{k}_z(t) = \frac{1}{t - z} \tag{2.2}$$

in the sense that

$$\langle f_+,\ k_z \rangle_{L_2} = f_+(z) \quad \text{and} \quad \langle f_-,\ \tilde{k}_z \rangle_{L_2} = f_-(z)/\bar{z} \tag{2.3}$$

for every $f_+ \in H_2^+$, $f_- \in H_2^-$, and $z \in \mathbb{D}$. More generally, the kernels

$$k_{j,z}(t) := \frac{1}{j!} \frac{\partial^j}{\partial \bar{z}^j} k_z(t) = \frac{t^j}{(1 - t\bar{z})^{j+1}}, \tag{2.4}$$

$$\tilde{k}_{j,z}(t) := \frac{1}{j!} \frac{\partial^j}{\partial z^j} \tilde{k}_z(t) = \frac{1}{(t - z)^{j+1}} \tag{2.5}$$

serve to evaluate derivatives:

$$\langle f_+, k_{j,z}\rangle_{L_2} = \frac{1}{j!} f_+^{(j)}(z), \qquad \langle f_-, \tilde{k}_{j,z}\rangle_{L_2} = \frac{1}{j!}\left(\frac{f_-(z)}{\bar{z}}\right)^{(j)}. \tag{2.6}$$

Now we introduce the vector-valued functions

$$K_z(t) = \begin{bmatrix} K_{z,+}(t) \\ K_{z,-}(t) \end{bmatrix} = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -w(z)^* \end{bmatrix} \cdot k_z(t), \tag{2.7}$$

$$\widetilde{K}_z(t) = \begin{bmatrix} \widetilde{K}_{z,+}(t) \\ \widetilde{K}_{z,-}(t) \end{bmatrix} = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} -w(z) \\ 1 \end{bmatrix} \cdot \tilde{k}_z(t), \tag{2.8}$$

defined for $z \in \mathbb{D}$ and $t \in \mathbb{T}$ and more generally, the vector-valued functions

$$K_z^{(j)}(t) := \frac{1}{j!}\frac{\partial^j}{\partial \bar{z}^j} K_z(t) \quad \text{and} \quad \widetilde{K}_z^{(j)}(t) := \frac{1}{j!}\frac{\partial^j}{\partial z^j} \widetilde{K}_z(t) \tag{2.9}$$

for $j \in \mathbb{Z}_+$. For $j = 0$ they coincide with (2.7) and (2.8). Upon differentiating (2.7) and (2.8) with respect to $\bar{z}$ and z, respectively, and making use of (2.4) and (2.5) we come to the following explicit formulas for $K_z^{(j)}$ and $\widetilde{K}_z^{(j)}$:

$$K_z^{(j)}(t) = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} k_{j,z}(t) \\ -\sum_{\ell=0}^{j} w_{j-\ell}(z)^* k_{\ell,z}(t) \end{bmatrix}, \tag{2.10}$$

$$\widetilde{K}_z^{(j)}(t) = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} -\sum_{\ell=0}^{j} w_{j-\ell}(z) \tilde{k}_{\ell,z}(t) \\ \tilde{k}_{j,z}(t) \end{bmatrix}, \tag{2.11}$$

where $w_\ell(z)$ are the Taylor coefficients from the expansion

$$w(\zeta) = \sum_{\ell=0}^{\infty} w_\ell(z)(\zeta - z)^\ell, \quad w_\ell(z) = \frac{w^{(\ell)}(z)}{\ell!}. \tag{2.12}$$

Formulas (2.10) and (2.11) define $K_z^{(j)}(t)$ and $\widetilde{K}_z^{(j)}(t)$ on the unit circle. The analytic (conjugate-analytic) continuations of their components to the unit disk are given as follows:

$$K_{z,+}^{(j)}(\zeta) = k_{j,z}(\zeta) - w(\zeta)\sum_{\ell=0}^{j} w_{j-\ell}(z)^* k_{\ell,z_i}(\zeta), \tag{2.13}$$

$$K_{z,-}^{(j)}(\zeta) = \bar{\zeta}\left(w(\zeta)^* \tilde{k}_{j,z}(\zeta)^* - \sum_{\ell=0}^{j} w_{j-\ell}(z)^* \tilde{k}_{\ell,z}(\zeta)^*\right), \tag{2.14}$$

$$\widetilde{K}_{z,+}^{(j)}(\zeta) = w(\zeta)\tilde{k}_{j,z}(\zeta) - \sum_{\ell=0}^{j} w_{j-\ell}(z)\tilde{k}_{\ell,z}(\zeta), \tag{2.15}$$

$$\widetilde{K}^{(j)}_{z,-}(\zeta)=\bar{\zeta}\left(k_{j,z}(\zeta)^*-w(\zeta)^*\sum_{\ell=0}^{j}w_{j-\ell}(z)k_{\ell,z}(\zeta)^*\right). \tag{2.16}$$

Lemma 2.2. *For every $j\in\mathbb{Z}_+$ and $z\in\mathbb{D}$, the functions $K^{(j)}_z$ and $\widetilde{K}^{(j)}_z$ belong to H^w. Furthermore, for every $f=\left[\begin{smallmatrix} f_+\\ f_-\end{smallmatrix}\right]\in H^w$, we have*

$$\left\langle f,\ K^{(j)}_z\right\rangle_{H^w}=\frac{1}{j!}\frac{d^j}{dz^j}f_+(z),\qquad \left\langle f,\ \widetilde{K}^{(j)}_z\right\rangle_{H^w}=\frac{1}{j!}\frac{d^j}{d\bar{z}^j}\left(\frac{f_-(z)}{\bar{z}}\right). \tag{2.17}$$

Proof. First we note that by formulas (2.10) and (2.11), the functions

$$\begin{bmatrix}1 & w(t)\\ w(t)^* & 1\end{bmatrix}^{[-1/2]}K^{(j)}_z(t)\quad\text{and}\quad \begin{bmatrix}1 & w(t)\\ w(t)^* & 1\end{bmatrix}^{[-1/2]}\widetilde{K}^{(j)}_z(t)$$

are bounded a.e. on $\mathbb{T}$ for every fixed $z\in\mathbb{D}$ and, therefore, $K^{(j)}_z$ and $\widetilde{K}^{(j)}_z$ belong to L^w. Furthermore, since $w\in H^\infty$, $k_{\ell,z}\in H^2_+$ and $\tilde{k}_{\ell,z}\in H^2_-$, it is readily seen from the formulas (2.13) and (2.16) that $K^{(j)}_{z,+}\in H_2^+$ and that $\widetilde{K}^{(j)}_{z,-}\in H_2^-$. Upon substituting the Taylor expansion (2.12) for w into (2.15) we arrive at

$$\widetilde{K}^{(j)}_{z,+}(\zeta)=\sum_{\ell=j+1}^{\infty}w_\ell(z)(\zeta-z)^{\ell-j-1} \tag{2.18}$$

which implies that $\widetilde{K}^{(j)}_{z,+}\in H_2^+$. By a similar argument, it follows from (2.14) that $K^{(j)}_{z,-}\in H_2^-$. Thus, the top components of $K^{(j)}_z$ and $\widetilde{K}^{(j)}_z$ belong to H_2^+, the bottom components are elements of H_2^- and therefore, $K^{(j)}_z$ and $\widetilde{K}^{(j)}_z$ belong to H^w. Furthermore, by the formula (2.7) for K_z and (2.1) for the inner product in L^w,

$$\langle f,\ K_z\rangle_{H^w}=\left\langle\begin{bmatrix}f_+\\ f_-\end{bmatrix},\ \begin{bmatrix}1\\ -w(z)^*\end{bmatrix}k_z\right\rangle_{L^2\oplus L^2}=\langle f_+,\ k_z\rangle_{L^2}+\left\langle f_-,\ w(z)^*k_z\right\rangle_{L^2}.$$

Since f_- belongs to H_2^-, by Definition 2.1, and $w(z)^*k_z$ belongs to H_2^-, the second term on the right-hand side equals zero, while the first term equals $f_+(z)$, by (2.3). Thus,

$$\langle f,\ K_z\rangle_{H^w}=f_+(z)\quad\text{and}\quad \langle f,\ \widetilde{K}_z\rangle_{H^w}=\frac{f_-(z)}{\bar{z}}, \tag{2.19}$$

where the second relation is verified in much the same way as the first one. Reproducing properties (2.17) follow from (2.19) upon differentiating the integrals with respect to parameters z and $\bar{z}$. □

