Integr. equ. oper. theory 62 (2008), 301–349
0378-620X/030301-49, *published online* October 8, 2008
DOI 10.1007/s00020-008-1626-1

Integral Equations and Operator Theory

Interpolation Problems for Schur Multipliers on the Drury-Arveson Space: from Nevanlinna-Pick to Abstract Interpolation Problem

Joseph A. Ball and Vladimir Bolotnikov

Abstract. We survey various increasingly more general operator-theoretic formulations of generalized left-tangential Nevanlinna-Pick interpolation for Schur multipliers on the Drury-Arveson space. An adaptation of the methods of Potapov and Dym leads to a chain-matrix linear-fractional parametrization for the set of all solutions for all but the last of the formulations for the case where the Pick operator is invertible. The last formulation is a multivariable analogue of the Abstract Interpolation Problem formulated by Katsnelson, Kheifets and Yuditskii for the single-variable case; we obtain a Redheffer-type linear-fractional parametrization for the set of all solutions (including in degenerate cases) via an adaptation of ideas of Arov and Grossman.

Mathematics Subject Classification (2000). Primary 47A57; Secondary 47A48, 47B32, 47B50, 32A05.

Keywords. Abstract Interpolation Problem, linear-fractional map, Stein equation, left tangential interpolation with operator argument.

1. Introduction

A multivariable generalization of the Szegö kernel $k(x,y) = (1 - x\overline{y})^{-1}$ is the positive kernel

$$k_d(z,\zeta) = \frac{1}{1 - \langle z,\, \zeta\rangle}$$

on $\mathbb{B}^d \times \mathbb{B}^d$ where $\mathbb{B}^d = \left\{z = (z_1, \dots, z_d) \in \mathbb{C}^d \colon \langle z, z\rangle < 1\right\}$ is the unit ball of the d-dimensional Euclidean space $\mathbb{C}^d$. By $\langle z, \zeta\rangle = \sum_{j=1}^d z_j\overline{\zeta}_j$ we mean the standard inner product in $\mathbb{C}^d$. The reproducing kernel Hilbert space (RKHS) $\mathcal{H}(k_d)$ associated with k_d via Aronszajn's construction [8] is a natural multivariable analogue of the Hardy space H^2 of the unit disk and coincides with H^2 if $d = 1$. In what follows, the symbol $\mathcal{L}(\mathcal{U},\mathcal{Y})$ stands for the algebra of bounded linear operators mapping $\mathcal{U}$ into $\mathcal{Y}$, and we abbreviate $\mathcal{L}(\mathcal{U},\mathcal{U})$ to $\mathcal{L}(\mathcal{U})$. For $\mathcal{Y}$ an auxiliary Hilbert space, we consider the tensor product Hilbert space $\mathcal{H}_{\mathcal{Y}}(k_d) := \mathcal{H}(k_d)\otimes\mathcal{Y}$ whose elements can be viewed as $\mathcal{Y}$-valued functions in $\mathcal{H}(k_d)$. Then $\mathcal{H}_{\mathcal{Y}}(k_d)$ can be characterized as follows:

$$\mathcal{H}_{\mathcal{Y}}(k_d) = \left\{ f(z) = \sum_{\mathbf{n}\in\mathbb{Z}_+^d} f_{\mathbf{n}} z^{\mathbf{n}} : \|f\|^2 = \sum_{\mathbf{n}\in\mathbb{Z}_+^d} \frac{\mathbf{n}!}{|\mathbf{n}|!}\cdot\|f_{\mathbf{n}}\|_{\mathcal{Y}}^2 < \infty \right\}. \tag{1.1}$$

Here and in what follows, we use standard multivariable notations: for multi-integers $\mathbf{n} = (n_1,\dots,n_d)\in\mathbb{Z}_+^d$ and points $z = (z_1,\dots,z_d)\in\mathbb{C}^d$ we set

$$|\mathbf{n}| = n_1+n_2+\dots+n_d, \qquad \mathbf{n}! = n_1!n_2!\dots n_d!, \qquad z^{\mathbf{n}} = z_1^{n_1}z_2^{n_2}\dots z_d^{n_d}.$$

