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Integral Equations and Operator Theory

On Negative Inertia of Pick Matrices Associated with Generalized Schur Functions

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Abstract. It is known [6] that for every function f in the generalized Schur class $\mathcal{S}_\kappa$ and every nonempty open subset Ω of the unit disk $\mathbb{D}$, there *exist* points $z_1, \dots, z_n \in \Omega$ such that the $n \times n$ *Pick matrix* $\left[\frac{1-f(z_i)f(z_j)^*}{1-z_i\bar{z}_j}\right]_{j,i=1}^n$ has κ negative eigenvalues. In this paper we discuss existence of an integer n_0 such that *any* Pick matrix based on $z_1, \dots, z_n \in \Omega$ with $n \geq n_0$ has κ negative eigenvalues. Definitely, the answer depends on Ω. We prove that if $\Omega = \mathbb{D}$, then such a number n_0 does not exist unless f is a ratio of two finite Blaschke products; in the latter case the minimal value of n_0 can be found. We show also that if the closure of Ω is contained in $\mathbb{D}$ then such a number n_0 exists for every function f in $\mathcal{S}_\kappa$.

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1. Introduction

Let κ be a fixed nonnegative integer. We say that a meromorphic function f on the unit disk $\mathbb{D}$ belongs to the *generalized Schur class* $\mathcal{S}_\kappa$ if the kernel

$$\mathbf{k}_f(z, \zeta) := \frac{1 - f(z)f(\zeta)^*}{1 - z\bar{\zeta}} \tag{1.1}$$

has κ negative squares on $\mathbb{D} \cap \rho(f)$, where $\rho(f)$ stands for the domain of analyticity of f; in formulas:

$$\mathrm{sq}_-(\mathbf{k}_f) = \kappa. \tag{1.2}$$

In more detail, the definition can be given as follows: for every choice of an integer n and of n points $z_1, \dots, z_n \in \mathbb{D} \cap \rho(f)$, the *Pick matrix*

$$P_n(f; z_1, \dots, z_n) := \left[\frac{1 - f(z_i)f(z_j)^*}{1 - z_i\bar{z}_j}\right]_{j,i=1}^n \tag{1.3}$$

has at most κ negative eigenvalues (counted with multiplicities), and at least one such Pick matrix has exactly κ negative eigenvalues; in formulas:

$$\text{sq}_-P_n(f;\, z_1,\dots,z_n)=\kappa. \tag{1.4}$$

The class $\mathcal{S}_0$ is the classical Schur class consisting of functions f such that the kernel in (1.1) is positive (that is, has no negative squares) on $\mathbb{D}$. This turns out to be equivalent to the property for f to be analytic and less than one in modulus on $\mathbb{D}$.

The classes $\mathcal{S}_\kappa$ appeared implicitly in [12] in connection with interpolation problems (see discussion in [1, Chapter 19]), and were studied by Krein and Langer [6]. It was shown in particular, that a function f belongs to the class $\mathcal{S}_\kappa$ if and only if it admits a representation of the form

$$f(z)=\frac{S(z)}{B(z)}, \tag{1.5}$$

for some Schur function $S\in\mathcal{S}_0$ and a Blaschke product B of degree κ, such that the zero set of S and the zero set of B are disjoint. This representation in turn, leads to a characterization of $\mathcal{S}_\kappa$ as the class of all functions f such that

1. f is meromorphic in $\mathbb{D}$ and has κ poles inside $\mathbb{D}$ counted with multiplicities.
2. f is bounded on an annulus $\{z:\ r<|z|<1\}$ for some $r\in(0,\,1)$.
3. Boundary nontangential limits $f(t):=\lim_{z\to t}f(z)$ exist for almost all $t\in\mathbb{T}$ and satisfy $|f(t)|\le 1$.

By definition of the class $\mathcal{S}_\kappa$, for every function $f\in\mathcal{S}_\kappa$ there exists an $n\times n$ Pick matrix with exactly κ negative eigenvalues (squares). If points $z_1,\dots,z_\kappa$ are close enough to the poles of the function f (if ω is a pole of f of multiplicity ℓ, we take ℓ points z_i's around ω), then the corresponding Pick matrix $P_\kappa(f;\, z_1,\dots,z_\kappa)$ is negative definite (see e.g., [3]). In other words, *there are* $\kappa\times\kappa$ Pick matrices having κ negative eigenvalues. On the other hand, *not every* $\kappa\times\kappa$ Pick matrix of $f\in\mathcal{S}_\kappa$ has to have this property. In this paper we consider the following

Question 1: Does there exist an integer n_0 such that

$$\text{sq}_-P_{n_0}(f;\, z_1,\dots,z_{n_0})=\kappa \tag{1.6}$$

for *every choice* of n_0 points $z_1,\dots,z_{n_0}\in\mathbb{D}\cap\rho(f)$?

Note that if such an n_0 exists, then all the Pick matrices $P_n(f;\, z_1,\dots,z_n)$ of a size $n\ge n_0$ will have exactly κ negative eigenvalues; this follows by the Cauchy's interlacing theorem (see e.g., [2, p. 59]) and inequality $\text{sq}_-P_n(f;\, z_1,\dots,z_n)\le\kappa$.

The answer to Question 1 is given by the following

Theorem 1.1. *Let $f\in\mathcal{S}_\kappa$. Then there exists an integer n_0 such that* (1.4) *holds for* every $n\ge n_0$ *and for every choice of n points $z_1,\dots,z_n\in\mathbb{D}\cap\rho(f)$ if and only if f is a ratio of two finite Blaschke products with no common zeroes:*

$$f(z)=\frac{\widetilde{B}(z)}{B(z)},\quad \deg B=\kappa,\ \ \deg\widetilde{B}=\widetilde{\kappa}. \tag{1.7}$$

In this case,

$$n_{\min} = \deg B + \deg \widetilde{B} = \kappa + \widetilde{\kappa}. \tag{1.8}$$

In other words, if the numerator S in the Krein–Langer representation (1.5) of f is not a finite Blaschke product, then there exist Pick matrices $P_n(f)$ of an arbitrarily large size having less than κ negative eigenvalues. In fact we shall show more: in this case there exist *positive definite* Pick matrices of arbitrarily large sizes.

It turns out that Pick matrices of "large" sizes with negative inertia less than κ necessarily invoke points z_j close to the boundary $\mathbb{T}$ of the unit disk. This is a consequence of the following

Theorem 1.2. *Given a function $f \in \mathcal{S}_\kappa$ and given a compact infinite subset $\Omega \subset \mathbb{D}$, there exists an integer $n_0 = n_0(f, \Omega)$ such that* (1.4) *holds for every $n \geq n_0$ and for every choice of n points $z_1, \dots, z_n \in \Omega \cap \rho(f)$.*

Remark 1.3. *Note that the conclusion of Theorem* 1.2 *holds for any infinite set Ω such that its closure $\overline{\Omega}$ is a subset of $\mathbb{D}$.*

The paper is organized as follows: in Section 2 we show that for every generalized Schur function which is not a ratio of two finite Blaschke products, there exist positive definite Pick matrices of arbitrarily large sizes (we call this the "Necessity" part of Theorem 1.1). In Section 3 we shall get formulas for the inertia of Pick matrices corresponding to ratios of two finite Blaschke products (we call this the "Sufficiency" part of Theorem 1.1). Theorem 1.2 will be proved in Section 5. The key tool for the proof of Theorem 1.2 is the notion of a Carathéodory matrix which will be discussed along with some elementary properties in Section 4. Finally, Theorems 1.1, 1.2 and Remark 1.3 give a partial answer to a general question which is posed and illustrated by two examples in Section 6.

2. Theorem 1.1: Necessity

In this section we prove the necessity part of Theorem 1.1. We start with a preliminary

Lemma 2.1. *Let $\Lambda \subset \mathbb{D}$ be a discrete set, let $\mathcal{E}$ be a function analytic in $\mathbb{D} \setminus \Lambda$ and let $\mathcal{E} \in \mathcal{S}_\kappa$, i.e.,*

$$\mathrm{sq}_- \left(\frac{1 - \mathcal{E}(z)\mathcal{E}(\zeta)^*}{1 - z\bar{\zeta}} \right) = \kappa \quad (z, \zeta \in \mathbb{D} \setminus \Lambda). \tag{2.1}$$

Assume also that

$$|\mathcal{E}(z)| \geq 1 \quad (z \in \mathbb{D} \setminus \Lambda). \tag{2.2}$$

Then

$$\mathcal{E}(z) = \frac{1}{B(z)} \quad (z \in \mathbb{D} \setminus \Lambda), \tag{2.3}$$

where B is a finite Blaschke product of degree κ.

Proof. Due to (2.2), $0 < \left|\frac{1}{\mathcal{E}(z)}\right| \le 1$ for every $z \in \mathbb{D} \setminus \Lambda$. Therefore, the function $\frac{1}{\mathcal{E}(z)}$ can be uniquely extended to a Schur function S on the whole $\mathbb{D}$ with all zeroes falling in Λ. Then the function $\frac{1}{S}$ is meromorphic on $\mathbb{D}$, it may have poles only in Λ and it coincides with $\mathcal{E}$ on $\mathbb{D} \setminus \Lambda$. Since $\mathbb{D} \setminus \Lambda$ is an open set and the restriction of the meromorphic function $\frac{1}{S}$ on this set is in the class $\mathcal{S}_\kappa$, then (see [3]) the function $\frac{1}{S}$ is in the class $\mathcal{S}_\kappa$ on its domain of definition. Therefore, it admits a Krein–Langer representation

$$\frac{1}{S(z)} = \frac{S_1(z)}{B_1(z)}$$

for some Schur function $S_1 \in \mathcal{S}_0$ and a Blaschke product B_1 of degree κ, having no common zeroes. Thus, $B_1(z) = S(z)S_1(z)$ everywhere in $\mathbb{D}$ and therefore, S and S_1 are divisors of B_1, that is, finite Blaschke products. Since S_1 and B_1 do not have common zeroes, S_1 is a unimodular constant and therefore, S is a Blaschke product of degree κ, which leads us to (2.3). □

Let $f \in \mathcal{S}_\kappa$. We will be dealing with a special property of f that we formulate below and which, for the sake of shortness, we will call the **Property**.

Property. *There exists an integer $n \ge 0$ such that*

1. *For every $\ell > n$ and every choice of ℓ points $\zeta_1, \dots, \zeta_\ell \in \mathbb{D} \cap \rho(f)$ the Pick matrix $P_\ell(f;\, \zeta_1, \dots, \zeta_\ell)$ is not positive definite.*
2. *There exist points $z_1, \dots, z_n \in \mathbb{D} \cap \rho(f)$ such that the corresponding Pick matrix is positive definite:*

$$P := P_n(f;\, z_1, \dots, z_n) = \left[\frac{1 - f(z_i)f(z_j)^*}{1 - z_i \bar{z}_j}\right]_{j,i=1}^{n} > 0. \tag{2.4}$$

Note that the integer n specified in the **Property** is unique (if exists): it is just the *minimal* integer satisfying the first part of the **Property**.

Lemma 2.2. *Let $f \in \mathcal{S}_\kappa$ satisfy the* **Property**. *Then f is a ratio of two finite Blaschke products.*

Proof. First we consider the case when the integer n in the **Property** is zero. In this case the number

$$P_1(f;\, z) = \frac{1 - |f(z)|^2}{1 - |z|^2}$$

is not positive for every $z \in \mathbb{D}$, which means that $|f(z)| \ge 1$ on $\mathbb{D}$. Thus, f meets the conditions of Lemma 2.1 and therefore, $f = \frac{1}{B}$ for some Blaschke product of degree κ.

Now let $n > 0$. The technique we are going to apply in the remaining part of the proof was developed by V. P. Potapov and I. V. Kovalishina in 70's. Fix a positive integer k and points $\zeta_1, \dots, \zeta_k \in \mathbb{D} \cap \rho(f)$. We pick also points $z_1, \dots, z_n \in \mathbb{D} \cap \rho(f)$ such that (2.4) holds. The matrix

$$P_{n+k}(f;\, z_1, \dots, z_n, \zeta_1, \dots, \zeta_k)$$

has at most κ negative eigenvalues as a Pick matrix of $f \in \mathcal{S}_\kappa$. On the other hand, by the **Property**, this matrix is not positive definite. The Pick matrix can be written in the block form corresponding to two groups of points $\{z_1, \dots, z_n\}$ and $\{\zeta_1, \dots, \zeta_k\}$ as

$$P_{n+k}(f;\, z_1, \dots, z_n, \zeta_1, \dots, \zeta_k) = \begin{bmatrix} P & \Psi^* \\ \Psi & P_k(f;\, \zeta_1, \dots, \zeta_k) \end{bmatrix}, \tag{2.5}$$

where Ψ is the $k \times n$ matrix with the entries

$$\Psi_{ij} = \frac{1 - f(\zeta_i)f(z_j)^*}{1 - \zeta_i \bar{z}_j} \quad (i = 1, \dots, k;\;\; j = 1, \dots, n).$$

Since P is positive definite, it follows that the matrix

$$R = [R_{ij}]_{i,j=1}^{k} := P_k(f;\, \zeta_1, \dots, \zeta_k) - \Psi P^{-1} \Psi^*, \tag{2.6}$$

the Schur complement of the block P in the matrix (2.5), is not positive definite and has at most κ negative eigenvalues:

$$R \not> 0 \quad \text{and} \quad \mathrm{sq}_- R \le \kappa. \tag{2.7}$$

The next step is to represent the matrix R in certain factorized form. To this end, let us consider the matrices

$$T = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} f(z_1) \\ \vdots \\ f(z_n) \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.8}$$

and the 2×2 matrix valued rational function

$$\begin{aligned} \Theta(z) &= \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix} \\ &= I_2 + (z-1) \begin{bmatrix} E^* \\ C^* \end{bmatrix} (I - zT^*)^{-1} P^{-1} (I - T)^{-1} \begin{bmatrix} E & -C \end{bmatrix}, \end{aligned} \tag{2.9}$$

where P is given by (2.4). It is seen from definitions (2.4) and (2.8) that P satisfies the Stein identity

$$P - TPT^* = EE^* - CC^*. \tag{2.10}$$

Straightforward calculations using just the latter identity lead us to relations

$$J - \Theta(z) J \Theta(\zeta)^* = (1 - z\bar{\zeta}) \begin{bmatrix} E^* \\ C^* \end{bmatrix} (I - zT^*)^{-1} P^{-1} (I - \bar{\zeta}T)^{-1} \begin{bmatrix} E & C \end{bmatrix} \tag{2.11}$$

and

$$\det \Theta(z) = \prod_{j=1}^{k} \frac{(z - z_j)(1 - \overline{z_j})}{(1 - z\overline{z_j})(1 - z_j)}. \tag{2.12}$$

holding for every choice of points z and ζ at which Θ is analytic (the proof can be found in many sources; see e.g., [4, p. 771]). It turns out that the entries R_{ij} in the matrix (2.6) can be written as

$$R_{ij} = \begin{bmatrix} 1 & -f(\zeta_i) \end{bmatrix} \frac{\Theta(\zeta_i)J\Theta(\zeta_j)^*}{1-\zeta_i\bar{\zeta}_j} \begin{bmatrix} 1 \\ -f(\zeta_j)^* \end{bmatrix}. \tag{2.13}$$

The proof of (2.13) can be found in [4, pp. 773-774]. Note that the function

$$d(z) := f(z)\Theta_{21}(z) - \Theta_{11}(z)$$

does not vanish identically. Indeed since Θ_{11} and Θ_{21} are rational functions with $\Theta_{11}(1) = 1$ and $\Theta_{21}(1) = 0$, the assumption $d(z) \equiv 0$ would imply that the function $f(z) = \dfrac{\Theta_{11}(z)}{\Theta_{21}(z)}$ of the class $\mathcal{S}_\kappa$ is rational and has a pole at $z = 1$. The latter cannot happen by the characterization (1)–(3) of the class $\mathcal{S}_\kappa$ mentioned in the introduction.