Lemma 2.3. *Let $K^{(j)}_z$ and $\widetilde{K}^{(j)}_z$ be the functions defined in* (2.9)*, and let z and ζ be two points in $\mathbb{D}$. Then*

$$\left\langle K^{(j)}_\zeta,\ K^{(i)}_z\right\rangle_{H^w}=\frac{1}{i!j!}\frac{\partial^{i+j}}{\partial z^i\partial\bar{\zeta}^j}\left(\frac{1-w(z)w(\zeta)^*}{1-z\bar{\zeta}}\right), \tag{2.20}$$

$$\left\langle \widetilde{K}_{\zeta}^{(j)},\ \widetilde{K}_{z}^{(i)}\right\rangle_{H^w} = \frac{1}{i!j!}\frac{\partial^{i+j}}{\partial\bar{z}^i\partial\zeta^j}\left(\frac{1-w(z)^*w(\zeta)}{1-\bar{z}\zeta}\right), \tag{2.21}$$

$$\left\langle \widetilde{K}_{z}^{(i)},\ K_{z}^{(j)}\right\rangle_{H^w} = w_{i+j+1}(z). \tag{2.22}$$

Proof. By the first formula in (2.19) and by definition (2.7),

$$\langle K_\zeta,\ K_z\rangle_{H^w} = K_{\zeta,+}(z) = \frac{1-w(z)w(\zeta)^*}{1-z\bar{\zeta}}.$$

On the other hand, by the first formula in (2.9),

$$\left\langle K_{\zeta}^{(j)},\ K_{z}^{(j)}\right\rangle_{H^w} = \frac{1}{i!j!}\frac{\partial^{i+j}}{\partial z^i\partial\bar{\zeta}^j}\langle K_\zeta,\ K_z\rangle_{H^w}$$

and substituting the first of the two last formulas into the second gives (2.20). The proof of (2.21) is quite similar. Making use of the formula (2.18) for $\widetilde{K}_{z,+}^{(i)}$, we get, again by the first reproducing property in (2.17), that

$$\begin{aligned}\left\langle \widetilde{K}_{z}^{(i)},\ K_{\zeta}^{(j)}\right\rangle_{H^w} &= \frac{1}{j!}\frac{\partial^j}{\partial z^j}\widetilde{K}_{z,+}^{(i)}(\zeta) = \frac{1}{j!}\frac{\partial^j}{\partial z^j}\left(\sum_{\ell=i+1}^{\infty} w_\ell(z)(\zeta-z)^{\ell-i-1}\right)\\ &= \sum_{\ell=i+j+1}^{\infty}\binom{\ell-i-1}{j} w_\ell(z)(\zeta-z)^{\ell-i-j-1},\end{aligned}$$

which implies (2.22), since

$$\lim_{\zeta\to z}\sum_{\ell=i+j+1}^{\infty}\binom{\ell-i-1}{j} w_\ell(z)(z-\zeta)^{\ell-i-j-1} = w_{i+j+1}(z). \qquad \square$$

Remark 2.4. Upon setting $\ell = j = n$ and $\zeta = z$ in formulas (2.20) and (2.21) in Lemma 2.3 we get

$$\left\| K_z^{(n)}\right\|_{H^w}^2 = \left\| \widetilde{K}_z^{(n)}\right\|_{H^w}^2 = d_{w,n}(z), \tag{2.23}$$

where $d_{w,n}(z)$ is given by (1.6), and thus, condition (1.7) tells us that

$$\liminf_{z\to t_0}\left\| K_z^{(n)}\right\|_{H^w} = \liminf_{z\to t_0}\left\| \widetilde{K}_z^{(n)}\right\|_{H^w} < \infty.$$

Remark 2.5. Formulas (2.20) allows us to rewrite the defining formula (1.1) for $\mathbf{P}_n^w(z)$ as

$$\mathbf{P}_n^w(z) = \left[\left\langle K_z^{(j)},\ K_z^{(i)}\right\rangle_{H^w}\right]_{i,j=0}^n \tag{2.24}$$

and thus, to realize the Schwarz–Pick matrix as the Gram matrix of the system of the functions $\{K_z^{(j)}\}_{j=0}^n$.

We conclude this section with three lemmas needed in the subsequent analysis. The first lemma gives a convenient representation of kernels $K_z^{(j)}$ and $\widetilde{K}_z^{(j)}$ as orthogonal projections of certain simple elements in L^w onto H^w.

Lemma 2.6. *Let $w \in \mathcal{S}$, let $z \in \mathbb{D}$, $j \in \mathbb{Z}_+$ and let $K_z^{(j)}$ and $\widetilde{K}_z^{(j)}$ be the functions defined in* (2.10) *and* (2.11)*, respectively. Then*

$$K_z^{(j)} = P_{H^w}\begin{bmatrix} 1 & w \\ w^* & 1 \end{bmatrix}\begin{bmatrix} k_{j,z} \\ 0 \end{bmatrix}, \qquad \widetilde{K}_z^{(j)} = P_{H^w}\begin{bmatrix} 1 & w \\ w^* & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \tilde{k}_{j,z} \end{bmatrix}, \tag{2.25}$$

where P_{H^w} denotes the orthogonal projection of L^w onto H^w and where $k_{j,z}$ and $\tilde{k}_{j,z}$ are the kernels defined in (2.4) *and* (2.5).

Proof. The function

$$g := \begin{bmatrix} 1 & w \\ w^* & 1 \end{bmatrix}\begin{bmatrix} k_{j,z} \\ 0 \end{bmatrix}$$

obviously belongs to L^w. We use the formula (2.1) to compute the L^w inner product between g and an arbitrary function $f = \left[\begin{smallmatrix} f_+ \\ f_- \end{smallmatrix}\right] \in H^w$:

$$\langle f,\ g\rangle_{L^w} = \left\langle \begin{bmatrix} f_+ \\ f_- \end{bmatrix},\ \begin{bmatrix} k_{j,z} \\ 0 \end{bmatrix} \right\rangle_{L^2} = \langle f_+,\ k_{j,z}\rangle_{L_2} = \frac{1}{j!} f_+^{(j)}(z).$$

Since $f \in H^w$, we have $P_{H^w} f = f$ and hence,

$$\langle f,\ P_{H^w} g\rangle_{H^w} = \langle P_{H^w} f,\ g\rangle_{L^w} = \langle f,\ g\rangle_{L^w} = \frac{1}{j!} f_+^{(j)}(z).$$

The first reproducing property in (2.17) now gives $\langle f,\ P_{H^w} g\rangle_{H^w} = \langle f,\ K_z^{(j)}\rangle_{H^w}$ and since f is arbitrary, the first equality in (2.25) follows. The proof of the second equality is quite similar. □

Lemma 2.7. *If $g_1 \in L_2$ and $g_2 \in H_2^+$, then*

$$P_{H^w}\begin{bmatrix} 1 & w \\ w^* & 1 \end{bmatrix}\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = P_{H^w}\begin{bmatrix} 1 & w \\ w^* & 1 \end{bmatrix}\begin{bmatrix} g_1 \\ 0 \end{bmatrix}, \tag{2.26}$$

which is straightforward from the definition of H^w and inner procuct L^w.

Lemma 2.8. *Let w be a Schur function and let h be an element of the space L^w. Then for every $t_0 \in \mathbb{T}$, $z \in \mathbb{D}$ and $n \geqslant 0$, the function*

$$h_z(t) = \left(\frac{1 - t\bar{t}_0}{1 - t\bar{z}}\right)^n h(t) \tag{2.27}$$

belongs to L^w and $\lim_{z \,\widehat{\to}\, t_0} \|h_z - h\|_{L^w} = 0$.