We will be particularly interested in the space of multipliers $\mathcal{M}_d(\mathcal{U},\mathcal{Y})$ defined as the space of all $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued analytic functions $z\mapsto F(z)$ on $\mathbb{B}^d$ such that the multiplication operator $M_F\colon f(z)\mapsto F(z)f(z)$ maps $\mathcal{H}_{\mathcal{U}}(k_d)$ into $\mathcal{H}_{\mathcal{Y}}(k_d)$. It follows by the closed graph theorem that for every $F\in\mathcal{M}_d(\mathcal{U},\mathcal{Y})$, the operator M_F is bounded. We denote by $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ the unit ball of $\mathcal{M}_d(\mathcal{U},\mathcal{Y})$:

$$\mathcal{S}_d(\mathcal{U},\mathcal{Y}) = \{S\in\mathcal{M}_d(\mathcal{U},\mathcal{Y})\colon\ \|M_S\|_{\rm op}\le 1\}.$$

We let $\mathbf{M}_{\boldsymbol{z}}$ denote the commuting d-tuple $\mathbf{M}_{\boldsymbol{z}} := (M_{z_1},\dots,M_{z_d})$ consisting of operators of multiplication by the coordinate functions of $\mathbb{C}^d$ on $\mathcal{H}(k_d)$ (called *the shift* (operator-tuple) of $\mathcal{H}_{\mathcal{Y}}(k_d)$), whereas we refer to the commuting d-tuple $\mathbf{M}_{\boldsymbol{z}}^* := (M_{z_1}^*,\dots,M_{z_d}^*)$ consisting of the adjoints of M_{z_j}'s (in the metric of $\mathcal{H}(k_d)$) as the *backward shift*. Then the space $\mathcal{M}_d(\mathcal{U},\mathcal{Y})$ can be characterized as those elements R of $\mathcal{L}(\mathcal{H}_{\mathcal{U}}(k_d),\mathcal{H}_{\mathcal{Y}}(k_d))$ which intertwine the shifts of $\mathcal{H}_{\mathcal{U}}(k_d)$ and $\mathcal{H}_{\mathcal{Y}}(k_d)$ (i.e., such that $M_{z_j}R = RM_{z_j}$ for $j=1,\dots,d$); if such an R is a contraction, then $R = M_S$ for some $S\in\mathcal{S}_d(\mathcal{U},\mathcal{Y})$. The simplest generalization of the classical Nevanlinna-Pick interpolation problem for the operator-valued case is:

Operator-valued Nevanlinna-Pick problem (see [7, 30, 2, 26]): *Given points* $\zeta^{(i)} = (\zeta_1^{(i)},\dots,\zeta_d^{(i)})$ *in* $\mathbb{B}^d$ *and operators* $X_i\in\mathcal{L}(\mathcal{U},\mathcal{Y})$ *for* $i=1,\dots,N$, *find* $S\in\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ *so that*

$$S(\zeta^{(i)}) = X_i \quad \textit{for} \quad i=1,\dots,N. \tag{1.2}$$

If S is any multiplier in $\mathcal{M}_d(\mathcal{U},\mathcal{Y})$, then an easy reproducing-kernel-space computation shows that

$$M_S^* k_{d,\zeta}\otimes y = k_{d,\zeta}\otimes S(\zeta)^* y \quad \text{for} \quad \zeta\in\mathbb{B}^d \text{ and } y\in\mathcal{Y},$$

where $k_{d,\zeta}(z) := k_d(z,\zeta)$ so that $k_{d,\zeta} \otimes y \in \mathcal{H}_{\mathcal{Y}}(k_d)$ and $k_{d,\zeta} \otimes S(\zeta)^* y \in \mathcal{H}_{\mathcal{U}}(k_d)$. Hence, if $S \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$ satisfies (1.2), then we see that

$$\sum_{i,j=1}^{N} \left\langle \frac{I_{\mathcal{Y}} - X_i X_j^*}{1 - \langle \zeta^{(i)}, \zeta^{(j)} \rangle} y_j, y_i \right\rangle$$
$$= \| \sum_{j=1}^{N} k_{d,\zeta_j} \otimes y_j \|^2_{\mathcal{H}_{\mathcal{Y}}(k_d)} - \| M_S^* \sum_{j=1}^{N} k_{d,\zeta_j} \otimes y_j \|^2_{\mathcal{H}_{\mathcal{U}}(k_d)} \geq 0$$