Thus, the function

$$\mathcal{E}(z) = \frac{\Theta_{12}(z) - f(z)\Theta_{22}(z)}{f(z)\Theta_{21}(z) - \Theta_{11}(z)} \tag{2.14}$$

is defined on $\mathbb{D}$ except for a discrete set Λ and is analytic on $\mathbb{D} \setminus \Lambda$ (to be more specific, we can choose Λ to be the set of all poles of the function f and all zeroes of the function d). We observe now that $\mathcal{E}$ meets conditions (2.1) and (2.2) of Lemma 2.1. Indeed, by (2.14),

$$\begin{bmatrix} 1 & -f(\zeta) \end{bmatrix} \Theta(\zeta) = -d(\zeta) \begin{bmatrix} 1 & -\mathcal{E}(\zeta) \end{bmatrix}$$

and thus, under the additional assumption that $\zeta_1, \ldots, \zeta_k \in \mathbb{D} \setminus \Lambda$, formula (2.13) can be written in terms of $\mathcal{E}$ as

$$\begin{aligned} R_{ij} &= d(\zeta_i) \cdot \begin{bmatrix} 1 & -\mathcal{E}(\zeta_i) \end{bmatrix} \frac{J}{1-\zeta_i\bar{\zeta}_j} \begin{bmatrix} 1 \\ -\mathcal{E}(\zeta_j)^* \end{bmatrix} \cdot d(\zeta_j)^* \\ &= d(\zeta_i) \cdot \frac{1-\mathcal{E}(\zeta_i)\mathcal{E}(\zeta_j)^*}{1-\zeta_i\bar{\zeta}_j} \cdot d(\zeta_j)^*. \end{aligned}$$

Thus,

$$R = \left[d(\zeta_i) \cdot \frac{1-\mathcal{E}(\zeta_i)\mathcal{E}(\zeta_j)^*}{1-\zeta_i\bar{\zeta}_j} \cdot d(\zeta_j)^* \right]_{i,j=1}^k. \tag{2.15}$$

Since $d(\zeta) \neq 0$ on $\mathbb{D} \setminus \Lambda$, it follows from (2.15) and the second relation in (2.7), that

$$\mathrm{sq}_- \left[\frac{1-\mathcal{E}(\zeta_i)\mathcal{E}(\zeta_j)^*}{1-\zeta_i\zeta_j^*} \right]_{i,j=1}^k \leq \kappa.$$

Since $\zeta_1, \ldots, \zeta_k$ are arbitrary points in $\mathbb{D} \setminus \Lambda$, the latter inequality means that the kernel

$$\mathbf{k}_{\mathcal{E}}(z,\zeta) = \frac{1-\mathcal{E}(z)\mathcal{E}(\zeta)^*}{1-z\bar{\zeta}}$$

has at most κ negative squares on $\mathbb{D} \setminus \Lambda$. Actually, this kernel has exactly κ negative squares. Indeed, let us pick points $\zeta_1, \dots, \zeta_k \in \mathbb{D} \setminus \Lambda$ so that the block $P_k(f; \zeta_1, \dots, \zeta_k)$ in (2.5) has κ negative eigenvalues (it can be done, since $f \in \mathcal{S}_\kappa$ and Λ is discrete). Then, since P is positive definite, it follows from (2.6) that $R \leq P_k(f; \zeta_1, \dots, \zeta_k)$ and thus,

$$\mathrm{sq}_- R \geq \mathrm{sq}_- P_k(f; \zeta_1, \dots, \zeta_k) = \kappa.$$

Therefore, $\mathrm{sq}_- R = \kappa$ which leads us to (2.1). Thus, $\mathcal{E}$ is in $\mathcal{S}_\kappa$.

On the other hand we saw that the **Property** implies that the matrix R associated to the function f is not positive definite for every choice of k and of k points $\zeta_1, \dots, \zeta_k \in \mathbb{D} \setminus \Lambda$. For $k = 1$, the latter reads: for every point $z \in \mathbb{D} \setminus \Lambda$,

$$|d(z)|^2 \cdot \frac{1 - |\mathcal{E}(z)|^2}{1 - |z|^2} \leq 0,$$

which is equivalent to (2.2), since $d(z) \neq 0$ on $\mathbb{D} \setminus \Lambda$ and $|z| < 1$.

Thus, $\mathcal{E}$ meets the conditions of Lemma 2.1 and therefore, it is of the form (2.3) for a Blaschke product B of degree κ. The zeros of B are in Λ, since all singularities of $\mathcal{E}$ are in Λ.

Substituting (2.3) into (2.14), we arrive at

$$f(z) = \frac{\Theta_{11}(z) + \Theta_{12}(z)B(z)}{\Theta_{21}(z) + \Theta_{22}(z)B(z)} \tag{2.16}$$

and the latter representation holds at every point $z \in \mathbb{D} \setminus \widetilde{\Lambda}$, where $\widetilde{\Lambda}$ is a discrete subset of $\mathbb{D}$. Thus, f is rational. Note that the denominator in (2.16) vanishes at no point of $\mathbb{T}$. Indeed, assuming that

$$\Theta_{21}(\zeta) + \Theta_{22}(\zeta)B(\zeta) = 0$$

for some $\zeta \in \mathbb{T}$, we conclude that

$$\Theta_{11}(\zeta) + \Theta_{12}(\zeta)B(\zeta) = 0,$$

since f cannot have poles on $\mathbb{T}$. The two last equalities imply

$$\Theta(\zeta) \begin{bmatrix} 1 \\ B(\zeta) \end{bmatrix} = 0$$

and therefore, $\det \Theta(\zeta) = 0$, which contradicts (2.12). Furthermore, f is unimodular on $\mathbb{T}$. Indeed, it follows from (2.11) that Θ is J-unitary on $\mathbb{T}$, i.e., that

$$\Theta(z)J\Theta(z)^* = J \quad (z \in \mathbb{T}).$$

A straightforward calculation based on the latter relation shows that for f of the form (2.16) and for every $z \in \mathbb{T}$,

$$1 - |f(z)|^2 = \frac{1 - |B(z)|^2}{|\Theta_{21}(z) + \Theta_{22}(z)B(z)|^2}.$$

Since $|B(z)| = 1$ for $|z| = 1$, the last relation implies that f is unimodular on $\mathbb{T}$. Thus, f is a rational function unimodular on $\mathbb{T}$ and therefore, it is a ratio of two finite Blaschke products. $\square$

As a consequence we obtain that for every function $f \in \mathcal{S}_\kappa$ which is not a ratio of two finite Blaschke products, **Property** does not hold, which means that for every such function there exist positive definite Pick matrices of arbitrarily large sizes.

3. Theorem 1.1: Sufficiency

Let H^2 be the Hardy space of square integrable functions on the unit circle $\mathbb{T}$ that admit analytic continuations inside the unit disk. It is a reproducing kernel Hilbert space with reproducing kernel

$$e_\lambda(t) = \frac{1}{1 - t\bar{\lambda}},$$

(the Cauchy kernel) in the sense that

$$\langle h, e_\lambda \rangle_{H^2} = h(\lambda) \quad \text{for every } \lambda \in \mathbb{D} \text{ and } h \in H^2. \tag{3.1}$$

Also we shall make use of the functions

$$e_{\lambda,k}(t) = \frac{t^k}{(1 - t\bar{\lambda})^{k+1}} \in H^2, \qquad \lambda \in \mathbb{D}, \quad k = 0, 1, \dots \tag{3.2}$$

Definitely, $e_{\lambda,0} \equiv e_\lambda$.

Let B be a finite Blaschke product of the form

$$B(z) = \prod_{j=1}^{\ell} \left(\frac{z - \lambda_j}{1 - z\bar{\lambda}_j} \right)^{r_j}, \tag{3.3}$$

where $r_1, \dots, r_\ell$ are positive integers such that $r_1 + \dots + r_\ell = \kappa = \deg B$ and $\lambda_1, \dots, \lambda_\ell$ are distinct points in $\mathbb{D}$. The next lemma contains some needed known facts that can be found, e.g., in [7].

Lemma 3.1. *Let B be the Blaschke product of the form* (3.3), *let $\mathcal{K}_B$ be the subspace of H^2 defined by*

$$\mathcal{K}_B := H^2 \ominus BH^2$$

and let $\mathbf{P}_{\mathcal{K}_B}$ denote the orthogonal projection of H^2 onto $\mathcal{K}_B$. Then

1. *The functions*

$$e_{\lambda_j,0}, \dots, e_{\lambda_j, r_j - 1}, \qquad j = 1, \dots, \ell, \tag{3.4}$$

defined via (3.2), *form a basis for $\mathcal{K}_B$; therefore,* $\dim K_B = \kappa$.

2. *A function g belongs to $\mathcal{K}_B$ if and only if it admits a representation*

$$g(t) = \frac{q(t)}{\prod_{j=1}^{\ell}(1 - t\bar{\lambda}_j)^{r_j}} \tag{3.5}$$

for some polynomial q of degree $\deg q \le \kappa - 1$.

3. *It holds that*

$$\left(\mathbf{P}_{\mathcal{K}_B} e_\lambda\right)(t) = \frac{1 - B(t)B(\lambda)^*}{1 - t\bar{\lambda}} \quad \textit{for every } \lambda \in \mathbb{D}. \tag{3.6}$$

To verify the first statement one can check that the functions (3.4) belong to $\mathcal{K}_B$ and that any H^2 function orthogonal to them has a zero of multiplicity at least r_j at λ_j. It follows directly from (3.2) that a function g is a linear combination of the functions (3.4) if and only if it is of the form (3.5), which proves the second statement. The last statement follows from the observation that $g_1 + g_2 = e_\lambda$ for every fixed $\lambda \in \mathbb{D}$, where the functions

$$g_1(z) = \frac{B(t)B(\lambda)^*}{1 - t\bar{\lambda}} \quad \text{and} \quad g_2(z) = \frac{1 - B(t)B(\lambda)^*}{1 - t\bar{\lambda}}$$

belong to BH^2 and $\mathcal{K}_B$, respectively.

Remark 3.2. *Let h and g be two functions on $\mathbb{D}$, let $z_1, \dots, z_n \in \mathbb{D}$ and let*

$$g(z_i) \neq 0 \quad (i = 1, \dots, n).$$

Then

$$P_n(h;\, z_1, \dots, z_n) - P_n(g;\, z_1, \dots, z_n) = G P_n\left(\frac{h}{g};\, z_1, \dots, z_n\right) G^* \tag{3.7}$$

where G is the diagonal matrix given by

$$G = \begin{bmatrix} g(z_1) & & 0 \\ & \ddots & \\ 0 & & g(z_n) \end{bmatrix}.$$

Proof. By definition (1.3) of the Pick matrix,

$$\begin{aligned}
&P_n(h;\, z_1, \dots, z_n) - P_n(g;\, z_1, \dots, z_n) \\
&= \left[\frac{1 - h(z_i)h(z_j)^*}{1 - z_i\bar{z}_j}\right]_{i,j=1}^n - \left[\frac{1 - g(z_i)g(z_j)^*}{1 - z_i\bar{z}_j}\right]_{i,j=1}^n \\
&= \left[\frac{g(z_i)g(z_j)^* - h(z_i)h(z_j)^*}{1 - z_i\bar{z}_j}\right]_{i,j=1}^n = G \left[\frac{1 - \dfrac{h(z_i)h(z_j)^*}{g(z_i)g(z_j)^*}}{1 - z_i\bar{z}_j}\right]_{i,j=1}^n G^* \\
&= G P_n\left(\frac{h}{g};\, z_1, \dots, z_n\right) G^*.
\end{aligned}$$

□

Since G is invertible, it follows from (3.7) that

$$\operatorname{sq}_- P_n\left(\frac{h}{g};\, z_1,\dots,z_n\right) = \operatorname{sq}_-\left(P_n(h;\, z_1,\dots,z_n) - P_n(g;\, z_1,\dots,z_n)\right)$$

which explains the importance of differences of two Pick matrices (associated with Schur functions) in the context of the generalized Schur class $\mathcal{S}_\kappa$. The next lemma treats the case when h and g are two finite Blaschke products with no common zeros.