Proof. Since the space L^w is invariant under multiplication by a bounded scalar function, it follows that h_z belongs to L^w. Furthermore, by (2.1) and (2.27),

$$\|h_z - h\|_{L^w}^2 = \int\limits_{\mathbb{T}} \left\| \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix}^{[-1/2]} h(t) \right\|_{\mathbb{C}^2}^2 \cdot \left| \left(\frac{1 - t\bar{t}_0}{1 - t\bar{z}} \right)^n - 1 \right|^2 m(dt).$$

Now the assertion follows by the Dominated Convergence theorem, since for every z in the nontangential neighborhood

$$\Gamma_a(t_0) = \{ z \in \mathbb{D}:\ |t_0 - z| < a(1 - |z|) \} \quad (a > 1), \tag{2.28}$$

of t_0, and for every $t \in \mathbb{T}$, we have

$$\left| \frac{1 - t\bar{t}_0}{1 - t\bar{z}} \right| = \left| 1 + t\frac{\bar{z} - \bar{t}_0}{1 - t\bar{z}} \right| \leqslant 1 + \left| \frac{z - t_0}{t - z} \right| < 1 + \frac{|t_0 - z|}{1 - |z|} \leqslant 1 + a,$$

and therefore,

$$\left| \left(\frac{1 - t\bar{t}_0}{1 - t\bar{z}} \right)^n - 1 \right| \leqslant (1 + a)^n + 1. \qquad \square$$

3. Boundary reproducing kernels

In this section we study boundary analogues $K_{t_0}^{(n)}$ and $\widetilde{K}_{t_0}^{(n)}$ (here $t_0 \in \mathbb{T}$) of reproducing kernels $K_z^{(n)}$ and $\widetilde{K}_z^{(n)}$, defined in (2.10) and (2.11). The central result is Theorem 3.1. As a byproduct of this theorem we will get the proof of (1) $\Rightarrow$ (2) in Theorem 1.2. We will need the boundary analogues of the kernels (2.4) and (2.5):

$$k_{j,t_0}(\zeta) = \frac{z^j}{(1 - \zeta\bar{t}_0)^{j+1}}, \qquad \tilde{k}_{j,t_0}(\zeta) = \frac{1}{(\zeta - t_0)^{j+1}}. \tag{3.1}$$

Theorem 3.1. *Let $w \in \mathcal{S}$, $t_0 \in \mathbb{T}$, $n \in \mathbb{Z}_+$ and let $\liminf_{z \to t_0} d_{w,n}(z) < \infty$. Then*

(1) *The following nontangential boundary limits exist*:

$$w_j(t_0) := \lim_{z \widehat{\to} t_0} w_j(z) \quad \textit{for } j = 0, \ldots, n \quad \left(w_j(z) := \frac{w^{(j)}(z)}{j!} \right). \tag{3.2}$$

(2) *The functions*

$$K_{t_0}^{(n)}(t) := \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} k_{n,t_0}(t) \\ -\sum_{\ell=0}^n w_{n-\ell}(t_0)^* k_{\ell,t_0}(t) \end{bmatrix}, \tag{3.3}$$

$$\widetilde{K}_{t_0}^{(n)}(t) := \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} -\sum_{\ell=0}^n w_{n-\ell}(t_0)\tilde{k}_{\ell,t_0}(t) \\ \tilde{k}_{n,t_0}(t) \end{bmatrix} \tag{3.4}$$

belong to the space H^w.

(3) *The kernels* $K_z^{(n)}$ *and* $\widetilde{K}_z^{(n)}$ *defined in* (2.10) *and* (2.11) *converge respectively to* $K_{t_0}^{(n)}$ *and* $\widetilde{K}_{t_0}^{(n)}$ *in norm of* H^w *as* z *tends to* t_0 *nontangentially*:

$$K_z^{(n)} \xrightarrow{H^w} K_{t_0}^{(n)} \quad \text{and} \quad \widetilde{K}_z^{(n)} \xrightarrow{H^w} \widetilde{K}_{t_0}^{(n)} \qquad (z \widehat{\to} t_0). \tag{3.5}$$

(4) *The following nontangential limit exists and*

$$\lim_{z \widehat{\to} t_0} d_{w,n}(z) = \left\| K_{t_0}^{(n)} \right\|_{H^w}^2 < \infty. \tag{3.6}$$

Proof. By Remark 2.4, the assumption (1.7) guarantees that there exists a sequence $\{z_i\}$ of points in $\mathbb{D}$ approaching t_0 such that the sequences $\|K_{z_i}^{(n)}\|_{H^w}$ and $\|\widetilde{K}_{z_i}^{(n)}\|_{H^w}$ are bounded. Since every bounded set in a Hilbert space is weakly compact, there is a subsequence of $\{z_i\}$ (which still will be denoted by $\{z_i\}$), such that the sequences $\{K_{z_i}^{(n)}\}$ and $\{\widetilde{K}_{z_i}^{(n)}\}$ weakly converge in H^w. Let $F, \widetilde{F} \in H^w$ stand for these weak limits:

$$F = \begin{bmatrix} F_+ \\ F_- \end{bmatrix} = \operatorname{w-lim}_{z_i \to t_0} K_{z_i}^{(n)} \quad \text{and} \quad \widetilde{F} = \begin{bmatrix} \widetilde{F}_+ \\ \widetilde{F}_- \end{bmatrix} = \operatorname{w-lim}_{z_i \to t_0} \widetilde{K}_{z_i}^{(n)}. \tag{3.7}$$

We will establish more explicit formulas for F and $\widetilde{F}$. We start with F. Since it belongs to H^w, we can use reproducing properties (2.19) to get

$$F_+(\zeta) = \langle F,\ K_\zeta \rangle_{H^w} = \lim_{z_i \to t_0} \langle K_{z_i}^{(n)},\ K_\zeta \rangle_{H^w} = \lim_{z_i \to t_0} K_{z_i,+}^{(n)}(\zeta), \tag{3.8}$$

$$\frac{F_-(\zeta)}{\bar{\zeta}} = \langle F,\ \widetilde{K}_\zeta \rangle_{H^w} = \lim_{z_i \to t_0} \langle K_{z_i}^{(n)},\ \widetilde{K}_\zeta \rangle_{H^w} = \lim_{z_i \to t_0} \frac{K_{z_i,-}^{(n)}(\zeta)}{\bar{\zeta}}, \tag{3.9}$$

$|\zeta| < 1$, which can be written, on account of (2.13) and (2.14) as

$$F_+(\zeta) = \lim_{z_i \to t_0} \left(k_{n,z_i}(\zeta) - w(\zeta) \sum_{\ell=0}^{n} w_{n-\ell}(z_i)^* k_{\ell,z_i}(\zeta) \right), \tag{3.10}$$

$$\frac{F_-(\zeta)}{\bar{\zeta}} = \lim_{z_i \to t_0} \left(w(\zeta)^* \tilde{k}_{n,z_i}(\zeta)^* - \sum_{\ell=0}^{n} w_{n-\ell}(z_i)^* \tilde{k}_{\ell,z_i}(\zeta)^* \right). \tag{3.11}$$

It follows from (3.11) and the formula (2.5) for $\tilde{k}_{\ell,z}$ that

$$(\bar{\zeta} - \bar{t}_0)^{n+1} \frac{F_-(\zeta)}{\bar{\zeta}} = w(\zeta)^* - \lim_{z_i \to t_0} \sum_{\ell=0}^{n} w_\ell(z_i)^* (\bar{\zeta} - \bar{z}_i)^\ell \tag{3.12}$$

and thus, the limit on the right-hand side exists for every $|\zeta| < 1$. Since the coefficients of a polynomial of degree n are determined by its values at $n+1$ points and depend on these values continuously, the existence of the latter limit implies that the sequences $\{w_\ell(z_i)\}$ converge for $\ell = 0, \ldots, n$. Letting

$$w_\ell := \lim_{z_i \to t_0} w_\ell(z_i) \quad (\ell = 0, \ldots, n) \tag{3.13}$$

we can rewrite (3.10) and (3.11) as

$$F_+(\zeta) = k_{n,t_0}(\zeta) - w(\zeta)\sum_{\ell=0}^{n} w_{n-\ell}^* k_{\ell,t_0}(\zeta), \tag{3.14}$$

$$F_-(\zeta) = \bar{\zeta}\left(w(\zeta)^*\tilde{k}_{n,t_0}(\zeta)^* - \sum_{\ell=0}^{n} w_{n-\ell}^*\tilde{k}_{\ell,t_0}(\zeta)^*\right). \tag{3.15}$$

Since $F \in H^w$, we have $F_- \in H_-^2$ and therefore, the function $f(z) := \overline{F_-(z)}/z$ belongs to H_+^2. By a well-known property of H_+^2 functions, $\lim_{z \widehat{\to} t_0}(z - t_0)f(z) = 0$ which can be written, on account of the formula (3.15) as

$$\lim_{z \widehat{\to} t_0}(z - t_0)\left(w(z)\tilde{k}_{n,t_0}(z) - \sum_{\ell=0}^{n} w_{n-\ell}\tilde{k}_{\ell,t_0}(z)\right) = 0$$

and rewritten, by the definition (3.1) of $\tilde{k}_{\ell,t_0}$ as

$$w(z) = \sum_{\ell=0}^{n}(z - t_0)^\ell w_\ell + o\big((z - t_0)^n\big) \quad (z \widehat{\to} t_0).$$

The latter equality implies (see, e.g., [2, Corollary 7.9]) that the nontangential limits (3.2) exist and are equal to the numbers w_ℓ's introduced in (3.13).