for any finite collection $y_1, \dots, y_N$ of vectors in $\mathcal{Y}$, and hence we see that a necessary condition for solutions to exist is that the Pick matrix $\left[\frac{I_{\mathcal{Y}} - X_i X_j^*}{1 - \langle \zeta_i, \zeta_j \rangle} \right]_{i,j=1}^{N}$ be positive semidefinite. In view of the fact that the Drury-Arveson kernel k_d is a *complete Pick kernel*, it follows that this necessary condition is also sufficient (see e.g. [2]).

A more natural problem for the vector-valued setting is the following more general form of the operator-valued Nevanlinna-Pick problem:

Left-tangential Nevanlinna-Pick interpolation problem (LNPP) (see [26, 52, 53]: *Given points* $\zeta^{(i)} = (\zeta_1^{(i)}, \dots, \zeta_d^{(i)}) \in \mathbb{B}^d$ *and vectors* $a_i \in \mathcal{Y}$ *and* $c_i \in \mathcal{U}$ *for* $i = 1, \dots, N$, *find all functions* $S \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$ *such that*

$$a_i^* S(\zeta^{(i)}) = c_i^* \quad \text{for} \quad i = 1, \dots, N. \tag{1.3}$$

There are various ways to study higher multiplicity versions of **LNPP**; three of which which we study here we call (1) the *Sarason Interpolation Problem* (**SIP**), (2) the *Commutant Lifting Problem* (**CLP**), and (3) the *strongly stable Operator Argument Problem* (**ssOAP**). For example, for the case of **ssOAP**, the data set consists of a set of the form $(\mathbf{T}, E, N)$ where $\mathbf{T} = (T_1, \dots, T_d)$ is a commutative d-tuple of operators on a state space $\mathcal{X}$ satisfying a strong stability hypothesis, $E \colon \mathcal{X} \to \mathcal{Y}$ and $N \colon \mathcal{X} \to \mathcal{U}$ are output operators such that the associated observability operators $\mathcal{O}_{E,\mathbf{T}} \colon \mathcal{X} \to \mathcal{H}_{\mathcal{Y}}(k_d)$ and $\mathcal{O}_{N,\mathbf{T}} \colon \mathcal{X} \to \mathcal{H}_{\mathcal{U}}(k_d)$ make sense, and the interpolation conditions on a Schur multiplier S assume the compact form

$$\mathcal{O}_{E,\mathbf{T}}^* M_S = \mathcal{O}_{N,\mathbf{T}}^*. \tag{1.4}$$

One way to study these problems is to study instead the corresponding problem on the asymmetric Fock space $\ell^2(\mathcal{F}_d)$ (where $\mathcal{F}_d$ is the free semigroup on d letters)—see [49, 50, 30, 7, 51, 52, 53, 17], and then use the result of [7, 30] (see also [18]) that the Drury-Arveson multiplier space is exactly the image of the free-semigroup algebra after applying a point-evaluation map associated with points in the unit ball $\mathbb{B}^d$; indeed many of the results concerning this problem were first arrived at in this way. It is also possible to study these problems directly, as in [2, 26, 3, 5, 6]. While the work in [50, 7, 51, 52, 53] relies on Commutant Lifting techniques, that of [30] is based on the original duality/Fejér-Riesz-type factorization approach of Sarason [55], that of [2, 26] on the "lurking isometry" technique, and that of [5, 6] on an adaptation of the Schur algorithm. It is also possible to approach these problems by using the grassmannian Kreĭn-space approach of Ball-Helton (see [35]). A broad survey of multivariable interpolation problems going beyond the context of the Drury-Arveson space is given in [23].