Lemma 3.3. *Let*

$$B(z) = \prod_{j=1}^{\ell}\left(\frac{z-\lambda_j}{1-z\bar{\lambda}_j}\right)^{r_j} \quad \textit{and} \quad \widetilde{B}(z) = \prod_{j=1}^{\widetilde{\ell}}\left(\frac{z-w_j}{1-z\bar{w}_j}\right)^{\widetilde{r}_j}, \tag{3.8}$$

be two finite Blaschke products with no common zeros and with $\deg B = \kappa$ *and* $\deg \widetilde{B} = \widetilde{\kappa}$*, and let* $z_1,\dots,z_n$ *be arbitrary* n *points in* $\mathbb{D}$*.*

1. *If* $n = \kappa + \widetilde{\kappa}$*, then the difference of the Pick matrices*
$$\mathbb{P}_n := P_n(\widetilde{B};\, z_1,\dots,z_n) - P_n(B;\, z_1,\dots,z_n) \tag{3.9}$$
is invertible.
2. *If* $n \ge \kappa + \widetilde{\kappa}$*, then*
$$\operatorname{sq}_- \mathbb{P}_n = \kappa \quad \textit{and} \quad \operatorname{sq}_+ \mathbb{P}_n = \widetilde{\kappa}. \tag{3.10}$$
3. *If* $n < \kappa + \widetilde{\kappa}$*, then the points* $z_1,\dots,z_n$ *can be chosen so that* $\operatorname{sq}_- \mathbb{P}_n < \kappa$*.*

Proof. Let

$$\mathcal{K} = \operatorname{span}\{e_{z_1},\dots,e_{z_n}\} = \operatorname{span}\left\{\frac{1}{1-t\bar{z}_1},\ \dots,\ \frac{1}{1-t\bar{z}_n}\right\}. \tag{3.11}$$

Note that by (3.6) and the reproducing property (3.1),

$$\begin{aligned}\left\langle \mathbf{P}_{\mathcal{K}_B} e_{z_j},\ \mathbf{P}_{\mathcal{K}_B} e_{z_i}\right\rangle_{H^2} &= \left\langle \mathbf{P}_{\mathcal{K}_B} e_{z_j},\ e_{z_i}\right\rangle_{H^2} \\ &= \left(\mathbf{P}_{\mathcal{K}_B} e_{z_j}\right)(z_i) = \frac{1-B(z_i)B(z_j)^*}{1-z_i\bar{z}_j}\end{aligned}$$

and thus, by definition (1.3) of the Pick matrix,

$$P_n(B;\, z_1,\dots,z_n) = \left[\left\langle \mathbf{P}_{\mathcal{K}_B} e_{z_j},\ \mathbf{P}_{\mathcal{K}_B} e_{z_i}\right\rangle_{H^2}\right]_{i,j=1}^n. \tag{3.12}$$

Similarly,

$$P_n(\widetilde{B};\, z_1,\dots,z_n) = \left[\left\langle \mathbf{P}_{\mathcal{K}_{\widetilde{B}}} e_{z_j},\ \mathbf{P}_{\mathcal{K}_{\widetilde{B}}} e_{z_i}\right\rangle_{H^2}\right]_{i,j=1}^n, \tag{3.13}$$

where $\mathbf{P}_{\mathcal{K}_{\widetilde{B}}}$ stands for the orthogonal projection of H^2 onto the subspace

$$\mathcal{K}_{\widetilde{B}} = H^2 \ominus \widetilde{B}H^2.$$

Therefore, the matrix $\mathbb{P}_n$ defined as in (3.9) is nonsingular if and only if the quadratic form

$$D(x,y) := \left\langle \mathbf{P}_{\mathcal{K}_{\widetilde{B}}} x,\ \mathbf{P}_{\mathcal{K}_{\widetilde{B}}} y\right\rangle_{H^2} - \left\langle \mathbf{P}_{\mathcal{K}_B} x,\ \mathbf{P}_{\mathcal{K}_B} y\right\rangle_{H^2} \tag{3.14}$$

is not degenerate on $\mathcal{K}$. Note also that since the spaces $\mathcal{K}_B$ and $\mathcal{K}_{\widetilde{B}}$ are of dimensions κ and $\widetilde{\kappa}$ respectively (by Statement 1 in Lemma 3.3), it follows from (3.12) and (3.13) that for every n,

$$\operatorname{rank} P_n(B;\, z_1,\ldots,z_n) \le \kappa \quad \text{and} \quad \operatorname{rank} P_n(\widetilde{B};\, z_1,\ldots,z_n) \le \widetilde{\kappa}$$

and, since the latter Pick matrices are positive semidefinite, we have for their difference

$$\operatorname{sq}_- \mathbb{P}_n \le \kappa \quad \text{and} \quad \operatorname{sq}_+ \mathbb{P}_n \le \widetilde{\kappa} \quad \text{for every } n. \tag{3.15}$$

To prove the first statement in the lemma, assume that $n = \kappa + \widetilde{\kappa}$ and that the form D is degenerate, i.e., that there exists $x \in \mathcal{K}$ such that $D(x,y) = 0$ for every element $y \in \mathcal{K}$. Then we have, by (3.14),

$$\begin{aligned} 0 &= \left\langle \mathbf{P}_{\mathcal{K}_{\widetilde{B}}} x,\, \mathbf{P}_{\mathcal{K}_{\widetilde{B}}} y \right\rangle_{H^2} - \left\langle \mathbf{P}_{\mathcal{K}_B} x,\, \mathbf{P}_{\mathcal{K}_B} y \right\rangle_{H^2} \\ &= \left\langle \mathbf{P}_{\mathcal{K}_{\widetilde{B}}} x,\, y \right\rangle_{H^2} - \left\langle \mathbf{P}_{\mathcal{K}_B} x,\, y \right\rangle_{H^2} \\ &= \left\langle \left(\mathbf{P}_{\mathcal{K}_{\widetilde{B}}} - \mathbf{P}_{\mathcal{K}_B}\right) x,\, y \right\rangle_{H^2} \quad \text{for every } \ y \in \mathcal{K}. \end{aligned}$$

Upon letting $y = e_{z_j}$ in the latter equality, we conclude, by the reproducing property (3.1), that

$$\left(\mathbf{P}_{\mathcal{K}_{\widetilde{B}}} x - \mathbf{P}_{\mathcal{K}_B} x\right)(z_j) = 0 \quad \text{for} \ \ j = 1,\ldots,n. \tag{3.16}$$

The functions $\mathbf{P}_{\mathcal{K}_B} x$ and $\mathbf{P}_{\mathcal{K}_{\widetilde{B}}} x$ belong to the spaces $\mathcal{K}_B$ and $\mathcal{K}_{\widetilde{B}}$, respectively, and therefore, by the second statement in Lemma 3.3, they are of the form

$$(\mathbf{P}_{\mathcal{K}_B} x)(t) = \frac{q(t)}{\prod_{j=1}^{\ell}(1 - t\bar{\lambda}_j)^{r_j}} \quad \text{and} \quad (\mathbf{P}_{\mathcal{K}_{\widetilde{B}}} x)(t) = \frac{\widetilde{q}(t)}{\prod_{j=1}^{\widetilde{\ell}}(1 - t\bar{w}_j)^{\widetilde{r}_j}} \tag{3.17}$$

for some polynomials q and $\widetilde{q}$ with

$$\deg q \le \kappa - 1 \quad \text{and} \quad \deg \widetilde{q} \le \widetilde{\kappa} - 1. \tag{3.18}$$

Recall that the denominators in (3.17) are polynomials of degree κ and $\widetilde{\kappa}$, respectively, and thus, it follows readily from (3.17) that

$$(\mathbf{P}_{\mathcal{K}_{\widetilde{B}}} x)(t) - (\mathbf{P}_{\mathcal{K}_B} x)(t) = \frac{r(t)}{p(t)}, \tag{3.19}$$

where

$$p(t) = \prod_{j=1}^{\ell}(1 - t\bar{\lambda}_j)^{r_j} \cdot \prod_{j=1}^{\widetilde{\ell}}(1 - t\bar{w}_j)^{\widetilde{r}_j}$$

and

$$r(t) = q(t)\prod_{j=1}^{\widetilde{\ell}}(1 - t\bar{w}_j)^{\widetilde{r}_j} - \widetilde{q}(t)\prod_{j=1}^{\ell}(1 - t\bar{\lambda}_j)^{r_j}$$

are polynomials with $\deg p = \kappa + \widetilde{\kappa} = n$ and (according to (3.18))

$$\deg r \le \kappa + \widetilde{\kappa} - 1 = n - 1. \tag{3.20}$$

By (3.16), the rational function $\frac{r}{p}$ has n distinct zeros which together with (3.20) implies that $r \equiv 0$. Thus, we have from (3.19)

$$\mathbf{P}_{\mathcal{K}_B} x \equiv \mathbf{P}_{\mathcal{K}_{\widetilde{B}}} x. \tag{3.21}$$

By the first statement in Lemma 3.3, the spaces $\mathcal{K}_B$ and $\mathcal{K}_{\widetilde{B}}$ are spanned by functions (3.4) and by functions

$$e_{w_j,0}, \ldots, e_{w_j,\widetilde{r}_j-1}, \qquad j = 1, \ldots, \widetilde{\ell},$$

respectively. Since all the functions in (3.4) are linearly independent and since $\lambda_i \neq w_j$ for $i = 1, \ldots, \ell$ and $j = 1, \ldots, \widetilde{\ell}$, it follows that $\mathcal{K}_B \cap \mathcal{K}_{\widetilde{B}} = \{0\}$. Thus, relation (3.21) implies

$$\mathbf{P}_{\mathcal{K}_B} x = 0 \quad \text{and} \quad \mathbf{P}_{\mathcal{K}_{\widetilde{B}}} x = 0.$$

Therefore, x is orthogonal to both of $\mathcal{K}_B$ and $\mathcal{K}_{\widetilde{B}}$ and thus,

$$x \in BH^2 \cap \widetilde{B}H^2.$$

Since B and $\widetilde{B}$ have no common zeros, it follows that

$$x \in (B \cdot \widetilde{B})H^2.$$

In particular, x has at least $\kappa + \widetilde{\kappa} = n$ zeros (counted with multiplicities). On the other hand, x belongs to $\mathcal{K}$ and therefore, by definition (3.11),

$$x(t) = \sum_{j=1}^{n} \frac{\alpha_j}{1 - t\bar{z}_j} = \frac{q(t)}{\prod_{j=1}^{n}(1 - t\bar{z}_j)},$$

where q is a polynomial with $\deg q \leq n - 1$. Therefore, if $x \neq 0$, it cannot have more than $n - 1$ zeros. Therefore, $x = 0$ and the form D is nondegenerate on $\mathcal{K}$. Therefore, the matrix $\mathbb{P}_n$ is invertible. Moreover, we have

$$\kappa + \widetilde{\kappa} = n = \operatorname{rank} \mathbb{P}_n = \operatorname{sq}_- \mathbb{P}_n + \operatorname{sq}_+ \mathbb{P}_n,$$

which together with bounds (3.15) implies (3.10) for the case when $n = \kappa + \widetilde{\kappa}$.

Now let $n > \kappa + \widetilde{\kappa} =: n_1$ and let $\mathbb{P}_{n_1}$ be the top $n_1 \times n_1$ principal submatrix of $\mathbb{P}_n$:

$$\mathbb{P}_{n_1} = P_{n_1}(B;\, z_1, \ldots, z_{n_1}) - P_{n_1}(\widetilde{B};\, z_1, \ldots, z_{n_1}).$$

By the preceding analysis, $\mathbb{P}_{n_1}$ is invertible and

$$\operatorname{sq}_- \mathbb{P}_{n_1} = \kappa \quad \text{and} \quad \operatorname{sq}_+ \mathbb{P}_{n_1} = \widetilde{\kappa}.$$

Therefore, since $\mathbb{P}_{n_1}$ is a principal submatrix of $\mathbb{P}_n$,

$$\operatorname{sq}_- \mathbb{P}_n \geq \operatorname{sq}_- \mathbb{P}_{n_1} = \kappa, \quad \operatorname{sq}_+ \mathbb{P}_n \geq \operatorname{sq}_+ \mathbb{P}_{n_1} = \kappa$$

which together with bounds (3.15) complete the proof of (3.10).

Finally, if $\kappa + \widetilde{\kappa} > n > \widetilde{\kappa}$, we first fix the points $z_{\widetilde{\kappa}+1}, \ldots, z_n$ and choose then the points $z_1, \ldots, z_{\widetilde{\kappa}}$ sufficiently close to the zeros $w_1, \ldots, w_{\widetilde{\ell}}$ of $\widetilde{B}$ (taking $\widetilde{r}_j$ points around w_j) to make the top $\widetilde{\kappa}\times\widetilde{\kappa}$ principal submatrix of $\mathbb{P}_n$ positive definite. Then $\mathrm{sq}_+\,\mathbb{P}_n = \widetilde{\kappa}$ and therefore,

$$\mathrm{sq}_-\,\mathbb{P}_n = \operatorname{rank}\mathbb{P}_n - \mathrm{sq}_+\,\mathbb{P}_n \le n - \widetilde{\kappa} < \kappa.$$

If $n \le \widetilde{\kappa}$, one can use a similar choice to get $\mathbb{P}_n > 0$. □

Theorem 3.4. *Let f be the ratio of two finite Blaschke products*

$$f(z) = \frac{\widetilde{B}(z)}{B(z)}, \quad \deg B = \kappa, \;\; \deg \widetilde{B} = \widetilde{\kappa}$$

with no common zeros, let $z_1, \dots, z_n$ be points in $\mathbb{D}\cap\rho(f)$ and let $P_n(f;\, z_1, \dots, z_n)$ be the corresponding Pick matrix. Then

1. *If $n = \kappa + \widetilde{\kappa}$, then $P_n(f;\, z_1, \dots, z_n)$ is invertible.*
2. *If $n \ge \kappa + \widetilde{\kappa}$, then*

$$\mathrm{sq}_-\, P_n(f;\, z_1, \dots, z_n) = \kappa \quad \textit{and} \quad \mathrm{sq}_+\, P_n(f;\, z_1, \dots, z_n) = \widetilde{\kappa}.$$

3. *If $n < \kappa + \widetilde{\kappa}$, then the points $z_1, \dots, z_n$ can be chosen so that*

$$\mathrm{sq}_-\, P_n(f;\, z_1, \dots, z_n) < \kappa.$$

Proof. Since $z_1, \dots, z_n$ are the points of analyticity of f, then $B(z_i) \ne 0$ for $i = 1, \dots, n$ and we can apply Remark 3.2 with $h = \widetilde{B}$ and $g = B$ to obtain

$$\begin{aligned} \mathbb{P}_n \;&:=\; P_n(\widetilde{B};\, z_1, \dots, z_n) - P_n(B;\, z_1, \dots, z_n) \\ &=\; G P_n\left(\frac{\widetilde{B}}{B};\, z_1, \dots, z_n\right) G^* \\ &=\; G P_n\left(f;\, z_1, \dots, z_n\right) G^*, \end{aligned}$$

where G is the diagonal matrix defined by

$$G = \begin{bmatrix} B(z_1) & & 0 \\ & \ddots & \\ 0 & & B(z_n) \end{bmatrix}.$$

Thus, $P_n\left(f;\, z_1, \dots, z_n\right)$ is congruent to $\mathbb{P}_n$ and all the statements of the theorem follow from the corresponding statements in Lemma 3.3. □

Now the "sufficiency" part in Theorem 1.1 is immediate: indeed, the second statement in Theorem 3.4 implies that $n_{\min} \le \kappa + \widetilde{\kappa}$ (see **Question 1** for the definition of the integer $n_{\min}$), whereas the third statement implies $n_{\min} \ge \kappa + \widetilde{\kappa}$ and the two last inequalities lead us to (1.8).