Upon setting $\zeta = t \in \mathbb{T}$ in (3.14) and (3.15) and taking into account that $\bar{t}\cdot\tilde{k}_{j,t_0}(t)^* = k_{j,t_0}(t)$ for $t \in \mathbb{T}$, we get the following expression for F:

$$F(t) = \begin{bmatrix} F_+(t) \\ F_-(t) \end{bmatrix} = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix}\begin{bmatrix} k_{n,t_0}(t) \\ -\sum_{\ell=0}^{n} w_{n-\ell}^* k_{\ell,t_0}(t) \end{bmatrix}. \tag{3.16}$$

Since, as we have just seen, the numbers $w_0, \ldots, w_n$ are equal respectively to the nontangential boundary limits $w_0(t_0), \ldots, w_n(t_0)$ from (3.2), the expression on the right-hand side of (3.16) is identical with that in (3.3). Thus, $F = K_{t_0}^{(n)}$ and the desired membership $K_{t_0}^{(n)} \in H^w$ follows, since F belongs to H^w by construction (3.7).

Now we introduce the auxiliary function

$$h_z(t) = K_{t_0}^{(n)}(t)\cdot\left(\frac{1 - t\bar{t}_0}{1 - t\bar{z}}\right)^{n+1} = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix}\begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix},$$

where, as it is readily seen from (3.16),

$$g_1(t) = k_{n,t_0}(t)\cdot\left(\frac{1 - t\bar{t}_0}{1 - t\bar{z}}\right)^{n+1} = \frac{t_n}{(1 - t\bar{z})^{n+1}} = k_{n,z}(t), \tag{3.17}$$

$$g_2(t) = -\sum_{\ell=0}^{n} w_{n-\ell}^* k_{\ell,t_0}(t)\left(\frac{1 - t\bar{t}_0}{1 - t\bar{z}}\right)^{n+1} = -\frac{\sum_{\ell=0}^{n} w_\ell^* t^{n-\ell}(1 - t\bar{t}_0)^\ell}{(1 - t\bar{z})^{n+1}}. \tag{3.18}$$

Setting $h = K_{t_0}^{(n)}$ in Lemma 2.8 (which we can do since $K_{t_0}^{(n)} \in H^w$), we conclude that $h_z \xrightarrow{L^w} K_{t_0}^{(n)}$ as z tends to t_0 nontangentially. Therefore,

$$P_{H^w} h_z \xrightarrow{H^w} P_{H^w} K_{t_0}^{(n)} = K_{t_0}^{(n)} \quad (z \widehat{\to} t_0). \tag{3.19}$$

On the other hand, since $g_2 \in H_+^2$ (which is clearly seen from (3.18)), we have by Lemma 2.7

$$P_{H^w} h_z = P_{H^w} \begin{bmatrix} 1 & w \\ w^* & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = P_{H^w} \begin{bmatrix} 1 & w \\ w^* & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ 0 \end{bmatrix}. \tag{3.20}$$

Upon taking into account the special form (3.17) of g_1 and invoking the first formula in (2.25) we conclude from (3.20) that $P_{H^w} h_z = K_z^{(n)}$. Then (3.19) turns into $K_z^{(n)} \xrightarrow{H^w} K_{t_0}^{(n)}$ which proves the first convergence in (3.5). Repeating the same arguments for $\widetilde{F}$ in (3.7) shows that $\widetilde{F}$ is equal to the kernel $\widetilde{K}_{t_0}^{(n)}$ given by (3.4) and that the kernels $\widetilde{K}_z^{(n)}$ converge to $\widetilde{K}_{t_0}^{(n)}$ in norm of H^w as z approaches t_0 nontangentially. This completes the proof of the three first statements in the theorem. Finally, by (3.5) and (2.23),

$$\lim_{z \widehat{\to} t_0} d_{w,n}(z) = \lim_{z,\zeta \to t_0} \| K_z^{(n)} \|_{H^w}^2 = \| \lim_{z \to t_0} K_z^{(n)} \|_{H^w}^2 = \| K_{t_0}^{(n)} \|_{H^w}^2 < \infty$$

which proves (3.6). □

Remark 3.2. The limits in (1.7) and (1.8) are equal.

Proof. Inequality

$$\liminf_{z \to t_0} d_{w,n}(z) \leqslant \lim_{z \widehat{\to} t_0} d_{w,n}(z)$$

is obvious since the first limit allows z to approach t_0 unrestrictedly in $\mathbb{D}$, while the second limit is nontangential. To prove the reverse inequality, assume that $\{z_j\}$ is a sequence that leads to the limit inferior in (1.7), so that the sequence of numbers

$$\| K_{z_j}^{(n)} \|_{H^w}^2 = d_{w,n}(z_j)$$

converges to the limit inferior. In particular, the sequence is bounded. Then there exists a subsequence of the sequence $\{z_j\}$ (that is still denoted by $\{z_j\}$) such that $K_{z_j}^{(n)}$ converges to $K_{t_0}^{(n)}$ weakly in H^w. Then

$$\| K_{t_0}^{(n)} \|_{H^w}^2 \leqslant \lim_{z_j \to t_0} \| K_{z_j}^{(n)} \|_{H^w}^2 = \liminf_{z \to t_0} d_{w,n}(z).$$

Since the limit in (1.8) equals $\| K_{t_0}^{(n)} \|_{H^w}^2$ by (3.6), then, by the latter inequality, it does not exceed $\liminf_{z \to t_0} d_{w,n}(z)$, which completes the proof. □

Remark 3.3. Formulas (3.3) and (3.4) for the boundary kernels $K_{t_0}^{(n)}$ and $\widetilde{K}_{t_0}^{(n)}$ can be viewed as the result of replacing z by t_0 and $w_0(z), \dots, w_n(z)$ by $w_0(t_0), \dots, w_n(t_0)$ in formulas (2.10) and (2.11) for the corresponding interior kernels. The proof of Theorem 3.1 shows that the space H^w contains any boundary analogues of the kernels $K_z^{(n)}$ and $\widetilde{K}_z^{(n)}$ for the point $t_0 \in \mathbb{T}$ at all if and only if condition (1.7) holds true. If it does not, the functions $K_{t_0}^{(n)}$ and $\widetilde{K}_{t_0}^{(n)}$ defined in (3.3) and (3.4) do not belong to H^w no matter the boundary limits $w_0(t_0), \dots, w_n(t_0)$ are used in these formulas or any other numbers.

If the condition (1.7) holds, we can use formulas (3.3) and (3.4) to define the boundary kernels $K_{t_0}^{(j)}$ and $\widetilde{K}_{t_0}^{(j)}$ for $j = 0, \dots, n$. The next result is a useful addition to Theorem 3.1.

Theorem 3.4. *Let $w \in \mathcal{S}$, $t_0 \in \mathbb{T}$, $n \in \mathbb{Z}_+$ and let us assume that condition* (1.7) *holds. Then the kernels $K_{t_0}^{(j)}$ and $\widetilde{K}_{t_0}^{(j)}$ defined via formulas* (3.3) *and* (3.4) *belong to the space H^w for $j = 0, \dots, n$ and*

$$K_z^{(j)} \xrightarrow{H^w} K_{t_0}^{(j)}, \quad \widetilde{K}_z^{(j)} \xrightarrow{H^w} \widetilde{K}_{t_0}^{(j)} \quad \text{as } z \,\widehat{\to}\, t_0, \tag{3.21}$$

where the kernels $K_z^{(j)}$ and $\widetilde{K}_z^{(j)}$ are defined in (2.10) *and* (2.11).