In the present paper, we review the well-known solution criteria for these problems and how all these problems can be considered as mutually equivalent once one understands how to transform an admissible data set of one type to admissible data sets of each of the other two types; for the one-variable case these ideas are addressed in detail in the books [21, 36, 37]. We obtain a chain-matrix type linear-fractional parametrization for the set of all solutions of such a problem for the case where the Pick operator is strictly positive definite by adapting the methods of Dym [33] and Potapov [45]. That these ideas can be adapted to this multivariable setting was already observed by the second author in [28]. We mention that recently Popescu (see [53, Theroem 2.3]) has obtained a parametrization formula for the set of all solutions in the context of the noncommutative commutant lifting theorem of Popescu [49] as well as a maximum entropy principle for this context (see Theorem 2.8 there). By the symmetrization technique mentioned above, this result in principle gives a parametrization for the set of solutions of interpolation problems discussed here, once one transforms the interpolation data to commutant-lifting data.

We mention that the more complicated bitangential problem is studied in [13, 14] as well as in [16, 15] in a more general setting; when specialized to the left-tangential case, the analysis there gives a Redheffer linear-fractional parametrization for a particular case of the **ssOAP**-type problem (where it is assumed that the joint spectrum of $\mathbf{T}$ is contained in the open unit ball from which it follows in particular that $\mathbf{T}$ is strongly stable).

It turns out that the problem **ssOAP** still makes sense without the *strongly stable* hypothesis; we refer to a problem of this type as a (not necessarily strongly stable) Operator Argument interpolation Problem **OAP**. The same methodology used here for **ssOAP** applies equally well to the **OAP** case; the associated reproducing kernel Hilbert space $\mathcal{H}(K_{\mathfrak{A}}^{\mathbf{J},J})$ (see Theorem 2.3 below) constructed from the interpolation data is contained contractively rather than isometrically in the ambient Drury-Arveson Kreĭn space.

We also introduce here a still more general interpolation problem, called the *analytic Abstract Interpolation Problem* (**aAIP**), where the assumption that the observability operators $\mathcal{O}_{E,\mathbf{T}}$ and $\mathcal{O}_{N,\mathbf{T}}$ map into the Drury-Arveson space is removed. Instead, in this more general formulation it is only assumed that the $\mathcal{O}_{E,\mathbf{T}}$ and $\mathcal{O}_{N,\mathbf{T}}$ map the state space into holomorphic vector-valued functions on the ball. For this setting the formulation (1.4) of the interpolation conditions does not make sense and one uses instead the following formulation: *a Schur-multiplier S is said to solve the interpolation problem* **aAIP** *if the operator* $F^S = \begin{bmatrix} I & -M_S \end{bmatrix} \begin{bmatrix} \mathcal{O}_{E,\mathbf{T}} \\ \mathcal{O}_{N,\mathbf{T}} \end{bmatrix}$ *maps the state space into the de Branges-Rovnyak space $\mathcal{H}(K_S)$ associated with S.* We show that the same solution procedure as for the previous cases still applies, despite the fact that the associated reproducing kernel Hilbert space $\mathcal{H}(K_{\mathfrak{A}}^{\mathbf{J},J})$ is no longer contained in the Drury-Arveson space. It was shown in [27] that the boundary Nevanlinna-Pick problem on the ball is a particular instance of this problem **aAIP** which does not fit into the **OAP** framework. The boundary Nevanlinna-Pick problem for the Drury-Arveson space setting has also been studied in [4, 14].

Our final interpolation problem is a more implicit version of **aAIP** which we call the Abstract Interpolation Problem (**AIP**). The problem formulation calls for finding not only a Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ but also a map F from the part of the state space specified by the problem data into the de Branges-Rovnyak space $\mathcal{H}(K_S)$. When specialized to the single-variable case $d = 1$, this problem coincides with a particular case (the "left-sided version") of the Abstract Interpolation Problem introduced by Katsnelson, Kheifets and Yuditskii [40] and further studied in [41, 42, 43, 44, 46, 25]. We show how the approach of Arov-Grossman [10, 11] can be used to obtain a Redheffer-type linear-fractional description for the set of all solution pairs (F, S). This approach has already been used in other several-variable contexts in [13, 14, 15].