4. Carathéodory matrices

Let f be a function analytic at a point $\omega \in \mathbb{D}$. Then we can define the *Carathéodory matrix*

$$C_n(f;\,\omega) = \left[\frac{1}{i!\,j!}\frac{\partial^{i+j}}{\partial\omega^i\partial\bar{\omega}^j}\left(\frac{1-f(\omega)f(\omega)^*}{1-\omega\bar{\omega}}\right)\right]_{i,j=0}^{n-1}, \tag{4.1}$$

for any $n \geq 0$. It is known that f belongs to the class $\mathcal{S}_\kappa$ (i.e., the kernel $\mathbf{k}_f(z,\zeta)$ given by (1.1) has κ negative squares on $\mathbb{D}\cap\rho(f)$) if and only if the following property holds: for every $\omega \in \mathbb{D}\cap\rho(f)$, there exists an integer n_ω such that

$$\text{sq}_- C_n(f;\,\omega) = \kappa \quad \text{for all } \; n \geq n_\omega. \tag{4.2}$$

Actually, it was shown by M. G. Krein and H. Langer in [6] that any meromorphic function f satisfying (4.2) even at a single point $\omega \in \mathbb{D}\cap\rho(f)$ belongs to $\mathcal{S}_\kappa$. Conversely, if $f \in \mathcal{S}_\kappa$, then continuity arguments (see Lemma 4.4) yield that

$$\text{sq}_- C_n(f;\,\omega) \leq \kappa$$

for every integer $n \geq 0$ and every point $\omega \in \mathbb{D}\cap\rho(f)$. If for some ω (4.2) holds with $\kappa' < \kappa$ then, by the mentioned above result of M. G. Krein and H. Langer $f \in \mathcal{S}_{\kappa'}$, which is a contradiction.

Since property (4.2) of Carathéodory matrices characterizes the class $\mathcal{S}_\kappa$, a question similar to **Question 1** can be raised in the context of Carathéodory matrices (see **Question 2** at the end of Section 5). Perhaps, this question is of certain independent interest. However, we do not know the answer and we will not discuss the question here; we rather use Carathéodory matrices as a tool to prove Theorem 1.2. In this section we present some auxiliary results concerning Carathéodory matrices needed for that proof. For the rest of the section, f will be assumed to be a function (not necessarily of the class $\mathcal{S}_\kappa$) analytic at a point $\omega \in \mathbb{D}$.

Remark 4.1. It follows from the the definition (4.1) that $C_n(f;\,\omega)$ depends only on the n first Taylor coefficients of f at ω. Therefore, the Taylor polynomial

$$p_{n-1}(z) = \sum_{k=0}^{n-1}(z-\omega)^k\frac{f^{(k)}(\omega)}{k!}$$

has the same Carathéodory matrix of order ℓ at ω as f does for every $\ell \leq n$. Therefore, studying the Carathéodory matrices of function f at point ω we may replace the function with the Taylor polynomial of the proper degree. Sometimes we will use the same notation f for this polynomial.

Remark 4.2. We shall also need to write all the Carathéodory matrices as matrices of quadratic forms with respect to certain bases. Let

$$b_n(z) := \left(\frac{z-\omega}{1-z\bar{\omega}}\right)^n$$

and let $\mathcal{K}_{b_n} := H^2 \ominus b_n H^2$. By Lemma 3.1, the functions

$$e_{\omega,0}, \dots, e_{\omega,n-1} \tag{4.3}$$

defined via (3.2) form a basis for $\mathcal{K}_{b_n}$. let T_{b_n} be the compressed shift operator restricted to $\mathcal{K}_{b_n}$:

$$T_{b_n} h = \mathbf{P}_{\mathcal{K}_{b_n}}(th) \quad (h \in \mathcal{K}_{b_n},\ t \in \mathbb{T}),$$

where $\mathbf{P}_{\mathcal{K}_{b_n}}$ is the orthogonal projection of H^2 onto $\mathcal{K}_{b_n}$. The adjoint of T_{b_n} is the restricted backward shift operator $T^*_{b_n} = P_+ \bar{t}|_{\mathcal{K}_{b_n}}$, where P_+ stands for the orthogonal projection of the Lebesgue space L^2 onto H^2. By H^∞ functional calculus (see, e.g., [7]), for every $\varphi \in H^\infty$

$$\varphi(T_{b_n}) = \mathbf{P}_{\mathcal{K}_{b_n}} \varphi|_{\mathcal{K}_{b_n}}$$

and

$$\varphi(T_{b_n})^* = P_+ \overline{\varphi}|_{\mathcal{K}_{b_n}} \ . \tag{4.4}$$

The matrix of the sesquilinear form

$$\left\langle \left(\mathbb{I}_{\mathcal{K}_{b_n}} - \varphi(T_{b_n})\varphi(T_{b_n})^*\right) x, y \right\rangle_{L^2}, \quad (x, y \in \mathcal{K}_{b_n}), \tag{4.5}$$

with respect to the basis (4.3) for $\mathcal{K}_{b_n}$ is equal to the Carathéodory matrix $C_n(\varphi;\, \omega)$. The proof of this fact can be found, for instance, in [7] (Lecture VIII) and [8, Chapter 2]. We will give here a brief explanation. Since $e_{\omega,k} = \frac{1}{k!}\frac{\partial^k}{\partial \overline{\omega}^k} e_{\omega,0}$, we get

$$\varphi(T_{b_n})^* e_{\omega,k} = P_+ \overline{\varphi} e_{\omega,k} = \frac{1}{k!}\frac{\partial^k}{\partial \overline{\omega}^k} P_+ \overline{\varphi} e_{\omega,0} = \frac{1}{k!}\frac{\partial^k}{\partial \overline{\omega}^k}\left(\overline{\varphi(\omega)} e_{\omega,0}\right).$$

Then

$$\begin{aligned} \left\langle \varphi(T_{b_n})^* e_{\omega,j},\ \varphi(T_{b_n})^* e_{\omega,i} \right\rangle_{L^2} &= \frac{1}{i!\,j!}\frac{\partial^{i+j}}{\partial \omega^i \partial \overline{\omega}^j}\left(\varphi(\omega)\overline{\varphi(\omega)}\langle e_{\omega,0},\ e_{\omega,0}\rangle\right) \\ &= \frac{1}{i!\,j!}\frac{\partial^{i+j}}{\partial \omega^i \partial \overline{\omega}^j}\frac{\varphi(\omega)\overline{\varphi(\omega)}}{1-\omega\overline{\omega}}. \end{aligned} \tag{4.6}$$

Upon setting $\phi \equiv 1$ in the latter equality we get

$$\left\langle e_{\omega,j},\ e_{\omega,i} \right\rangle_{L^2} = \frac{1}{i!\,j!}\frac{\partial^{i+j}}{\partial \omega^i \partial \overline{\omega}^j}\frac{1}{1-\omega\overline{\omega}} \tag{4.7}$$

and now, subtracting (4.6) from (4.7) we see that the ij-th entry in the matrix of the sesquilinear form (4.5) with respect to the basis (4.3) for $\mathcal{K}_{b_n}$ coincides with the corresponding entry in the Carathéodory matrix $C_n(\varphi;\, \omega)$.

Lemma 4.3. *Let f be analytic at $\omega \in \mathbb{D}$ and such that $f(\omega) \neq 0$, let*

$$b_r(z) := \left(\frac{z-\omega}{1-z\bar{\omega}}\right)^r \tag{4.8}$$

and let $C_n(f;\, \omega)$ and $C_n(b;\, \omega)$ be the Carathéodory matrices of order $n > r$. Then

$$\mathrm{sq}_-\left(C_n(f;\, \omega) - C_n(b_r;\, \omega)\right) = r + \mathrm{sq}_- C_{n-r}(f;\, \omega). \tag{4.9}$$

Proof. By Remark 4.1, we can replace the function f with the Taylor polynomial p_{n-1}, but we will keep notation f for it. Besides the space $\mathcal{K}_{b_n}$ we shall also use its subspaces

$$\begin{aligned} \mathcal{K}_{b_r} &= H^2 \ominus b_r H^2 = \operatorname{span}\{e_{\omega,0},\dots,e_{\omega,r-1}\}, && (4.10)\\ \mathcal{K}_2 &= \operatorname{span}\{e_{\omega,r},\dots,e_{\omega,n-1}\}. && (4.11)\end{aligned}$$

Note that

$$\mathcal{K}_{b_n} = \mathcal{K}_{b_r} \dot{+} \mathcal{K}_2. \tag{4.12}$$

The latter decomposition is not orthogonal. Orthogonal complement of $\mathcal{K}_{b_r}$ in $\mathcal{K}_{b_n}$ is $b_r\mathcal{K}_{b_{n-r}}$, where

$$b_{n-r}(z) = \left(\frac{z-\omega}{1-z\bar{\omega}}\right)^{n-r}$$

and

$$\mathcal{K}_{b_{n-r}} = H^2 \ominus b_{n-r}H^2 = \operatorname{span}\{e_{\omega,0},\dots,e_{\omega,n-r-1}\}. \tag{4.13}$$

By Remark 4.2, the matrices $C_n(f;\omega) - C_n(b_r;\omega)$ and $C_{n-r}(f;\omega)$ in (4.9) are the matrices of quadratic forms associated with the operators

$$b_r(T_{b_n})b_r(T_{b_n})^* - f(T_{b_n})f(T_{b_n})^* \quad \text{and} \quad \mathbb{I}_{\mathcal{K}_{b_{n-r}}} - f(T_{b_{n-r}})f(T_{b_{n-r}})^*,$$

with respect to the bases (4.3) and (4.13) for $\mathcal{K}_{b_n}$ and $\mathcal{K}_{b_{n-r}}$, respectively. These forms can be written as

$$\mathbf{L}_1(x,x) := \|b_r(T_{b_n})^*x\|^2 - \|f(T_{b_n})^*x\|^2 \qquad (x\in\mathcal{K}_{b_n}) \tag{4.14}$$

and

$$\mathbf{L}_2(y,y) := \|y\|^2 - \left\|f(T_{b_{n-r}})^*y\right\|^2 \qquad (y\in\mathcal{K}_{b_{n-r}}). \tag{4.15}$$

Thus, (4.9) is equivalent to

$$\operatorname{sq}_-\mathbf{L}_1 = r + \operatorname{sq}_-\mathbf{L}_2. \tag{4.16}$$

We break the proof into steps.

Step 1. *The mapping $\Gamma = P_+\overline{b_r}$ maps $\mathcal{K}_2$ onto $\mathcal{K}_{b_{n-r}}$ and vanishes on $\mathcal{K}_{b_r}$. Moreover,*

$$\Gamma x_2 = \left(\mathbb{I}_{\mathcal{K}_{b_{n-r}}} - \omega T^*_{b_{n-r}}\right)^{-r}\widetilde{\Gamma}x_2 \tag{4.17}$$

for every $x_2\in\mathcal{K}_2$, where $\widetilde{\Gamma}:\ \mathcal{K}_2\to\mathcal{K}_{b_{n-r}}$ is the linear operator uniquely defined by equalities

$$\widetilde{\Gamma}e_{\omega,k} = e_{\omega,k-r} \quad \text{for} \ \ k=r,r+1,\dots,n-1. \tag{4.18}$$

Proof of Step 1. By definitions (3.2) and (4.8) of $e_{\omega,k}$ and b_r, we get

$$\begin{aligned}\overline{b_r(t)}e_{\omega,k}(t) &= \frac{(1-t\bar{\omega})^r}{(t-\omega)^r}\cdot\frac{t^k}{(1-t\bar{\omega})^{k+1}}\\ &= \frac{t^r}{(t-\omega)^r}\cdot\frac{t^{k-r}}{(1-t\bar{\omega})^{k-r+1}}\\ &= (1-\bar{t}\omega)^{-r}e_{\omega,k-r}(t)\end{aligned}$$

for $k = r, \dots, n-1$. Therefore,

$$P_+\overline{b_r(t)}e_{\omega,k}(t) = P_+\left((1-\bar{t}\omega)^{-r}e_{\omega,k-r}(t)\right)$$

for $k = r, \dots, n-1$. Since $e_{\omega,k-r} \in \mathcal{K}_{b_{n-r}}$ for $k = r, \dots, n-1$, and in virtue of (4.4) applied to the function $\phi(t) = (1-t\bar{\omega})^{-r}$, the latter equality reads as

$$P_+\overline{b_r}e_{\omega,k} = \left(\mathbb{I}_{\mathcal{K}_{b_{n-r}}} - \omega T^*_{b_{n-r}}\right)^{-r} e_{\omega,k-r} \quad (r \le k \le n-1),$$

which in turn, on account of definition (4.18) of $\widetilde{\Gamma}$, can be written as

$$P_+\overline{b_r}e_{\omega,k} = \left(\mathbb{I}_{\mathcal{K}_{b_{n-r}}} - \omega T^*_{b_{n-r}}\right)^{-r} \widetilde{\Gamma}e_{\omega,k} \quad (r \le k \le n-1). \tag{4.19}$$