Proof. We will prove the part concerning the kernels $K_{t_0}^{(j)}$. Using the following recursive relation

$$K_{t_0}^{(j-1)}(t) = \frac{1}{t}\left((1 - t\bar{t}_0)K_{t_0}^{(j)}(t) + \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix}\begin{bmatrix} 0 \\ w_j(t_0)^* \end{bmatrix}\right), \tag{3.22}$$

verification of which is straightforward, we can show that

$$K_{t_0}^{(j)} \in H^w \quad \Rightarrow \quad K_{t_0}^{(j-1)} \in H^w. \tag{3.23}$$

Indeed, if $K_{t_0}^{(j)} \in H^w$, then in particular, $K_{t_0}^{(j)} \in L^w$ and it follows from (3.22) that $K_{t_0}^{(j-1)} \in L^w$. Furthermore, by (3.22), $K_{t_0,+}^{(j-1)}(\zeta) = f_+(\zeta)/\zeta$, where

$$f_+(\zeta) := (1 - \zeta\bar{t}_0)K_{t_0,+}^{(j)}(\zeta) + w(\zeta)w_j(t_0)^*.$$

The function f_+ belongs to H_2^+, since $K_{t_0,+}^{(j)} \in H_2^+$. Since

$$f_+(0) = K_{t_0,+}^{(j)}(0) + w(0)w_j(t_0)^* = -w(0)w_j(t_0)^* + w(0)w_j(t_0)^* = 0,$$

then $K_{t_0,+}^{(j-1)}(\zeta) = f_+(\zeta)/\zeta \in H_2^+$ as well. Comparing the bottom components in (3.22) we get

$$K_{t_0,-}^{(j-1)}(\zeta) = (\bar{\zeta} - \bar{t}_0)K_{t_0,-}^{(j)}(\zeta) + \bar{\zeta}w_j(t_0)^*$$

and thus, the assumption $K_{t_0,-}^{(j)} \in H_2^-$ implies that $K_{t_0,-}^{(j-1)} \in H_2^-$. Therefore, $K_{t_0}^{(j-1)} \in H^w$ which completes the proof of (3.23). Since by Theorem 3.1 $K_{t_0}^{(n)} \in H^w$, the inverse induction arguments show that $K_{t_0}^{(j)} \in H^w$ for every $j = 0, \dots, n$. Then it follows by a virtue of Theorem 3.1 that the first series of convergences in (3.21) holds true. The part concerning the kernels $\widetilde{K}_{t_0}^{(j)}$ is proved in much the same way. □

The next remark explains the role of $K_{t_0}^{(n)}$ and $\widetilde{K}_{t_0}^{(n)}$ as boundary reproducing kernels: they reproduce boundary limits of the derivatives of the components of H^w functions. In the case when $n=0$ the result can be found in [7,8] in a slightly different form.

Remark 3.5. Let $w \in \mathcal{S}$ and let us assume that condition (1.7) holds. Then for every function $f = \left[\begin{smallmatrix} f_+ \\ f_- \end{smallmatrix}\right] \in H^w$, the following nontangential limits exist and are reproduced by the kernels $K_{t_0}^{(j)}$ and $\widetilde{K}_{t_0}^{(j)}$:

$$\lim_{z \widehat{\rightarrow} t_0} \frac{1}{j!}\frac{d^j}{dz^j} f_+(z) = \langle f, K_{t_0}^{(j)} \rangle_{H^w}, \qquad \lim_{z \widehat{\rightarrow} t_0} \frac{1}{j!}\frac{d^j}{d\bar{z}^j}\left(\frac{f_-(z)}{\bar{z}}\right) = \langle f, \widetilde{K}_{t_0}^{(j)} \rangle_{H^w}$$

for $j = 0, \ldots, n$.

For the proof, it suffices to use reproducing properties (2.17) of $K_z^{(n)}$ and $\widetilde{K}_z^{(n)}$ and to take advantage of (3.21).

In conclusion we will show that under assumption (1.7), the boundary kernels $K_{t_0}^{(j)}$ and $\widetilde{K}_{t_0}^{(j)}$ satisfy certain linear relations.

Theorem 3.6. *Let $w \in \mathcal{S}$ and let us assume that condition* (1.7) *holds. Then*

(1) *The nontangential boundary limits*

$$w_j = w_j(t_0) := \lim_{z \widehat{\rightarrow} t_0} w_j(z) \quad (j = 0, \ldots, n) \tag{3.24}$$

(*that exist by Theorem* 3.1) *are subject to the matrix equality*

$$\begin{bmatrix} w_0 & w_1 & \ldots & w_n \\ 0 & w_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & w_1 \\ 0 & \ldots & 0 & w_0 \end{bmatrix} \boldsymbol{\Psi}_n(t_0) \begin{bmatrix} w_0^* & w_1^* & \ldots & w_n^* \\ 0 & w_0^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & w_1^* \\ 0 & \ldots & 0 & w_0^* \end{bmatrix} = \boldsymbol{\Psi}_n(t_0), \tag{3.25}$$

where $\boldsymbol{\Psi}_n(t_0)$ is the upper triangular matrix with the entries $\Psi_{j\ell}$ defined in (1.5). *In particular, $|w_0| = 1$.*

(2) *The kernels $K_{t_0}^{(j)}$ and $\widetilde{K}_{t_0}^{(j)}$ defined via formulas* (3.3) *and* (3.4) *for $j = 0, \ldots, n$, satisfy relations*

$$\sum_{i=0}^{j} \widetilde{K}_{t_0}^{(i)} g_{i,j} = K_{t_0}^{(j)} \quad (j = 0, \ldots, n), \tag{3.26}$$

where $g_{i,j}$ are the numbers given by

$$g_{i,j} := \sum_{\ell=0}^{j-i} \Psi_{i,j-\ell} w_\ell^* \quad \textit{for } 0 \leqslant i \leqslant j \leqslant n. \tag{3.27}$$

Proof. First we note that the kernels (3.1) satisfy relations

$$k_{j,t_0}(\zeta) = -\sum_{i=0}^{j} \Psi_{ij} \tilde{k}_{i,t_0}(\zeta) \quad (j \in \mathbb{Z}_+,\ \zeta \neq t_0). \tag{3.28}$$

Verification is straightforward and rests on definitions (3.1) and (1.5) (see [2, Proposition 10.4] for detail). Using these relations we have

$$\begin{aligned}\sum_{\ell=0}^{j} w_{j-\ell}^* k_{\ell,t_0}(\zeta) &= -\sum_{\ell=0}^{j} w_{j-\ell}^* \sum_{i=0}^{\ell} \Psi_{i,\ell} \tilde{k}_{i,t_0}(\zeta) = -\sum_{i=0}^{j} \left(\sum_{\ell=i}^{j} \Psi_{i,\ell} w_{j-\ell}^* \right) \tilde{k}_{i,t_0}(\zeta) \\ &= -\sum_{i=0}^{j} \left(\sum_{\ell=0}^{j-i} \Psi_{i,j-\ell} w_\ell^* \right) \tilde{k}_{i,t_0}(\zeta) = -\sum_{i=0}^{j} g_{ij} \tilde{k}_{i,t_0}(\zeta),\end{aligned} \tag{3.29}$$

where the first equality is obtained upon replacing the kernels k_{ℓ,t_0} by the corresponding expressions from (3.28), the second equality is the result of changing the order of summation, the third equality is just the substitution $\ell := j - \ell$ and the last equality holds by definition (3.27). Now we plug in (3.28) and (3.29) into (3.3) to express the kernel $K_{t_0}^{(j)}$ in terms of $\tilde{k}_{i,t_0}$'s rather than k_{i,t_0}'s:

$$K_{t_0}^{(j)}(t) = \begin{bmatrix} K_{t_0,+}^{(j)}(t) \\ K_{t_0,-}^{(j)}(t) \end{bmatrix} = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \sum_{i=0}^{j} \begin{bmatrix} -\Psi_{ij} \\ g_{ij} \end{bmatrix} \tilde{k}_{i,t_0}(t). \tag{3.30}$$