The paper is organized as follows. Section 2 collects various preliminary material on Schur multipliers, reproducing kernel Hilbert spaces and linear-fractional maps which will be needed in the sequel. Section 3 studies the three problems **SIP**, **CLP** and **ssOAP** and lays out the well-known solution criteria and the data manipulations showing the mutual equivalences among them. Section 4 extends the theory to the setting of **OAP**. Finally Section 5 studies the analytic Abstract Interpolation Problem and Section 6 handles the most general version of our problems, the Abstract Interpolation Problem. Section 3 is essentially a review of known material to set the context while Sections 4, 5, 6 present new results on more general types of interpolation problems for the Drury-Arveson Schur-multiplier class.

2. Schur multipliers, reproducing kernel Hilbert spaces, and linear-fractional transformations

In this section we collect miscellaneous preliminary results needed for the work in the sequel. The following result appears in [26, 2, 34].

Theorem 2.1. *Let S be a $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function analytic in $\mathbb{B}^d$. The following are equivalent:*

1. *S belongs to $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$.*
2. *The kernel $K_S(z, \zeta) = \dfrac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1 - \langle z, \zeta \rangle}$ is positive definite on $\mathbb{B}^d \times \mathbb{B}^d$ or equivalently, there exists an auxiliary Hilbert space $\mathcal{H}$ and an analytic $\mathcal{L}(\mathcal{H}, \mathcal{Y})$-valued function $H(z)$ on $\mathbb{B}^d$ so that*

$$\frac{I - S(z)S(\zeta)^*}{1 - \langle z, \zeta \rangle} = H(z)H(\zeta)^*. \tag{2.1}$$

3. *There is a unitary operator*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}^d \\ \mathcal{Y} \end{bmatrix} \tag{2.2}$$

such that

$$S(z) = D + C\left(I_{\mathcal{H}} - Z(z)A\right)^{-1} Z(z)B, \tag{2.3}$$

where

$$Z(z) = \begin{bmatrix} z_1 I_{\mathcal{H}} & \cdots & z_d I_{\mathcal{H}} \end{bmatrix}. \tag{2.4}$$

The representation (2.3) is called *a unitary realization* of $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. More formally, we shall say that a collection $\mathcal{C}$ of the form $\mathcal{C} = \{\mathcal{H},\ \mathcal{U},\ \mathcal{Y},\ \mathbf{U}\}$ with $\mathbf{U}$ of the form (2.2) is a (d-variable) *colligation* with connecting operator equal to $\mathbf{U}$ and with associated *characteristic function* equal to $S(z)$ given by (2.3). If $\mathbf{U}$ is unitary, we say that the colligation is unitary. We say that the unitary colligation $\mathcal{C}$ is *closely connected* if the smallest subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that $\mathcal{H}_0$ is reducing for $A_1, \dots, A_d$ and $\mathcal{H}_0$ contains $\operatorname{Ran} C^*$ as well as $\operatorname{Ran} B_j$ for $j = 1, \dots, d$ is the whole space $\mathcal{H}$. In case $\mathcal{H}_0$ is not the whole space $\mathcal{H}$, then

$$\mathbf{U}_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D \end{bmatrix} = \begin{bmatrix} A|_{\mathcal{H}_0} & B \\ C|_{\mathcal{H}_0} & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_0^d \\ \mathcal{Y} \end{bmatrix}$$

is again unitary and gives rise to the same characteristic function $S(z)$. Hence there is no loss of generality in Theorem 2.1 if we assume that the unitary colligation $\mathcal{C}$ with connecting operator $\mathbf{U}$ in (2.2) is closely connected.
Note that for S of the form (2.3), relation (2.1) holds with

$$H(z) = C\left(I_{\mathcal{H}} - Z(z)A\right)^{-1}. \tag{2.5}$$

Note also that formulas (2.3) and (2.5) can be written directly in terms of the unitary operator $\mathbf{U}$ as follows:

$$S(z) = \mathcal{P}_{\mathcal{Y}} \mathbf{U} \left(I_{\mathcal{H}\oplus\mathcal{U}} - \mathcal{P}_{\mathcal{H}}^* Z(z) \mathcal{P}_{\mathcal{H}^d} \mathbf{U}\right)^{-1} |_{\mathcal{U}}, \tag{2.6}$$