Equality (4.19) means that (4.17) holds for x_2 equals $e_{\omega,r}, \dots, e_{\omega,n-1}$. Since $\mathcal{K}_2$ is the linear span of those vectors, (4.17) follows. The map $\widetilde{\Gamma}$ is a bijection between $\mathcal{K}_2$ and $\mathcal{K}_{b_{n-r}}$ since (by (4.18)) it maps a basis $\{e_{\omega,r}, \dots, e_{\omega,n-1}\}$ for $\mathcal{K}_2$ onto the basis $\{e_{\omega,0}, \dots, e_{\omega,n-r-1}\}$ for $\mathcal{K}_{b_{n-r}}$. Since $T^*_{b_{n-r}}$ is a contraction and since $|\omega| < 1$, the operator $\mathbb{I}_{\mathcal{K}_{b_{n-r}}} - \omega T^*_{b_{n-r}}$ is invertible on $\mathcal{K}_{b_{n-r}}$. Therefore, the operator

$$\Gamma|_{\mathcal{K}_2} = \left(\mathbb{I}_{\mathcal{K}_{b_{n-r}}} - \omega T^*_{b_{n-r}}\right)^{-r} \widetilde{\Gamma}|_{\mathcal{K}_2}$$

maps $\mathcal{K}_2$ bijectively onto $\mathcal{K}_{b_{n-r}}$. Finally, for $x_1 \in \mathcal{K}_{b_r}$

$$\overline{b_r}x_1 \in H^2_- := L^2 \ominus H^2$$

and thus,

$$\Gamma x_1 = P_+\overline{b_r}x_1 = 0 \quad \text{for every} \;\; x_1 \in \mathcal{K}_{b_r}. \tag{4.20}$$

Step 2. *Let x be an element of $\mathcal{K}_{b_n}$ decomposed as*

$$x = x_1 + x_2 \quad \textit{where } x_1 \in \mathcal{K}_{b_r} \textit{ and } x_2 \in \mathcal{K}_2 \tag{4.21}$$

(existence and uniqueness of such a decomposition follows from (4.12)*) and let* $\mathbf{L}_1$ *be the quadratic form defined in* (4.14). *Then*

$$\mathbf{L}_1(x,x) = -\left\|f(T_{b_r})^*x_1 + \mathbf{P}_{\mathcal{K}_{b_r}}\overline{f}x_2\right\|^2 + \|\Gamma x_2\|^2 - \left\|f(T_{b_{n-r}})^*\Gamma x_2\right\|^2. \tag{4.22}$$

Proof of Step 2. By (4.4),

$$b_r(T_{b_n})^*x = P_+\overline{b_r}x = \Gamma x = \Gamma x_1 + \Gamma x_2$$

and since Γ vanishes on $\mathcal{K}_{b_r}$,

$$b_r(T_{b_n})^*x = \Gamma x_2. \tag{4.23}$$

Next,

$$f(T_{b_n})^*x = P_+\overline{f}x = P_+\overline{f}x_1 + P_+\overline{f}x_2. \tag{4.24}$$

Since $x_1 \in \mathcal{K}_{b_r}$, we have (by the formula (4.4))

$$P_+\overline{f}x_1 = f(T_{b_r})^*x_1 \in \mathcal{K}_{b_r}.$$

Furthermore, since (see [7], The Projection Lemma in Lecture II, p.34)

$$P_+ = \mathbf{P}_{\mathcal{K}_{b_r}} + \mathbf{P}_{b_rH^2} = \mathbf{P}_{\mathcal{K}_{b_r}} + b_rP_+\overline{b_r},$$

we also have

$$P_+\overline{f}x_2 = \mathbf{P}_{\mathcal{K}_{b_r}}\overline{f}x_2 + b_rP_+\overline{b}_r\overline{f}x_2.$$

Substituting the latter into (4.24) we get

$$f(T_{b_n})^*x = \big(f(T_{b_r})^*x_1 + \mathbf{P}_{\mathcal{K}_{b_r}}\overline{f}x_2\big) + b_rP_+\overline{b}_r\overline{f}x_2.$$

Since

$$f(T_{b_r})^*x_1 + \mathbf{P}_{\mathcal{K}_{b_r}}\overline{f}x_2 \in \mathcal{K}_{b_r} \quad \text{and} \quad b_rP_+\overline{b}_r\overline{f}x_2 \in b_rH^2 = H^2 \ominus \mathcal{K}_{b_r},$$

it follows by the Pythagorean theorem, that

$$\begin{aligned} \|f(T_{b_n})^*x\|^2 &= \left\|f(T_{b_r})^*x_1 + \mathbf{P}_{\mathcal{K}_{b_r}}\overline{f}x_2\right\|^2 + \left\|b_rP_+\overline{b}_r\overline{f}x_2\right\|^2 \\ &= \left\|f(T_{b_r})^*x_1 + \mathbf{P}_{\mathcal{K}_{b_r}}\overline{f}x_2\right\|^2 + \left\|P_+\overline{b}_r\overline{f}x_2\right\|^2. \end{aligned} \tag{4.25}$$

The latter equality holds since b_r is an inner function. Now we focus on the second term on the right hand side in (4.25). Let P_- be the orthogonal projection of L^2 onto $H^2_- = L^2 \ominus H^2$. Then clearly,

$$P_+\overline{f}P_-\overline{b_r}x_2 = 0$$

and therefore,

$$\begin{aligned} P_+\overline{b_r}\ \overline{f}x_2 &= P_+\overline{f}\ \overline{b_r}x_2 \\ &= P_+\overline{f}\,(P_+ + P_-)\,\overline{b_r}x_2 \\ &= P_+\overline{f}P_+\overline{b_r}x_2 \\ &= P_+\overline{f}\Gamma x_2 \end{aligned} \tag{4.26}$$

where Γ is the operator introduced in Step 1. Since $x_2 \in \mathcal{K}_2$, it follows by Step 1, that $\Gamma x_2 \in \mathcal{K}_{b_{n-r}}$ and thus, by formula (4.4),

$$P_+\overline{f}\Gamma x_2 = f(T_{b_{n-r}})^*\Gamma x_2.$$

Substituting the latter relation into (4.26) and then (4.26) into (4.25) we arrive at

$$\|f(T_{b_n})^*x\|^2 = \left\|f(T_{b_r})^*x_1 + \mathbf{P}_{\mathcal{K}_{b_r}}\overline{f}x_2\right\|^2 + \left\|f(T_{b_{n-r}})^*\,\Gamma x_2\right\|^2. \tag{4.27}$$

Now we substitute (4.23) and (4.27) into (4.14) and get (4.22).

Step 3 (completion of the proof). Since the spectrum of the operator T_{b_r} consists of a single point ω and since $f(\omega) \neq 0$, the operator $f(T_{b_r})^*$ is invertible on $\mathcal{K}_{b_r}$ (by the Spectrum Mapping Theorem). Since moreover, $\mathcal{K}_{b_r}$ and $\mathcal{K}_2$ are linearly independent, the formula

$$x_1 + x_2 \mapsto \big(f(T_{b_r})^*x_1 + \mathbf{P}_{\mathcal{K}_{b_r}}f(T_{b_n})^*x_2\big) + x_2$$

defines a nonsingular transformation of $\mathcal{K}_{b_n} = \mathcal{K}_{b_r} \dot{+} \mathcal{K}_2$ onto itself. Therefore (and in view of (4.22)),

$$\begin{aligned} \mathrm{sq}_-\mathbf{L}_1(x,x) &= \mathrm{sq}_-\left(-\left\|f(T_{b_r})^*x_1 + \mathbf{P}_{\mathcal{K}_{b_r}}\overline{f}x_2\right\|^2 + \|\Gamma x_2\|^2 - \left\|f(T_{b_{n-r}})^*\Gamma x_2\right\|^2\right) \\ &= \mathrm{sq}_-\left(-\|x_1\|^2 + \|\Gamma x_2\|^2 - \left\|f(T_{b_{n-r}})^*\Gamma x_2\right\|^2\right). \end{aligned}$$

Using again linear independence of the subspaces $\mathcal{K}_{b_r}$ and $\mathcal{K}_2$, we get

$$\mathrm{sq}_-\mathbf{L}_1(x,x) = \mathrm{sq}_-\left(-\|x_1\|^2\right) + \mathrm{sq}_-\left(\|\Gamma x_2\|^2 - \left\|f(T_{b_{n-r}})^*\Gamma x_2\right\|^2\right).$$

Since x_1 runs over $\mathcal{K}_{b_r}$,

$$\mathrm{sq}_-\left(-\|x_1\|^2\right) = \dim \mathcal{K}_{b_r} = r$$

and since Γ maps $\mathcal{K}_2$ onto $\mathcal{K}_{b_{n-r}}$, the vector $y = \Gamma x_2$ runs over $\mathcal{K}_{b_{n-r}}$ as x_2 runs over $\mathcal{K}_2$. Thus,

$$\mathrm{sq}_-\mathbf{L}_1(x,x) = r + \mathrm{sq}_-\left(\|y\|^2 - \left\|f(T_{b_{n-r}})^* y\right\|^2\right) = r + \mathrm{sq}_-\mathbf{L}_2(y,y)$$

which proves (4.16) and completes the proof of the lemma. □

Lemma 4.4. *Let f be analytic at $\omega \in \mathbb{C}$, let $n \geq 0$ be a fixed integer and let $C_n(f;\,\omega)$ be the Carathéodory matrix given by* (4.1). *There exists a neighborhood $\mathcal{U}_\omega$ of ω such that*

$$\mathrm{sq}_- C_n(f;\,\omega) \leq \mathrm{sq}_- P_n(f;\,z_1,\ldots,z_n) \tag{4.28}$$

and

$$\mathrm{sq}_+ C_n(f;\,\omega) \leq \mathrm{sq}_+ P_n(f;\,z_1,\ldots,z_n) \tag{4.29}$$

for any choice of n points $z_1,\ldots,z_n \in \mathcal{U}_\omega$, where $P_n(f;\,z_1,\ldots,z_n)$ is the Pick matrix defined in (1.3).

Proof. Let $\mathcal{D}_\omega$ be a disk centered at ω on which f is analytic: that is, let $\mathcal{D}_\omega \subset \rho(f)$. Pick any points $z_1,\ldots,z_n \in \mathcal{D}_\omega$ and consider the lower triangular $n \times n$ matrix $\Phi_{z_1,\ldots,z_n} = [\Phi_{i,j}]_{i,j=1}^n$ with the entries

$$\Phi_{i,j} = \begin{cases} \dfrac{1}{\phi_i'(z_j)} & \text{if } i \geq j, \\ 0 & \text{if } i < j, \end{cases} \quad , \quad \text{where } \phi_i(z) = \prod_{j=1}^{i}(z - z_j). \tag{4.30}$$

Then for every function $v(z)$ analytic at ω, with the Taylor series

$$v(z) = \sum_{k=0}^{\infty}(z-\omega)^k v_k,$$

it holds that

$$\lim_{z_1,\ldots,z_n \to \omega} \Phi_{z_1,\ldots,z_n} \begin{bmatrix} v(z_1) \\ v(z_2) \\ \vdots \\ v(z_n) \end{bmatrix} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix}$$

(see [5] for the proof, also see [3, Lemma 3.1]). Now it follows from definitions (1.3) and (4.1) that

$$C_n(f;\,\omega) = \lim_{z_1,\ldots,z_n \to \omega} \Phi_{z_1,\ldots,z_n} \cdot P_n(f;\,z_1,\ldots,z_n) \cdot \Phi^*_{z_1,\ldots,z_n}. \tag{4.31}$$

Therefore, there exists a neighborhood $\mathcal{U}_\omega \subset \mathcal{D}_\omega$ of ω such that

$$\begin{aligned} \mathrm{sq}_\pm C_n(f;\,\omega) &\leq \mathrm{sq}_\pm \left[\Phi_{z_1,\ldots,z_n} \cdot P_n(f;\, z_1,\ldots,z_n) \cdot \Phi^*_{z_1,\ldots,z_n}\right] \\ &= \mathrm{sq}_\pm P_n(f;\, z_1,\ldots,z_n) \end{aligned}$$

for every choice of n points $z_1,\ldots,z_n \in \mathcal{U}_\omega$. The latter equality holds since $\Phi_{z_1,\ldots,z_n}$ is nonsingular. □

Corollary 4.5. *Let f and g be analytic at $\omega \in \mathbb{C}$, let $n \geq 0$ be a fixed integer and let $C_n(f;\,\omega)$ and $C_n(g;\,\omega)$ be the corresponding Carathéodory matrices. There exists a neighborhood $\mathcal{U}_\omega$ of ω such that*

$$\mathrm{sq}_-\left(C_n(f;\,\omega) - C_n(g;\,\omega)\right) \leq \mathrm{sq}_-\left(P_n(f;\, z_1,\ldots,z_n) - P_n(g;\, z_1,\ldots,z_n)\right)$$

for any choice of n points $z_1,\ldots,z_n \in \mathcal{U}_\omega$.

Proof. Similar to relation (4.31), we have also

$$C_n(g;\,\omega) = \lim_{z_1,\ldots,z_n \to \omega} \Phi_{z_1,\ldots,z_n} \cdot P_n(g;\, z_1,\ldots,z_n) \cdot \Phi^*_{z_1,\ldots,z_n}$$

which, being subtracted from (4.31), leads us to

$$\begin{aligned} &C_n(f;\,\omega) - C_n(g;\,\omega) \\ &= \lim_{z_1,\ldots,z_n \to \omega} \Phi_{z_1,\ldots,z_n}\left(P_n(f;\, z_1,\ldots,z_n) - P_n(g;\, z_1,\ldots,z_n)\right)\Phi^*_{z_1,\ldots,z_n}. \end{aligned}$$

The rest follows as in Lemma 4.4. □

Note also the following "multiple" analogue of Remark 3.2.