By Theorem 3.1, the kernels $K_{t_0}^{(j)}$ belong to H^w for $j = 0, \ldots, n$; in particular, their top and bottom components belong to H_+^2 and to H_-^2, respectively. In virtue of the arguments following formula (3.15), the membership $K_{t_0,-}^{(j)} \in H_-^2$ implies the asymptotic relation

$$w(z) = \sum_{\ell=0}^{j} (z - t_0)^\ell w_\ell + o\big((z - t_0)^j\big) \quad \text{as } z \widehat{\to} t_0. \tag{3.31}$$

On the other hand, since $K_{t_0,+}^{(j)}$ belongs to H_+^2, we have

$$(z - t_0) K_{t_0,+}^{(j)}(z) \to 0 \quad \text{as } z \widehat{\to} t_0. \tag{3.32}$$

Making use of the formula (3.30) for $K_{t_0,+}^{(j)}$ and the definition (3.1) of $\tilde{k}_{i,t_0}$ we conclude from (3.32) that

$$(z-t_0)^{j+1}K_{t_0,+}^{(j)}(z)=w(z)\cdot\sum_{i=0}^{j}g_{ij}(z-t_0)^{j-i}-\sum_{i=0}^{j}\Psi_{i,j}(z-t_0)^{j-i}$$
$$=o\big((z-t_0)^j\big)\quad(z\widehat{\rightarrow}t_0).$$

Substituting (3.31) into the latter asymptotic equality and using r and i instead of $j-i$ lead us to

$$\left(\sum_{\ell=0}^{j}(z-t_0)^{\ell}w_\ell\right)\cdot\left(\sum_{r=0}^{j}g_{j-r,j}(z-t_0)^r\right)-\sum_{i=0}^{j}\Psi_{j-i,j}(z-t_0)^i=o\big((z-t_0)^j\big).$$

The expression on the left-hand side is a polynomial $p(z)=\sum_{i=0}^{2j}p_i(z-t_0)^i$ and the above condition implies that $p_i=0$ for $i=0,\dots,j$. Thus,

$$p_i=\sum_{\ell=0}^{i}w_\ell g_{j-i+\ell,j}-\Psi_{j-i,j}=0\quad\text{for } i=0,\dots,j,$$

which on account of (3.27), can be written equivalently as (using again $j-i$ instead of i)

$$\Psi_{i,j}=\sum_{\ell=0}^{j-i}w_\ell g_{i+\ell,j}=\sum_{\ell=0}^{i}w_\ell\sum_{r=0}^{j-i-\ell}\Psi_{i+\ell,j-r}w_r^*\quad(i=0,\dots,j).\tag{3.33}$$

The latter relations express equality of the ijth entries in the matrix identity (3.25) for $0\leqslant i\leqslant j\leqslant n$. Due to the upper triangular structure, all the remaining entries on the left-hand side and on the right-hand side of (3.25) are zeros; thus, equality (3.25) follows. Equality for the top diagonal entries in (3.25) reads: $w_0\Psi_{00}w_0^*=\Psi_{00}$ which is equivalent (since $\Psi_{00}=t_0\neq 0$) to $|w_0|=1$. This completes the proof of statement (1) of the theorem.

To verify (3.26), we will use the formulas (3.30) and (3.4) for the boundary kernels $K_{t_0}^{(j)}$ and $\widetilde{K}_{t_0}^{(j)}$. Due to the common left factor

$$\begin{bmatrix}1 & w\\ w^* & 1\end{bmatrix}$$

in these formulas, it suffices to verify equalities

$$\sum_{i=0}^{j}\begin{bmatrix}-\sum_{\ell=0}^{i}w_{i-\ell}\tilde{k}_{\ell,t_0}(t)\\ \tilde{k}_{i,t_0}(t)\end{bmatrix}g_{i,j}=\sum_{i=0}^{j}\begin{bmatrix}-\Psi_{ij}\\ g_{ij}\end{bmatrix}\tilde{k}_{i,t_0}(t)\tag{3.34}$$

for $j=0,\dots,n$. Equality of the bottom components is self-evident. The top components are also equal since

$$\sum_{i=0}^{j}g_{ij}\sum_{\ell=0}^{i}w_{i-\ell}\tilde{k}_{\ell,t_0}(t)=\sum_{\ell=0}^{j}\left(\sum_{i=0}^{j-\ell}w_i g_{i+\ell,j}\right)\tilde{k}_{\ell,t_0}(t)=\sum_{\ell=0}^{j}\Psi_{\ell j}\tilde{k}_{\ell,t_0}(t),$$

where the first equality is obtained by changing the order of summation and substituting $i := i - \ell$, and the second equality is justified by (3.33). □

Remark 3.7. Note that equality of the rightmost columns in (3.25) already implies the "whole" matrix identity (see [2, Theorem 10.5]). In other words, the matrix identity (3.25) is equivalent to the system of the following equalities (compare with (3.33)):

$$\sum_{\ell=0}^{i} w_\ell \sum_{r=0}^{n-i-\ell} \Psi_{i+\ell,n-r} w_r^* = \Psi_{i,n} \quad (i = 0, \dots, n). \tag{3.35}$$

Remark 3.8. It is curious that all the assertions in Theorem 3.6 follow from the assumption that

$$K_{t_0,+}^{(n)} \in H_+^2 \quad \text{and} \quad K_{t_0,-}^{(n)} \in H_-^2. \tag{3.36}$$

Indeed, the existence of the boundary limits (3.24) follows from the fact that $K_{t_0,-}^{(n)} \in H_-^2$. Furthermore, as it was shown in the proof of Theorem 3.4, conditions (3.36) guarantee that $K_{t_0,+}^{(j)} \in H_+^2$ and $K_{t_0,-}^{(j)} \in H_-^2$ for every $j = 0, \dots, n$. That is all we needed to get (3.25), which, in turn, implies (3.26). Note that (3.36) is weaker than (1.7), since (1.7) is equivalent to $K_{t_0}^{(n)} \in H^w$, which in turn is equivalent to $K_{t_0}^{(n)} \in L^w$. The latter yields (3.36) but does not follow from (3.36).

We also remark that relations (3.26) and (3.27) in Theorem 3.6 are of triangular form and can be rewritten in matrix notation as follows.

Remark 3.9. Let $\boldsymbol{\Psi}_n(t_0)$ be defined as in (1.5) and let $\mathbf{W}_n$ and $\mathbf{G}_n$ be the upper triangular matrices with the entries

$$W_{ij} = \begin{cases} w_{j-i}^*, & \text{if } j \geqslant i, \\ 0, & \text{if } j < i, \end{cases} \qquad G_{ij} = \begin{cases} g_{ij}, & \text{if } j \geqslant i, \\ 0, & \text{if } j < i, \end{cases} \qquad (i, j = 0, \dots, n), \tag{3.37}$$

where the numbers $w_0, \dots, w_n$ and g_{ij} are defined in (3.24) and (3.27) (note that $\mathbf{W}_n$ appears in (3.25) as the rightmost factor in the left-hand side expression). Then relations (3.26) in (3.27) can be written in the matrix form as

$$\begin{bmatrix} \widetilde{K}_{t_0}^{(0)} & \dots & \widetilde{K}_{t_0}^{(n)} \end{bmatrix} \mathbf{G}_n = \begin{bmatrix} K_{t_0}^{(0)} & \dots & K_{t_0}^{(n)} \end{bmatrix} \quad \text{and} \quad \mathbf{G}_n = \boldsymbol{\Psi}_n(t_0)\mathbf{W}_n, \tag{3.38}$$

respectively.

4. Proof of Theorem 1.2

In this section we complete the proof of Theorem 1.2. Recall that equivalence (1) ⇔ (2) and equality of the limits in (1.7) and (1.8) has been already proved in Theorem 3.1 and Remark 3.2. Now we will use the results obtained in Section 3 to prove (1) ⇒ (3), (1) ⇒ (4) and equality (1.10).