$$H(z) = \mathcal{P}_{\mathcal{Y}} \mathbf{U} \left(I_{\mathcal{H}\oplus\mathcal{U}} - \mathcal{P}_{\mathcal{H}}^* Z(z) \mathcal{P}_{\mathcal{H}^d} \mathbf{U}\right)^{-1} |_{\mathcal{H}}, \tag{2.7}$$

where $\mathcal{P}_{\mathcal{Y}}$ and $\mathcal{P}_{\mathcal{H}^d}$ are the orthogonal projections of the space $\mathcal{H}^d \oplus \mathcal{Y}$ onto $\mathcal{Y}$ and $\mathcal{H}^d$, respectively, and $P_{\mathcal{H}}^*$ is the inclusion map of $\mathcal{H}$ into $\mathcal{H} \oplus \mathcal{U}$.

Associated with any $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is the *de Branges-Rovnyak space* $\mathcal{H}(K_S)$, the reproducing kernel Hilbert space with reproducing kernel K_S (which is positive by Theorem 2.1). The original characterization of $\mathcal{H}(K_S)$, as the space of all functions $f \in \mathcal{H}_{\mathcal{Y}}(k_d)$ such that

$$\|f\|^2_{\mathcal{H}(K_S)} := \sup_{g \in \mathcal{H}_{\mathcal{U}}(k_d)} \left\{ \|f + Sg\|^2_{\mathcal{H}_{\mathcal{Y}}(k_d)} - \|g\|^2_{\mathcal{H}_{\mathcal{U}}(k_d)} \right\} < \infty, \tag{2.8}$$

is due to de Branges and Rovnyak [29] (for the case $d = 1$). In particular, it follows from (2.8) that $\|f\|_{\mathcal{H}(K_S)} \geq \|f\|_{\mathcal{H}_{\mathcal{Y}}(k_d)}$ for every $f \in \mathcal{H}(K_S)$, i.e., that $\mathcal{H}(K_S)$ is contained in $\mathcal{H}_{\mathcal{Y}}(k_d)$ contractively. On the other hand, the general complementation theory applied to the contractive operator M_S provides the characterization of $\mathcal{H}(K_S)$ as the operator range

$$\mathcal{H}(K_S) = \operatorname{Ran}(I - M_S M_S^*)^{\frac{1}{2}} \tag{2.9}$$

with the lifted norm

$$\|(I - M_S M_S^*)^{\frac{1}{2}} f\|_{\mathcal{H}(K_S)} = \|(I - \pi) f\|_{\mathcal{H}_{\mathcal{Y}}(k_d)} \tag{2.10}$$

for all $f \in \mathcal{H}_{\mathcal{Y}}(k_d)$ where π here is the orthogonal projection onto $\operatorname{Ker}(I - M_S M_S^*)^{\frac{1}{2}}$. Upon setting $f = (I - M_S M_S^*)^{\frac{1}{2}} h$ in (2.10) we get

$$\|(I - M_S M_S^*) h\|_{\mathcal{H}(K_S)} = \langle (I - M_S M_S^*) h,\, h \rangle_{\mathcal{H}_{\mathcal{Y}}(k_d)}. \tag{2.11}$$

More complete details concerning the spaces $\mathcal{H}(K_S)$ and related matters of realization and the model theory for commutative row contractions can be found in the recent series of papers [18, 19, 20].

We shall also have use of the Beurling-Lax theorem for the Drury-Arveson space. The following definition of *inner multiplier* is useful.