Remark 4.6. *Let h and g be two functions on $\mathbb{D}$, let $\omega \in \mathbb{D}$ and let $g(\omega) \neq 0 \quad (i = 1,\ldots,n)$. Then*

$$C_n(h;\,\omega) - C_n(g;\,\omega) = GC_n\left(\frac{h}{g};\,\omega\right)G^* \tag{4.32}$$

where G is the lower triangular toeplitz matrix given by

$$G = \begin{bmatrix} g(\omega) & 0 & \ldots & 0 \\ g'(\omega) & g(\omega) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{g^{(n-1)}(\omega)}{(n-1)!} & \ldots & g'(\omega) & g(\omega) \end{bmatrix}.$$

The proof is immediate and follows from the definition (4.1) of the Carathéodory matrix by the Leibnitz rule.

5. Proof of Theorem 1.2

The proof of Theorem 1.2 can be obtained by standard compactness arguments based on the following result.

Theorem 5.1. *Let f belong to $\mathcal{S}_\kappa$. Then for every point $\omega \in \mathbb{D}$, there exists an open neighborhood $\mathcal{U}_\omega \subset \mathbb{D}$ and a positive integer n_ω such that*

$$\mathrm{sq}_- P_n(f;\, z_1, \ldots, z_n) = \kappa \tag{5.1}$$

for every $n \geq n_\omega$ and every choice of n points $z_1, \ldots, z_n \in \mathcal{U}_\omega \cap \rho(f)$.

Proof. Let us assume first that $\omega \in \rho(f)$, i.e., that f is analytic at ω. Then we can define Carathéodory matrices $C_n(f;\, \omega)$ of any order $n \geq 0$ via formula (4.1). By the classical Krein–Langer result, there exists an integer n_ω such that

$$\mathrm{sq}_- C_n(f;\, \omega) = \kappa \quad \text{for all } \ n \geq n_\omega. \tag{5.2}$$

By Lemma 4.4, there exists a neighborhood $\mathcal{U}_\omega$ of ω such that

$$\mathrm{sq}_- C_n(f;\, \omega) \leq \mathrm{sq}_- P_n(f;\, z_1, \ldots, z_n)$$

for every choice of n points $z_1, \ldots, z_n \in \mathcal{U}_\omega$, where $P_n(f;\, z_1, \ldots, z_n)$ is the Pick matrix defined in (1.3). Taking into account that

$$\mathrm{sq}_- P_n(f;\, z_1, \ldots, z_n) \leq \kappa, \tag{5.3}$$

since $f \in \mathcal{S}_\kappa$, and making use of (5.2), we come to (5.1).

Now we consider the case when ω is a pole of f. Then f can be represented as

$$f(z) = \frac{S(z)}{b(z)B_1(z)}, \quad \text{where } \ b(z) = \left(\frac{z-\omega}{1-z\overline{w}}\right)^r \tag{5.4}$$

and B_1 is a Blaschke product of degree $\kappa - r$ that does not vanish at ω. Thus, the function

$$\widetilde{f}(z) = \frac{S(z)}{B_1(z)} \tag{5.5}$$

is analytic at ω and belongs to the class $\mathcal{S}_{\kappa-r}$, by the Krein–Langer characterization theorem. We apply the first part of the proof to the function $\widetilde{f}$ and conclude that there exists an integer n_0 such that

$$\mathrm{sq}_- C_{n_0}(\widetilde{f};\, \omega) = \kappa - r. \tag{5.6}$$

The number

$$n := n_0 + r$$

and the functions b and $\widetilde{f}$ meet the conditions of Lemma 4.3. Therefore, by (4.18),

$$\mathrm{sq}_- \left(C_n(\widetilde{f};\, \omega) - C_n(b;\, \omega)\right) = r + \mathrm{sq}_- C_{n_0}(\widetilde{f};\, \omega),$$

which together with (5.6) implies

$$\mathrm{sq}_- \left(C_n(\widetilde{f};\, \omega) - C_n(b;\, \omega)\right) = \kappa. \tag{5.7}$$

By Corollary 4.5, there exists a neighborhood $\mathcal{U}_\omega$ of ω such that

$$\mathrm{sq}_-\left(C_n(\widetilde{f};\,\omega)-C_n(b;\,\omega)\right)\le \mathrm{sq}_-\left(P_n(\widetilde{f};\,z_1,\dots,z_n)-P_n(b;\,z_1,\dots,z_n)\right) \tag{5.8}$$

for every choice of n points $z_1,\dots,z_n\in\mathcal{U}_\omega$. Since $B_1(\omega)\neq 0$, the neighborhood $\mathcal{U}_\omega$ can be chosen so that $B_1(z)\neq 0$ for every $z\in\mathcal{U}_\omega$. Then we can apply Remark 3.2 with $h=\widetilde{f}$ and $g=b$ to conclude that

$$\begin{aligned}\mathrm{sq}_-\left(P_n(\widetilde{f};\,z_1,\dots,z_n)-P_n(b;\,z_1,\dots,z_n)\right) &= \mathrm{sq}_-P_n\left(\frac{\widetilde{f}}{b};\,z_1,\dots,z_n\right)\\ &= \mathrm{sq}_-P_n(f;\,z_1,\dots,z_n).\end{aligned}$$

Upon combining the last relation with (5.7) and (5.8), we obtain

$$\mathrm{sq}_-P_n(f;\,z_1,\dots,z_n)\ge\kappa\quad\text{for every } z_1,\dots,z_n\in\mathcal{U}_\omega.$$

Taking into account the reverse inequality (5.3) which holds for any choice of $z_1,\dots,z_n\in\rho(f)$, we arrive at (5.1). □

Proof of Theorem 1.2. Let Ω be a compact subset of $\mathbb{D}$. For every point $\omega\in\Omega$ one can find $\mathcal{U}_\omega$ and n_ω indicated in Theorem 5.1. The set $\bigcup_{\omega\in\Omega}\mathcal{U}_\omega$ is an open covering of Ω. Pick a finite subcovering of Ω: let $\omega_1,\dots,\omega_m\in\Omega$ and $n_{\omega_1},\dots,n_{\omega_m}\in\mathbb{N}$ be such that

$$\Omega\subset\bigcup_{i=1}^{m}\mathcal{U}_{\omega_i} \tag{5.9}$$

and (5.1) holds for $n\ge n_{\omega_i}$ for every choice of $z_1,\dots,z_n\in\mathcal{U}_{\omega_i}\cap\rho(f)$ and $i=1,\dots,m$.

Let n be an integer such that

$$n\ge m\widehat{n},\quad\text{where }\ \widehat{n}:=\max_i n_{\omega_i}. \tag{5.10}$$

We shall show that (5.1) holds for every choice of points $z_1,\dots,z_n$ in $\Omega\cap\rho(f)$.

Indeed let $z_1,\dots,z_n$ be such points. By (5.9) and (5.10), there exist the index i ($1\le i\le m$) such that at least $\widehat{n}$ of points $z_1,\dots,z_n$ fall inside $\mathcal{U}_{\omega_i}$. After a suitable rearrangement we assume that these points are $z_1,\dots,z_{\widehat{n}}$. Since $\widehat{n}\ge n_1$, it follows that

$$\mathrm{sq}_-P_{\widehat{n}}(f;\,z_1,\dots,z_{\widehat{n}})=\kappa.$$

Since $\mathrm{sq}_-P_{\widehat{n}}(f;\,z_1,\dots,z_{\widehat{n}})$ is a principal submatrix of $P_n(f;\,z_1,\dots,z_n)$,

$$\mathrm{sq}_-P_n(f;\,z_1,\dots,z_n)\ge\mathrm{sq}_-P_{\widehat{n}}(f;\,z_1,\dots,z_{\widehat{n}})=\kappa.$$

On the other hand,

$$\mathrm{sq}_-P_n(f;\,z_1,\dots,z_n)\le\kappa,$$

since $f\in\mathcal{S}_\kappa$. Thus, every Pick matrix $P_n(f;\,z_1,\dots,z_n)$ of size $n\ge m\widehat{n}$ has κ negative eigenvalues, whenever $z_1,\dots,z_n\in\Omega$. The latter means that the integer n_0 in **Question 1** can be chosen so that $n_0\le m\widehat{n}$. □

Analogues of some of the results obtained above can be established in the context of Carathéodory matrices. For example, the analogue of Theorem 1.2 reads as follows.

Theorem 5.2. *Given a function $f \in \mathcal{S}_\kappa$ and given a compact subset $\Omega \in \mathbb{D}$, there exists an integer $n_0 = n_0(f, \Omega)$ such that*

$$\mathrm{sq}_- C_{n_0}(f;\, \omega) = \kappa \tag{5.11}$$

for every point $\omega \in \Omega \cap \rho(f)$.

Proof. The proof is actually contained in the proof of Theorem 1.2. For every point $\omega \in \Omega$ we choose an integer n_ω such that

$$\mathrm{sq}_- C_{n_\omega}(f;\, \omega) = \kappa$$

and therefore, $\mathrm{sq}_- C_n(f;\, \omega) = \kappa$ for every $n \geq n_\omega$. By continuity, there exists a neighborhood $\mathcal{U}_\omega$ of ω (depending on ω and n_ω) such that

$$\mathrm{sq}_- C_{n_\omega}(f;\, z) \geq \mathrm{sq}_- C_{n_\omega}(f;\, \omega) = \kappa \quad \text{for every } \ z \in \mathcal{U}_\omega. \tag{5.12}$$

Since $f \in \mathcal{S}_\kappa$, we have

$$\mathrm{sq}_- C_n(f;\, z) \leq \kappa$$

for every $n \geq 0$ and every $z \in \mathbb{D}$, which together with (5.12) implies

$$\mathrm{sq}_- C_{n_\omega}(f;\, z) = \kappa \quad \text{for every } \ z \in \mathcal{U}_\omega.$$

It remains to choose a finite open covering (5.9) of Ω and to note that (5.11) holds for every $\omega \in \Omega$ with $n_0 = \max_i \, n_{\omega_i}$. □

Theorem 1.1 also admits a partial analogue in the context of Carathéodory matrices:

Theorem 5.3. *Let f be the ratio of two finite Blaschke products*

$$f(z) = \frac{\widetilde{B}(z)}{B(z)}, \quad \deg B = \kappa, \ \ \deg \widetilde{B} = \widetilde{\kappa} \tag{5.13}$$

with no common zeros, let ω be a point in $\mathbb{D} \cap \rho(f)$ and let $C_n(f;\, \omega)$ be the corresponding Carathéodory matrix. Then

1. *If $n = \kappa + \widetilde{\kappa}$, then $C_n(f;\, \omega)$ is invertible.*
2. *If $n \geq \kappa + \widetilde{\kappa}$, then*

$$\mathrm{sq}_- \, C_n(f;\, \omega) = \kappa \quad \textit{and} \quad \mathrm{sq}_+ \, C_n(f;\, \omega) = \widetilde{\kappa}.$$

3. *If $n < \kappa + \widetilde{\kappa}$, then there is $\omega \in \mathbb{D} \cap \rho(f)$ such that* $\mathrm{sq}_- \, C_n(f;\, \omega) < \kappa$.

We omit the proof, which is based on the arguments close to those in the proofs of Lemma 3.3 and Theorem 3.4. Theorem 5.3 says, in particular, that for f of the form (5.13), one can find and integer n such that

$$\mathrm{sq}_- C_n(f;\, \omega) = \kappa \quad \text{for every } \ \omega \in \mathbb{D} \cap \rho(f) \tag{5.14}$$

and that the minimal value of such n is

$$n_0 = \deg B + \deg \widetilde{B}.$$

This gives the sufficiency part for the "Carathéodory" analogue of Theorem 1.1. However, we do not know if the complete analogue (that is, including necessity part) of Theorem 1.1 holds true. This gives rise to the following

Question 2: Does there exist a function $f \in \mathcal{S}_\kappa$ having property (5.14) and which is not a ratio of two finite Blaschke products?

Theorem 1.1 allows us to reformulate this question in the following equivalent form: *Does there exist a function $f \in \mathcal{S}_\kappa$ having positive definite Pick matrices of arbitrarily large orders and such that all Carathéodory matrices of order n_0 have κ negative eigenvalues?*

Roughly speaking, property (5.14) means that Pick matrices $P_n(f;\, z_1, \ldots, z_n)$ have κ negative eigenvalues if the points $z_1, \ldots, z_n$ are close to each other. However, Pick matrices based on relatively distant points do not have to enjoy this property.

6. General subdomains of $\mathbb{D}$.

Theorems 1.1 and 1.2 and Remark 1.3 give a partial answer to the following

Question 3: Given an infinite set $\Omega \subseteq \mathbb{D}$, characterize the set $\mathcal{P}_\kappa(\Omega)$ of all functions $f \in \mathcal{S}_\kappa$ with the following property:

There exists an integer n_0 such that $\operatorname{sq}_- P_n(f;\, z_1, \ldots, z_n) = \kappa$ *for every $n \geq n_0$ and every choice of points $z_1, \ldots, z_n \in \Omega$.*

More precisely, Theorem 1.1 states that if $\Omega = \mathbb{D}$, then $\mathcal{P}_\kappa(\Omega)$ coincides with $\mathcal{B}_\kappa$, the set of all ratios of two finite Blaschke products with no common zeros and degree of the denominator equal κ. On the other hand, Theorem 1.2 and Remark 1.3 claim that if $\overline{\Omega} \subset \mathbb{D}$ i.e., if $\overline{\Omega} \cap \mathbb{T} = \emptyset$, then $\mathcal{P}_\kappa(\Omega) = \mathcal{S}_\kappa$. In general,

$$\mathcal{B}_\kappa \subseteq \mathcal{P}_\kappa(\Omega) \subseteq \mathcal{S}_\kappa$$

(the left inclusion follows from Theorem 3.4 and the right inclusion is selfevident). Theorem 6.1 below asserts that the right inclusion is proper unless we are in the assumption of Theorem 1.2. Theorem 6.4 asserts that the left inclusion is proper if $\overline{\Omega}$ misses an arbitrarily small arc of $\mathbb{T}$ (compare with Theorem 1.1).