Once again we assume that the condition (1.7) holds for a Schur function w and a boundary point $t_0 \in \mathbb{T}$. Then the nontangential boundary limits

$$w_j = w_j(t_0) := \lim_{z \widehat{\to} t_0} \frac{w^{(j)}(z)}{j!} \tag{4.1}$$

exist for $j = 0, \dots, n$ (by Theorem 3.1), $|w_0| = 1$ (by Theorem 3.6(1)), the kernels $K_{t_0}^{(j)}$ and $\widetilde{K}_{t_0}^{(j)}$ defined via formulas (3.3) and (3.4) for $j = 0, \dots, n$ belong to the space H^w (by Theorem 3.4) and satisfy relations (3.26) (by Theorem 3.6); finally the kernels $K_z^{(j)}$ and $\widetilde{K}_z^{(j)}$ are defined in (2.10) and (2.11) converge to $K_{t_0}^{(j)}$ and $\widetilde{K}_{t_0}^{(j)}$:

$$K_z^{(j)} \xrightarrow{H^w} K_{t_0}^{(j)}, \quad \widetilde{K}_z^{(j)} \xrightarrow{H^w} \widetilde{K}_{t_0}^{(j)} \quad \text{as } z \widehat{\to} t_0, \tag{4.2}$$

by Theorem 3.4. Making use of (1.2), (2.24) and (4.2) we get

$$\mathbf{P}_n^w(t_0) := \lim_{z \widehat{\to} t_0} \mathbf{P}_n(z) = \lim_{z \widehat{\to} t_0} \left[\langle K_z^{(j)}, K_z^{(i)} \rangle_{H^w} \right]_{i,j=0}^n = \left[\langle K_{t_0}^{(j)}, K_{t_0}^{(i)} \rangle_{H^w} \right]_{i,j=0}^n \tag{4.3}$$

which proves the existence of the boundary Schwartz–Pick matrix and also shows that $\mathbf{P}_n^w(t_0)$ is the Gram matrix of the system of the functions $\{K_{t_0}^{(j)}\}_{j=0}^n$. This completes the proof of (1) $\Rightarrow$ (3) in Theorem 1.2. Now we will show that the nontangential limits (4.1) exist also for $j = n + 1, \dots, 2n + 1$. We take the advantage of (2.22) and (4.2) to get

$$w_{i+j+1}(t_0) := \lim_{z \widehat{\to} t_0} w_{i+j+1}(z) = \lim_{z \widehat{\to} t_0} \langle \widetilde{K}_z^{(i)}, K_z^{(j)} \rangle_{H^w} = \langle \widetilde{K}_{t_0}^{(i)}, K_{t_0}^{(j)} \rangle_{H^w} \tag{4.4}$$

for $i, j = 0, \dots n$. Letting i and j run through the set $\{1, \dots, n\}$ we conclude from (4.4) that the limits (4.1) indeed exist for $j = n + 1, \dots, 2n + 1$ and therefore, for every $j = 0, \dots, 2n + 1$. Using these limits we can define the matrix $\mathbb{P}_n^w(t_0)$ via the formula (1.4), i.e.,

$$\mathbb{P}_n^w(t_0) = \mathbf{H}_n \boldsymbol{\Psi}_n(t_0) \mathbf{W}_n, \tag{4.5}$$

where $\mathbf{H}_n = [w_{i+j+1}]_{i,j=0}^n$ and $\mathbf{W}_n$ is defined in (3.37). To complete the proof of (1) $\Rightarrow$ (4), it remains to show that $\mathbb{P}_n^w(t_0) \geqslant 0$. But this will follow from equality (1.10) since $\mathbf{P}_n^w(t_0) \geqslant 0$.

To prove (1.10), we fix two vector-columns

$$x = \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{C}^{n+1} \quad \text{and} \quad \text{let } e = \begin{bmatrix} e_0 \\ \vdots \\ e_n \end{bmatrix} := \mathbf{G}_n x, \tag{4.6}$$

where $\mathbf{G}_n$ is the matrix defined in (3.37). By formula (4.3),

$$y^* \mathbf{P}_n(t_0) x = \langle [\, K_{t_0}^{(0)} \ \dots \ K_{t_0}^{(n)} \,] x, \ [\, K_{t_0}^{(0)} \ \dots \ K_{t_0}^{(n)} \,] y \rangle_{H^w}.$$

Now we transform the latter expression, subsequently using (3.38), (4.6), (4.4), and again (4.6) to get

$$\begin{aligned}
y^*\mathbf{P}_n(t_0)x &= \big\langle \big[\, \widetilde{K}_{t_0}^{(0)} \ \dots \ \widetilde{K}_{t_0}^{(n)} \,\big]\mathbf{G}_n x,\ \big[\, K_{t_0}^{(0)} \ \dots \ K_{t_0}^{(n)} \,\big] y \big\rangle_{H^w} \\
&= \big\langle \big[\, \widetilde{K}_{t_0}^{(0)} \ \dots \ \widetilde{K}_{t_0}^{(n)} \,\big] e,\ \big[\, K_{t_0}^{(0)} \ \dots \ K_{t_0}^{(n)} \,\big] y \big\rangle_{H^w} \\
&= \sum_{i,j=0}^{n} \big\langle \widetilde{K}_{t_0}^{(i)} e_i,\ K_{t_0}^{(j)} y_j \big\rangle_{H^w} \\
&= \sum_{i,j=0}^{n} \bar{y}_j w_{i+j+1} e_i = y^*\mathbf{H}_n e = y^*\mathbf{H}_n\mathbf{G}_n x.
\end{aligned}$$

Since vectors x and y are arbitrary, it follows that $\mathbf{P}_n(t_0) = \mathbf{H}_n\mathbf{G}$ which on account of the second relation in (3.38) and (4.6) gives

$$\mathbf{P}_n(t_0) = \mathbf{H}_n\mathbf{G}_n = \mathbf{H}_n\boldsymbol{\Psi}_n(t_0)\mathbf{W}_n = \mathbb{P}_n^w(t_0)$$

which proves (1.10) and completes the proof of (1) ⇒ (4) in Theorem 1.2. Since the proof of (3) ⇒ (2) ⇒ (1) is self-evident, it suffices to prove (4) ⇒ (2) to close the loop. The proof presented below is based on arguments of interpolation nature.

Lemma 4.1. *Let f and w be two functions analytic on a "neighborhood" $\mathcal{U} = \{z \in \mathbb{D}: |z - t_0| < \varepsilon\}$ of $t_0 \in \mathbb{T}$ and let us assume that the nontangential boundary limits of their $2n+2$ first derivatives at t_0 exist and are equal*:

$$w_j(t_0) = f_j(t_0) \quad \text{for } j = 0, \dots, 2n+1. \tag{4.7}$$

Then $d_{w,n}(z) - d_{f,n}(z) = o(1)$ as $z \widehat{\rightarrow} t_0$.

Proof. Straightforward differentiation of the product

$$\frac{1}{(n!)^2} w(z) \frac{1}{1-|z|^2} w(z)^*$$

gives

$$\frac{1}{(n!)^2} \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \frac{|w(z)|^2}{1-|z|^2} = \sum_{i,j=0}^{n} w_{n-i}(z) \frac{u_{i,j}(z)}{(1-|z|^2)^{i+j+1}} w_{n-j}(z)^*, \tag{4.8}$$

where, as before, $w_j(z)$ stands for $\frac{1}{j!} w^{(j)}(z)$ and

$$u_{i,j}(z) = \sum_{\ell=0}^{\min(i,j)} \frac{(i+j-\ell)!}{(i-\ell)!(j-\ell)!l!} \bar{z}^{i-\ell} z^{j-\ell} \big(1-|z|^2\big)^{\ell}, \tag{4.9}$$

for $i, j = 0, \dots, n$. Making use of (4.8) and of a similar formula for f, we get

$$
\begin{aligned}
d_{w,n}(z) - d_{f,n}(z) &= \frac{1}{(n!)^2}\frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n}\frac{|f(z)|^2 - |w(z)|^2}{1-|z|^2} \\
&= \sum_{i,j=0}^{n} f_{n-i}(z)\frac{u_{i,j}(z)}{(1-|z|^2)^{i+j+1}} f_{n-j}(z)^* \\
&\quad - \sum_{i,j=0}^{n} w_{n-i}(z)\frac{u_{i,j}(z)}{(1-|z|^2)^{i+j+1}} w_{n-j}(z)^* \\
&= \sum_{i,j=0}^{n} \big(f_{n-i}(z) - w_{n-i}(z)\big)\frac{u_{i,j}(z)}{(1-|z|^2)^{i+j+1}} f_{n-j}(z)^* \\
&\quad - \sum_{i,j=0}^{n} w_{n-i}(z)\frac{u_{i,j}(z)}{(1-|z|^2)^{i+j+1}} \big(w_{n-j}(z)^* - f_{n-j}(z)^*\big). \qquad (4.10)
\end{aligned}
$$