Definition 2.2. *A contractive multiplier $\Theta \in \mathcal{S}_d(\mathcal{F}, \mathcal{Y})$ is called inner if the multiplication operator $M_\Theta : \mathcal{H}_{\mathcal{F}}(k_d) \to \mathcal{H}_{\mathcal{Y}}(k_d)$ is a partial isometry.*

If Θ is inner, then the associated de Branges-Rovnyak space $\mathcal{H}(K_\Theta)$ is isometrically included in $\mathcal{H}_{\mathcal{Y}}(k_d)$ and

$$\mathcal{H}(K_\Theta) = \mathcal{H}_{\mathcal{Y}}(k_d) \ominus \Theta \mathcal{H}_{\mathcal{F}}(k_d). \tag{2.12}$$

Moreover, the orthogonal projection $\mathcal{P}_{\mathcal{H}(K_\Theta)}$ of $\mathcal{H}_{\mathcal{Y}}(k_d)$ onto $\mathcal{H}(K_\Theta)$ is given by

$$\mathcal{P}_{\mathcal{H}(K_\Theta)} = I_{\mathcal{H}_{\mathcal{Y}}(k_d)} - M_\Theta M_\Theta^*. \tag{2.13}$$

Since the space $\Theta \mathcal{H}_{\mathcal{F}}(k_d)$ is shift invariant (i.e., M_{z_j}-invariant for $j = 1, \dots, d$), it follows from (2.12) that the space $\mathcal{H}(K_\Theta)$ is backward shift invariant (i.e., $M_{z_j}^*$-invariant for $j = 1, \dots, d$). The Beurling-Lax theorem for $\mathcal{H}_{\mathcal{Y}}(k_d)$ (see [12, 48, 18, 20] for the commutative setting and [50, 31, 18] for the noncommutative setting from which the commutative setting can be derived) asserts that *any shift invariant closed subspace $\mathcal{M}$ of $\mathcal{H}_{\mathcal{Y}}(k_d)$ necessarily has the form $\Theta \mathcal{H}_{\mathcal{F}}(k_d)$ for some inner multiplier $\Theta \in \mathcal{S}_d(\mathcal{F}, \mathcal{Y})$*; in this situation we say that Θ is a *Beurling-Lax representer* for the shift-invariant subspace $\mathcal{M}$. Therefore any backward-shift-invariant subspace $\mathcal{M}$ of $\mathcal{H}_{\mathcal{Y}}(k_d)$ has the form $\mathcal{M} = \mathcal{H}(K_\Theta)$.

It is convenient to introduce the following noncommutative multivariable functional-calculus notation from [18] even though here we are only interested in the commutative setting. We let $\mathcal{F}_d$ denote the free semigroup generated by the alphabet consisting of the letters $\{1, \dots, d\}$. Elements of $\mathcal{F}_d$ are words $v = i_N \cdots i_1$ where each $i_k \in \{1, \dots, d\}$. Given such a word $v = i_N \cdots i_1 \in \mathcal{F}_d$, we let $|v| = N$ denote the *length* of the word (i.e., the number of letters in v) and we let $\mathbf{a}(v) \in \mathbb{Z}_+^d$

be the *abelianization* of v, i.e., the d-tuple $(n_1, \dots, n_d)$ of nonnegative integers determined by

$$\mathbf{a}(v) = (n_1, \dots, n_d) \quad \text{where} \quad n_j = \#\{k\colon i_k = j\} \quad \text{for} \quad j = 1, \dots, d$$

(where in general $\#\Xi$ denotes the cardinality of the set Ξ). If $\mathbf{T} = (T_1, \dots, T_d)$ is a d-tuple of Hilbert-space operators and if $z = (z_1, \dots, z_d)$ is a collection of complex variables, we use the standard multivariable notation:

$$T^v = T_{i_N} \cdots T_{i_1}, \qquad z^v = z_{i_N} \cdots z_{i_1}$$

if $v = i_N \cdots i_1$. If $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$ and if $\mathbf{T}$ is a commutative operator d-tuple, we write

$$\mathbf{T}^{\mathbf{n}} = T_1^{n_1} \cdots T_d^{n_d}, \qquad z^{\mathbf{n}} = z_1^{n_1} \cdots z_d^{n_d}.$$

If $v \in \mathcal{F}_d$ and $\mathbf{T}$ is a commutative operator d-tuple, we then have the following connections between the noncommutative and commutative multivariable functional calculus:

$$\mathbf{T}^v = \mathbf{T}^{\mathbf{a}(v)}, \qquad z^v = z^{\mathbf{a}(v)}.$$

A useful combinatorial fact is the following: for a given $\mathbf{n} \in \mathbb{Z}_+^d$,

$$\#\{v \in \mathcal{F}_d \colon \mathbf{a}(v) = \mathbf{n}\} = \frac{|\mathbf{n}|!}{\mathbf{n}!}.$$