Theorem 6.1. *Let Ω be an open subset of $\mathbb{D}$. Then the inclusion $\mathcal{P}_\kappa(\Omega) \subset \mathcal{S}_\kappa$ is proper if and only if*

$$\overline{\Omega} \cap \mathbb{T} \neq \emptyset. \tag{6.1}$$

Proof. The "only if" part follows from Theorem 1.2 and Remark 1.3. To prove the "if" part we pick a point t_0 from $\overline{\Omega} \cap \mathbb{T}$ (without loss of generality we may assume that $t_0 = -1$) and show that the function

$$f(z) = \left(\frac{z+1}{2z}\right)^\kappa \tag{6.2}$$

belongs to $\mathcal{S}_\kappa \setminus \mathcal{P}_\kappa(\Omega)$. It belongs to $\mathcal{S}_\kappa$, since it admits the Krein–Langer factorization (1.5) with

$$S(z) = \left(\frac{z+1}{2}\right)^\kappa \quad \text{and} \quad B(z) = z^\kappa. \tag{6.3}$$

We shall show that for every fixed n,

$$C_n(f;\, \omega) > 0 \quad \text{whenever} \quad \left|\omega - \frac{1}{2n+1}\right| > \frac{2n}{2n+1} \tag{6.4}$$

Assuming for a moment that (6.4) is already proved, let us show that $f \notin \mathcal{P}_\kappa(\Omega)$. Indeed, choosing any ω such that the corresponding Carathéodory matrix $C_n(f;\, \omega)$ is positive, we conclude by the second statement in Lemma 4.4 (inequality (4.29)), that there exists a neighborhood $\mathcal{U}_\omega$ of ω such that the Pick matrix $P_n(f;\, z_1, \dots, z_n)$ is positive definite for every choice of n points $z_1, \dots, z_n \in \mathcal{U}_\omega$. Since Ω is open, we can choose $\mathcal{U}_\omega$ to be a subset of Ω. Thus, we have shown that for every fixed integer n there exists a positive definite Pick matrix $P_n(f;\, z_1, \dots, z_n)$ based on n points $z_1, \dots, z_n \in \Omega$. The latter means that $f \notin \mathcal{P}_\kappa(\Omega)$.

Thus, it remains to prove (6.4). To this end we once again use the construction from the proof of Lemma 4.3. Since $B(\omega) = \omega^\kappa \neq 0$ for $\omega \neq 0$, it follows by Remark 4.6 that $C_n(f;\, \omega)$ is congruent to the difference $C_n(S;\, \omega) - C_n(B;\, \omega)$ of Carathéodory matrices and thus, $C_n(f;\, \omega) > 0$ if and only if

$$C_n(S;\, \omega) - C_n(B;\, \omega) > 0. \tag{6.5}$$

Let, as in the proof of Lemma 4.3,

$$b_n(z) := \left(\frac{z-\omega}{1-z\bar{\omega}}\right)^n, \quad \mathcal{K}_{b_n} := H^2 \ominus b_n H_2, \quad T := T_{b_n} = \mathbf{P}_{\mathcal{K}_{b_n}} t\,|_{\mathcal{K}_{b_n}}.$$

Step 1. $C_n(f;\, \omega) > 0$ *if and only if the operator*

$$A_{n,\kappa} = T^\kappa (T^\kappa)^* - \frac{(T+I)^\kappa (T^*+I)^\kappa}{2^{2\kappa}} \;:\; \mathcal{K}_{b_n} \to \mathcal{K}_{b_n} \tag{6.6}$$

is positive definite.

Proof of Step 1. As in the proof of Lemma 4.3, we refer to the fact that $C_n(S;\, \omega)$ and $C_n(B;\, \omega)$ are the matrices of the quadratic forms associated with the operators

$$I - S(T)S(T)^* \quad \text{and} \quad I - B(T)B(T)^*$$

with respect to the basis (4.3) for $\mathcal{K}_{b_n}$. Therefore, condition (6.5) can be equivalently written as

$$A_{n,\kappa} := B(T)B(T)^* - S(T)S(T)^* > 0.$$

By definition (6.3) of functions B and S, this operator A_κ coincides with that in (6.6)

Step 2. *If* ω *is such that*

$$A_{n,1} = TT^* - \frac{(T+I)(T^*+I)}{4} > 0, \tag{6.7}$$

then the operator $A_{n,\kappa}$ *is also positive for every positive integer* κ.

Proof of Step 2. We prove it by induction. Let the operator $A_{n,\kappa}$ defined via (6.6) be positive, i.e., let

$$T^{\kappa}(T^{\kappa})^{*} > 2^{-2\kappa}(T+I)^{\kappa}(T^{*}+I)^{\kappa}.$$

Then, since T is invertible,

$$\begin{aligned} T^{\kappa+1}(T^{\kappa+1})^{*} = TT^{\kappa}(T^{\kappa})^{*}T^{*} &> 2^{-2\kappa}T(T+I)^{\kappa}(T^{*}+I)^{\kappa}T^{*} \\ &= 2^{-2\kappa}(T+I)^{\kappa}TT^{*}(T^{*}+I)^{\kappa} \\ &> 2^{-2\kappa-2}(T+I)^{\kappa+1}(T^{*}+I)^{\kappa+1}. \end{aligned}$$

The last inequality is due to the assumption (6.7). So, we got that $A_{n,\kappa+1}$ is positive.

Step 3. We will use the following known property of the operator T:

$$I - TT^{*} = \mathbf{e}\mathbf{e}^{*}, \tag{6.8}$$

where $\mathbf{e}: \ \mathbb{C} \to \mathcal{K}_{b_n}$ *is the operator of multiplication by the function*

$$\mathbf{e}(t) = 1 - b_n(t)\overline{b_n(0)} = \mathbf{P}_{\mathcal{K}_{b_n}} 1 \in \mathcal{K}_{b_n},$$

$\mathbf{e}^{*}$ *is its adjoint*

$$\mathbf{e}^{*}x = \langle x, \, \mathbf{e}\rangle = x(0) \qquad (x \in \mathcal{K}_{b_n}).$$

Step 4. *Since the spectrum of* T *consists of a single point* $\omega, |\omega| < 1$, *the operator is* $I - T$ *is invertible on* $\mathcal{K}_{b_n}$. *It holds that*

$$(I - T)^{-1}\mathbf{e} = \frac{1 - b_n\overline{b_n(1)}}{1 - t}. \tag{6.9}$$

Proof of Step 4. Since $I - T$ is invertible on $\mathcal{K}_{b_n}$, then $x = (I - T)^{-1}\mathbf{e}$ is the unique solution of the equation

$$(I - T)x = \mathbf{e} \tag{6.10}$$

in $\mathcal{K}_{b_n}$. In a detailed form the equation reads as

$$(1 - t)x - b_n P_{+}\overline{b_n}(1 - t)x = \mathbf{e} = 1 - b_n\overline{b_n(0)}. \tag{6.11}$$

Since $x \in \mathcal{K}_{b_n}$, it follows that $\overline{b_n}x \in H^2_{-}$ and hence,

$$P_{+}\overline{b_n}x = 0 \quad \text{and} \quad P_{+}\overline{b_n}tx = \alpha = \text{const}.$$

Then (6.11) takes the form

$$(1 - t)x - b_n \cdot \alpha = 1 - b_n\overline{b_n(0)}.$$

Thus,

$$x = \frac{1 - b_n\overline{b_n(0)} + b_n \cdot \alpha}{1 - t} \tag{6.12}$$

and since x is in H^2, we have also

$$1 - b_n(1)\overline{b_n(0)} + b_n(1) \cdot \alpha = 0.$$

The latter equality gives (since $|b_n(1)| = 1$) $\alpha = \overline{b_n(0)} - \overline{b_n(1)}$ which being substituted into (6.12) leads us to

$$x = \frac{1 - b_n\overline{b_n(1)}}{1-t} \ .$$

One can easily check that the above function is indeed in $\mathcal{K}_{b_n}$, but there is no need to do that. This follows from the above computation since we know that the equation (6.10) has a unique solution in $\mathcal{K}_{b_n}$ and since the obtained function is the only conceivable candidate for this solution. Formula (6.9) follows.

Step 5. *The operator* $A_{n,1}$ *introduced in* (6.7) *is positive if and only if*

$$\left\| \frac{1 - b_n\overline{b_n(1)}}{1-t} \right\|_{L_2}^2 < \frac{1}{2}. \tag{6.13}$$

Proof of Step 5. Making use of (6.8) we represent $A_{n,1}$ as

$$\begin{aligned} A_{n,1} \quad &:= \quad TT^* - \frac{I+T}{2} \cdot \frac{I+T^*}{2} \\ &= \quad \frac{1}{4}(I-T)(I-T^*) - \frac{1}{2}(I - TT^*) \\ &= \quad \frac{1}{4}(I-T)(I-T^*) - \frac{1}{2}\mathbf{e}\mathbf{e}^*. \end{aligned}$$

and conclude that $A_{n,1} > 0$ if and only if

$$I - 2(I-T)^{-1}\mathbf{e}\mathbf{e}^*(I-T^*)^{-1} > 0$$

or, equivalently, if and only if

$$1 - 2\mathbf{e}^*(I-T^*)^{-1}(I-T)^{-1}\mathbf{e} > 0.$$

The latter inequality is equivalent to

$$\left\|(I-T)^{-1}\mathbf{e}\right\|_{L_2}^2 < \frac{1}{2}$$

which can be written in the form (6.13), due to formula (6.9).

Step 6. *It holds that*

$$\left\| \frac{1 - b_n\overline{b_n(1)}}{1-t} \right\|_{L_2}^2 = b_n'(1)\overline{b_n(1)}. \tag{6.14}$$

Proof of Step 6. One of variations of the classical Carathéodory–Julia theorem on boundary derivatives claims that if w is a Schur function such that

$$\liminf_{r\to 1} \frac{1 - |w(r\beta)|^2}{1 - r^2} < \infty \tag{6.15}$$

where β is a point on $\mathbb{T}$, then there exist the radial boundary limits

$$w_0 = \lim_{r\to 1} w(r\beta), \qquad w_1 = \lim_{r\to 1} w'(r\beta) \tag{6.16}$$

and

$$\left\|\frac{1-w\overline{w_0}}{1-t}\right\|_{L_2}^2+\left\|\frac{1-|w|^2}{|t-\beta|^2}\right\|_{L_1}=w_1\overline{w_0}\ \beta. \tag{6.17}$$

We apply this result to a simple case when $w=b_n$ and $\beta=1$. Condition (6.15) is clearly satisfied, the limits (6.16) exist regardless condition (6.15): since b_n is analytic at $\beta=1$, it holds that $w_0=b_n(1)$ and $w_1=b_n'(1)$. Furthermore, since b_n is inner,

$$\left\|\frac{1-|b_n|^2}{|t-\beta|^2}\right\|_{L_1}=0$$

and thus, (6.17) reduces to (6.14).

Step 7. We complete now the proof of the theorem. Upon combining (6.13) and (6.14) we conclude that $A_{n,1}$ is positive definite if and only if

$$b_n'(1)\overline{b_n(1)}<\frac{1}{2}$$

which reduces, since

$$b_n(1)=\left(\frac{1-\omega}{1-\bar{\omega}}\right)^n \quad \text{and} \quad b_n'(1)=n\cdot\left(\frac{1-\omega}{1-\bar{\omega}}\right)^n\frac{1-|\omega|^2}{|1-\omega|^2},$$

to

$$n\cdot\frac{1-|\omega|^2}{|1-\omega|^2}<\frac{1}{2}.$$

Solving the latter quadratic inequality gives

$$\left|\omega-\frac{1}{2n+1}\right|>\frac{2n}{2n+1}. \tag{6.18}$$

Thus, $A_{n,1}$ is positive if and only if ω meets condition (6.18). Now (6.4) follows by Steps 1 and 2. □

We have completed the first part of this section (we proved that the inclusion $\mathcal{P}_\kappa(\Omega)\subseteq\mathcal{S}_\kappa$ is proper under the assumption that $\overline{\Omega}\cap\mathbb{T}\neq\emptyset$). The goal of the second part is proving Theorem 6.4, which deals with assumption that $\overline{\Omega}$ misses an arbitrarily small arc of $\mathbb{T}$ and asserts that then the inclusion $\mathcal{B}_\kappa\subseteq\mathcal{P}_\kappa(\Omega)$ is proper.

Recall that formula (4.1) defines the Carathéodory matrix $C_n(f;\omega)$ at any point $w\in\mathbb{D}$ at which f is analytic. It turns out that if $w\in\mathbb{T}$ and f is analytic at ω and unimodular on an arc of $\mathbb{T}$ around ω, then the notion of the Carathéodory matrix can be meaningfully extended to this boundary situation.

Proposition 6.2. *Let g be a function analytic on a simply connected domain $\mathcal{U}$ and let*

$$\Phi(z,\zeta):=\begin{cases}\dfrac{g(z)-g(\zeta)}{z-\zeta}, & \textit{if} \quad z\neq\zeta,\\ g'(z), & \textit{if} \quad z=\zeta.\end{cases} \tag{6.19}$$

Then $\Phi(z,\zeta)$ is analytic in z and ζ on $\mathcal{U}\times\mathcal{U}$ and

$$\frac{\partial^{i+j}}{\partial z^i \partial^j \zeta}\,\Phi(z,\zeta) = \begin{cases} \dfrac{1}{(z-\zeta)^{i+j+1}} \displaystyle\int_\zeta^z g^{(i+j+1)}(u)(u-\zeta)^i (z-u)^j\,du & \text{if} \quad z \neq \zeta, \\ \dfrac{i!\,j!}{(i+j+1)!}\, g^{(i+j+1)}(z) & \text{if} \quad z = \zeta. \end{cases}$$

Verification of this result is straightforward and will be omitted.