By (4.7),

$$
f_i(z) - w_i(z) = o\big((z-t_0)^{2n+1-i}\big) \quad \text{as } z \widehat{\to} t_0,
$$

and therefore, since $z - t_0 = O(1-|z|^2)$ when z approaches t_0 nontangentially,

$$
\frac{f_{n-i}(z) - w_{n-i}(z)}{(1-|z|^2)^{i+j+1}} = o\big((z-t_0)^{2n+1-(n-i)-(i+j+1)}\big) = o\big((z-t_0)^{n-j}\big), \quad z \widehat{\to} t_0,
$$

and the latter equalities hold for all $i, j = 0, \dots, n$. It remains to note that (as it is readily seen from (4.9) $u_{i,j}(z) = O(1)$ as $z \widehat{\to} t_0$ and now the desired assertion follows from (4.10). □

Proof of (4) ⇒ (2) in Theorem 1.2. Thus, we assume that the nontangential limits

$$
w_j = w_j(t_0) := \lim_{z \widehat{\to} t_0} \frac{w^{(j)}(z)}{j!} \quad (j = 0, \dots, 2n+1) \tag{4.11}
$$

exist and satisfy conditions (1.9). Then there exists a finite Blaschke product f such that

$$
f_j(t_0) = w_j \quad \text{for } j = 0, \dots, 2n+1. \tag{4.12}
$$

Indeed, equalities (4.12) can be considered as interpolation conditions for a boundary interpolation problem for Schur class functions with the data $t_0 \in \mathbb{T}$ and $w_0, \dots, w_{2n+1} \in \mathbb{C}$ satisfying conditions (1.9), that is

$$
|w_0| = 1 \quad \text{and} \quad \mathbb{P}_n^w(t_0) := \begin{bmatrix} w_1 & \cdots & w_{n+1} \\ \vdots & & \vdots \\ w_{n+1} & \cdots & w_{2n+1} \end{bmatrix} \boldsymbol{\Psi}_n(t_0) \begin{bmatrix} \bar{w}_0 & \dots & \bar{w}_n \\ & \ddots & \vdots \\ 0 & & \bar{w}_0 \end{bmatrix} \geqslant 0. \tag{4.13}
$$

This problem was studied in [6], [1, Section 21], [2, Section 13]. The results obtained there show in particular, that in case the matrix $\mathbb{P}_n^w(t_0)$ is positive definite, there are infinitely many Schur functions (and also infinitely many finite Blaschke products) f satisfying conditions (4.12). We will discuss this problem in more detail and we will describe all its solutions at another opportunity.

By (4.11) and (4.12), equalities $w_j(t_0) = f_j(t_0)$ hold for $j = 0, \dots, 2n+1$ and we apply Lemma 4.1 to conclude that $d_{w,n}(z) - d_{f,n}(z) = o(1)$ as z approaches t_0 nontangentially. Since f is a finite Blaschke product, $\lim_{z\to t_0} d_{f,n}(z)$ exists and is finite (see, e.g., [3, Proposition 6.2]). Therefore

$$\lim_{z \widehat{\to} t_0} d_{w,n}(z) = \lim_{z\to t_0} d_{f,n}(z) < \infty$$

and property (2) follows.

If $\mathbb{P}_n^w(t_0) \geqslant 0$ is singular, then w is a finite Blaschke product of degree equal the rank of $\mathbb{P}_n^w(t_0)$. Therefore, (2) holds as well. This completes the proof. □

5. Final remarks

Theorem 1.2 imposes conditions on (and establishes relations between) the quantities of two different types: the ratio $\frac{1-|w(z)|^2}{1-|z|^2}$ and its partial derivatives, on one hand (statements (1)–(3)), and angular boundary limits of derivatives of w, on another hand (statement (4)). Condition (1.7) is apparently the weakest condition of the first type that implies all other statements in Theorem 1.2. We will discuss briefly to what extent conditions in statement (4) on angular boundary derivatives can be relaxed in order to guarantee the condition (1.7) to hold true. Note that in the proof of (4) ⇒ (2) (at the end of Section 4) we did not use the fact that w is a Schur class function. In other words, condition (1.8) holds true for any function w analytic on $\mathbb{D}$ for which the angular boundary limits (4.11) exist and satisfy conditions (1.9). Actually, the positivity assumption about $\mathbb{P}_n^w(t_0)$ in (4.13) can be dropped.

Theorem 5.1. *Let w be analytic in a neighborhood $\{z \in \mathbb{D}\colon |z - t_0| < \varepsilon\}$ of $t_0 \in \mathbb{T}$. Let the nontangential limits* (4.11) *exist and let us assume that*

$$|w_0| = 1 \quad \textit{and} \quad \mathbb{P}_n^w(t_0) = \mathbb{P}_n^w(t_0)^*, \tag{5.1}$$

where $\mathbb{P}_n^w(t_0)$ is defined in (4.13). *Then condition* (1.8) *holds true.*

The proof will be presented elsewhere. Finally, we note that another higher order analogue of the Carathéodory–Julia theorem different from our Theorem 1.2 appears in [2, Section 9] in the context of matrix-valued Schur functions. In the present scalar valued case, the results from [2, Section 9] can be formulated as follows.

Theorem 5.2. *For $w \in \mathcal{S}$, $t_0 \in \mathbb{T}$ and $n \in \mathbb{Z}_+$, the following are equivalent*:

(1) $\sup_{z\in\Gamma_a(t_0)} d_{w,n}(z) < \infty$ *for some $\Gamma_a(t_0)$ of the form* (2.28).
(2) *The boundary Schwarz–Pick matrix $\mathbf{P}_n^w(t_0)$ exists.*
(3) *The following nontangential limits exist*:

$$w_j(t_0) := \lim_{z \widehat{\to} t_0} w_j(z) \quad \textit{for } j = 0, \dots, n;$$

$$\mathcal{P}_n^w(t_0) := \lim_{z \widehat{\to} t_0} \left[\frac{1}{i!} \frac{d^i}{dz^i} \left(k_{j,t_0}(z) - w(z) \sum_{\ell=0}^{j} \overline{w_{j-\ell}(t_0)} k_{\ell,t_0}(z) \right) \right]_{i,j=0}^{n} \tag{5.2}$$

and satisfy conditions (3.25) *and* $\mathcal{P}_n^w(t_0) \geqslant 0$.

Moreover, in this case, $\mathbf{P}_n^w(t_0) = \mathcal{P}_n^w(t_0)$.

Note that in case $n = 0$, condition (3.25) reduces to $|w_0(t_0)| = 1$, and the matrix (5.2) reduces to

$$\mathcal{P}_0^w(t_0) = \lim_{z \widehat{\to} t_0} \frac{1 - w(z)\overline{w_0(t_0)}}{1 - z\bar{t}_0}.$$

Now it is readily seen that in this case, Theorem 5.2(3) is identical with Theorem 1.1(3).

Combining Theorems 1.2 and 5.2 we conclude that if the boundary Schwarz–Pick matrix $\mathbf{P}_n^w(t_0)$ exists (i.e., if condition (1.7) is satisfied), the matrices $\mathcal{P}_n^w(t_0)$ and $\mathbb{P}_n^w(t_0)$ also exist and $\mathbf{P}_n^w(t_0) = \mathcal{P}_n^w(t_0) = \mathbb{P}_n^w(t_0)$. Thus, in this case, both $\mathcal{P}_n^w(t_0)$ and $\mathbb{P}_n^w(t_0)$ can be used to represent the boundary Schwarz–Pick matrix; however the matrix $\mathbb{P}_n^w(t_0)$ is much more convenient for computational purposes. If condition (1.7) is not satisfied, then the matrices $\mathcal{P}_n^w(t_0)$ and $\mathbb{P}_n^w(t_0)$ may exist or not and may be equal or not; we do not proceed in detail, since in this case both of them do not make much sense.

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