Corollary 6.3. *For every i, $j \geq 0$, the function $\frac{\partial^{i+j}}{\partial z^i \partial^j \zeta}\,\Phi(z,\zeta)$ is analytic on $\mathcal{U}\times\mathcal{U}$ and*

$$\left\| \frac{\partial^{i+j}\Phi}{\partial z^i \partial^j \zeta} \right\|_\infty \leq \left\| g^{(i+j+1)} \right\|_\infty,$$

where $\|g\|_\infty := \max_{z\in\mathcal{U}} |g(z)|$.

Let g be a Schur function on $\mathbb{D}$ that admits an analytic continuation across an arc of $\mathbb{T}$ and is unimodular on this arc. Let the region

$$U_\rho := \left\{ z \in \mathbb{C} : \ \theta_1 < \arg z < \theta_2, \ \rho < |z| < \frac{1}{\rho} \right\}, \quad (0 < \rho < 1) \tag{6.20}$$

be such that

1. $\overline{U_\rho}$ is contained in the domain of analyticity of g and
2. g is unimodular on $\overline{U_\rho} \cap \mathbb{T}$.

Then we have by the symmetry principle,

$$g(z)\overline{g(1/\bar{z})} = 1 \quad \text{for every} \ \ z \in \overline{U_\rho}. \tag{6.21}$$

Let us consider the function $C_{0,0}$ defined by the formula

$$C_{00}(z,\bar{z}) := \frac{1 - g(z)\overline{g(z)}}{1 - z\bar{z}} \tag{6.22}$$

for every $z \in \mathbb{D}$. However, if $z \in \mathbb{D} \cap \overline{U_\rho}$, then due to symmetry relation (6.21), this function can be represented as

$$C_{00}(z,\bar{z}) = \frac{g(z) - g(1/\bar{z})}{z - \frac{1}{\bar{z}}} \cdot \frac{\overline{g(z)}}{\bar{z}} = \Phi(z, 1/\bar{z}) \cdot \frac{\overline{g(z)}}{\bar{z}} \tag{6.23}$$

where Φ is given in (6.19). Formula (6.23) makes sense on the whole $\overline{U_\rho}$ and thus, $C_{00}(z,\bar{z})$ is now well defined on $\mathbb{D} \cup \overline{U_\rho}$ (it is defined on $\mathbb{D}$ and on $\overline{U_\rho}$ by formulas (6.22) and (6.23), respectively, and these formulas coincide on $\mathbb{D} \cap \overline{U_\rho}$).

We define now the Carathéodory matrix $C_n(g;\, z)$ for every $z \in \mathbb{D} \cup \overline{U_\rho}$ as the matrix with the ij-th entry

$$C_{ij}(z,\bar{z}) = \frac{1}{i!\,j!}\frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} C_{00}(z,\bar{z}). \tag{6.24}$$

It is readily seen from (6.22) that this definition coincides with (4.1) for every $z \in \mathbb{D}$. Let us show that C_{ij} is well defined on $\mathbb{D} \cup \overline{U_\rho}$.

For $z \in \mathbb{D}$ we have, by the Leibnitz's rule,

$$\begin{aligned} C_{ij}(z,\bar{z}) &= \frac{1}{i!\,j!} u_{ij}(z,\bar{z}) \cdot (1-|g(z)|^2) \\ &\quad -\frac{1}{i!\,j!} \sum_{\substack{k=0 \\ k+\ell<}}^{i} \sum_{\substack{\ell=0 \\ i+j}}^{j} \begin{pmatrix} i \\ k \end{pmatrix} \begin{pmatrix} j \\ \ell \end{pmatrix} u_{k\ell}(z,\bar{z}) g^{(i-k)}(z) \overline{g^{(j-\ell)}(z)} \end{aligned} \tag{6.25}$$

where

$$u_{k\ell}(z,\bar{z}) = \frac{\partial^{k+\ell}}{\partial z^k \partial \bar{z}^\ell} \, \frac{1}{1-z\bar{z}}.$$

For $z \in \overline{U_\rho}$, again using the Leibnitz's rule, we get from (6.23):

$$C_{ij}(z,\bar{z}) = \frac{1}{i!\,j!} \sum_{\ell=0}^{j} \begin{pmatrix} j \\ \ell \end{pmatrix} \left(\frac{\partial^{i+\ell}}{\partial z^i \partial \bar{z}^\ell} \, \Phi(z,1/\bar{z}) \right) \cdot \overline{\left(\frac{g(z)}{z}\right)^{(j-\ell)}}. \tag{6.26}$$

Since

$$\frac{\partial^\ell}{\partial \bar{z}^\ell} = \sum_{k=0}^{\ell} p_{k\ell}(1/\bar{z}) \, \frac{\partial^k}{\partial \left(\frac{1}{\bar{z}}\right)^k},$$

where $p_{0\ell} = \delta_{0\ell}$, the Kronecker symbol, and $p_{k\ell}$ is a polynomial of degree $k+\ell$ $(1 \le k \le \ell)$, it follows from (6.26) that

$$C_{ij}(z,\bar{z}) = \frac{1}{i!\,j!} \sum_{\ell=0}^{j} \sum_{k=0}^{\ell} \begin{pmatrix} j \\ \ell \end{pmatrix} p_{k\ell}(1/\bar{z}) \left(\frac{\partial^{i+k}}{\partial z^i \partial \left(\frac{1}{\bar{z}}\right)^k} \, \Phi(z,1/\bar{z}) \right) \cdot \overline{\left(\frac{g(z)}{z}\right)^{(j-\ell)}} \tag{6.27}$$

for $z \in \overline{U_\rho}$. It follows from (6.25) that $C_{ij}(z,\bar{z})$ is well defined and continuous on $\mathbb{D}$. The same conclusion for $\overline{U_\rho}$ follows from (6.27), by Proposition 6.2.

Theorem 6.4. *Let Ω be an open subset of $\mathbb{D}$ and let*

$$\overline{\Omega} \cap \mathbb{T} \neq \mathbb{T}. \tag{6.28}$$

Then the inclusion $\mathcal{B}_\kappa \subset \mathcal{P}_\kappa(\Omega)$ is proper.

Proof. Assumption (6.28) means that there exists a point $t_0 \in \mathbb{T}$ which does not belong to $\overline{\Omega}$. We consider the function

$$f_a(z) = \frac{S_a(z)}{B(z)}, \quad \text{where } S_a(z) = e^{a\frac{z+t_0}{z-t_0}} \text{ and } B(z) = z^\kappa \tag{6.29}$$

and show that it belongs to $\mathcal{P}_\kappa(\Omega) \setminus \mathcal{B}_\kappa$ for an appropriate choice of $a > 0$. Since S_a in (6.29) is a Schur function and $S_a(0) = e^{-a} \neq 0$, the function f_a belongs to the class $\mathcal{S}_\kappa$, by the Krein–Langer characterization. On the other hand, $f_a \notin \mathcal{B}_\kappa$, since S_a is a singular inner function (with the mass a at the point $z = t_0$). We shall show that there exists $a > 0$ such that the function f_a of the form (6.29) belongs to $\mathcal{P}_\kappa(\Omega)$. First we note that since Ω is open and $t_0 \notin \overline{\Omega}$, there is a neighborhood of t_0 disjoint with $\overline{\Omega}$. This neighborhood can be chosen to be of the form

$$V_{\rho,\alpha}(t_0) = \left\{ z \in \mathbb{C} : \ \rho < |z| < \frac{1}{\rho}, \ \ |\arg z - \arg t_0| < \alpha \right\} \quad (0 < \rho < 1, \ \ \alpha > 0)$$

for an appropriate choice of ρ and α. Define

$$U_{\rho,\alpha} := \left\{ z \in \mathbb{C} : \ \rho < |z| < \frac{1}{\rho} \right\} \setminus V_{\rho,\alpha}(t_0)$$

and let $\mathbb{D}_\rho$ denote the disk of radius ρ centered at the origin. Define the domain

$$\Omega_1 := \overline{\mathbb{D}_\rho} \cup \overline{U_{\rho,\alpha}} = \overline{\mathbb{D}_{1/\rho} \setminus V_{\rho,\alpha}}. \tag{6.30}$$

It is clear that $\Omega \subseteq \Omega_1 \cap \mathbb{D}$ and therefore, $\mathcal{P}_\kappa(\Omega_1 \cap \mathbb{D}) \subseteq \mathcal{P}_\kappa(\Omega)$ (this follows directly from the definition of the class $\mathcal{P}_\kappa(\Omega)$). Thus, to complete the proof, it suffices to show that the function f_a of the form (6.29) belongs to $\mathcal{P}_\kappa(\Omega_1 \cap \mathbb{D})$ for an appropriate choice of $a > 0$. This verification is broken into three steps.

Step 1. *There exists* $\delta > 0$ *such that*

$$C_\kappa(B;\, z) > \delta I_\kappa \quad \text{for every } \ z \in \overline{\mathbb{D}}. \tag{6.31}$$

Proof of Step 1. Since $B(z) = z^\kappa$, we have

$$\frac{1 - B(z)\overline{B(z)}}{1 - z\bar{z}} = \sum_{\ell=0}^{\kappa-1} z^\ell \bar{z}^\ell$$

and subsequently,

$$\begin{aligned} \frac{1}{i!\, j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \left(\frac{1 - B(z)\overline{B(z)}}{1 - z\bar{z}} \right) &= \sum_{\ell=\max(i,j)}^{\kappa-1} \begin{pmatrix} \ell \\ i \end{pmatrix} \begin{pmatrix} \ell \\ j \end{pmatrix} z^{\ell-i} \cdot \bar{z}^{\ell-j} \\ &= \sum_{\ell=0}^{\kappa-1} R_{i\ell} \overline{R_{j\ell}} \end{aligned}$$

for $i, j = 0, \ldots, \kappa - 1$, where

$$R_{i\ell} = \begin{cases} \begin{pmatrix} \ell \\ i \end{pmatrix} z^{\ell-i} & \text{if} \quad i \leq \ell, \\ \quad 0 & \text{if} \quad i > \ell. \end{cases}$$

Now it follows from the definition (4.1) of the Carathéodory matrix that

$$C_\kappa(B;\, z) = RR^*,$$

where R is the upper triangular matrix with the entries $R_{i\ell}$. Since $R_{ii} = 1$ for $i = 1, \ldots, \kappa - 1$, it follows that R is not singular and therefore the matrix $C_\kappa(B;\, z)$ is positive definite for every $z \in \mathbb{C}$. The above representation also implies that $C_\kappa(B;\, z)$ depends continuously on $z \in \mathbb{C}$. The rest follows by the standard compactness argument.

Step 2. *Let Ω_1 be the domain defined in* (6.30). *Then for every $\varepsilon > 0$ there exists $a_\varepsilon > 0$ such that*

$$\|C_\kappa(S_a;\ z)\| \le \varepsilon \quad \textit{for every } a < a_\varepsilon \textit{ and } z \in \Omega_1, \tag{6.32}$$

where S_a is the inner Schur function defined in (6.29).

Proof of Step 2. It suffices to show that the entries $C_{ij}(z, \bar{z})$ of the Carathéodory matrix $C_\kappa(S_a;\ z)$ tend to zero uniformly on Ω_1 as a tends to zero. To this end, we first note that since $\frac{\omega+t_0}{\omega-t_0}$ is bounded on Ω_1, it follows from the formula for definition of S_a that

$$\lim_{a\to 0} S_a(z) = 1 \quad \text{and} \quad \lim_{a\to 0} S_a^{(j)}(z) = 0 \quad \text{for } \ j \ge 1 \tag{6.33}$$

and convergences are uniform on Ω_1. Now the uniform estimate (6.32) follows for $\mathbb{D}_\rho$ from (6.25) and (6.33) and for $\overline{U_{\rho,\alpha}}$ it follows from (6.27) and (6.33) by Corollary 6.3. It is worthwhile to note that formula (6.27) is relevant since S_a is analytic and unimodular on $U_{\rho,\alpha} \cap \mathbb{T}$.

Step 3. Combining (6.31) and (6.32) we conclude that there exists $a > 0$ such that for every point $\omega \in \overline{\Omega}$,

$$C_\kappa(S_a;\ z) - C_\kappa(B;\ z) < 0.$$

By Remark 4.6, the latter difference is congruent to the Carathéodory matrix $C_\kappa(\frac{S_a}{B};\ z)$ of the ratio of S_a and B, that is, of f_a. Thus, there exists $a > 0$ such that for every point $z \in \overline{\Omega}$, the Carathéodory matrix $C_\kappa(f_a;\ z)$ is negative definite. Now we repeat the compactness arguments from the proof of Theorem 1.2 in Section 5 to conclude that there exists an integer n such that

$$\text{sq}\_P_n(f_a;\ z_1, \ldots, z_n) = \kappa \tag{6.34}$$

for every choice of n points $z_1, \ldots, z_n \in \Omega$. Thus, f_a belongs to $\mathcal{P}_\kappa(\Omega)$. $\square$

7. Concluding remarks

The problems discussed in the paper can be put in a more abstract framework as the following question.

Question 4: Given a Pontryagin space H, $\text{sq}\_H = \kappa$ and a family $\mathcal{G}$ of finite dimensional subspaces of H. Determine whether or not the following is true: there exists a positive integer n_0 such that for every $G \in \mathcal{G}$ with $\dim G \ge n_0$ it holds that $\text{sq}\_G = \kappa$.

Apparently the answer depends on how the subspaces in $\mathcal{G}$ are located relative to the negative cone in H. For instance, if $\mathcal{G}$ is a family of all finite dimensional subspaces of H and $\dim H = \infty$ then the answer is negative. If $\mathcal{G}$ is the family of all finite dimensional subspaces that contain a fixed maximal negative subspace then the answer is affirmative. We do not know how to approach this problem and what are the interesting families $\mathcal{G}$ to be considered.

In the present paper $\mathcal{G}$ can be viewed as a family of subspaces spanned by the reproducing kernels associated with an $\mathcal{S}_\kappa$ class function f for Ω a subset of D. Theorems 1.1, 1.2, 6.1, 6.4 give affirmative or negative answers depending on f and Ω. Question 2 addresses one more situation that we do not know the answer.

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