We shall need some J-analogues of results concerning ranges of observability operators and associated reproducing kernel Hilbert spaces given in [18]. Let $\mathbf{T} = (T_1, \cdots, T_d)$ be a d-tuple of operators in $\mathcal{L}(\mathcal{X})$ and let $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The pair $(C, \mathbf{T})$ is said to be *output-stable* if the associated observability operator

$$\mathcal{O}_{C,\mathbf{T}} \colon x \mapsto C(I_{\mathcal{X}} - z_1 T_1 - \cdots - z_d T_d)^{-1} x = C(I_{\mathcal{X}} - Z(z)T)^{-1} x$$

where

$$T = \begin{bmatrix} T_1 \\ \vdots \\ T_d \end{bmatrix} : \mathcal{X} \to \mathcal{X}^d \quad \text{and} \quad Z(z) = \begin{bmatrix} z_1 I_{\mathcal{X}} & \cdots & z_d I_{\mathcal{X}} \end{bmatrix},$$

maps $\mathcal{X}$ into $\mathcal{H}_{\mathcal{Y}}(k_d)$ and is bounded, i.e., $\mathcal{O}_{C,\mathbf{T}} \in \mathcal{L}(\mathcal{X}, \mathcal{H}_{\mathcal{Y}}(k_d))$. To obtain the Taylor expansion for $\mathcal{O}_{C,\mathbf{T}} x \in \mathcal{H}_{\mathcal{Y}}(k_d)$, we compute:

$$\begin{aligned} (\mathcal{O}_{C,\mathbf{T}} x)(z) &= C(I_{\mathcal{X}} - Z(z)T)^{-1} x = \sum_{N=0}^{\infty} C(Z(z)T)^N x \\ &= C \sum_{N=0}^{\infty} \sum_{v \in \mathcal{F}_d \colon |v| = N} \mathbf{T}^v z^v x = C \sum_{v \in \mathcal{F}_d} \mathbf{T}^v z^v x \\ &= C \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \sum_{v \in \mathcal{F}_d \colon \mathbf{a}(v) = \mathbf{n}} \mathbf{T}^v z^v x = \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} C \mathbf{T}^{\mathbf{n}} z^{\mathbf{n}} x. \end{aligned} \tag{2.14}$$

We now introduce the *observability gramian*

$$\mathcal{G}_{C,\mathbf{T}} := \mathcal{O}_{C,\mathbf{T}}^* \mathcal{O}_{C,\mathbf{T}}$$

whose representation in terms of strongly convergent power series

$$\mathcal{G}_{C,\mathbf{T}} = \sum_{\mathbf{n}\in\mathbb{Z}_+^d} \frac{|\mathbf{n}|!}{\mathbf{n}!}\mathbf{T}^{\mathbf{n}*}C^*C\mathbf{T}^{\mathbf{n}} = \sum_{v\in\mathcal{F}_d} \mathbf{T}^{v*}C^*C\mathbf{T}^v \tag{2.15}$$

follows from the power-series expansion (2.14) for the observability operator together with the characterization (1.1) of the $\mathcal{H}_{\mathcal{Y}}(k_d)$-norm. An important property of $\mathcal{G}_{C,\mathbf{T}}$ is that it satisfies the Stein equation

$$P - \sum_{j=1}^d T_j^* P T_j = C^*C \tag{2.16}$$

as can be seen by plugging in the series expansion (2.15).
It is useful to identify the special case where in addition the commutative d-tuple is *strongly stable*, i.e.,

$$\lim_{N\to\infty} \sum_{v\in\mathcal{F}_d\colon |v|=N} \|\mathbf{T}^v x\|_{\mathcal{X}}^2 = \lim_{N\to\infty} \sum_{\mathbf{n}\in\mathbb{Z}_+^d\colon |\mathbf{n}|=N} \frac{N}{\mathbf{n}!}\|\mathbf{T}^{\mathbf{n}} x\|_{\mathcal{X}}^2 = 0 \quad \text{for all} \quad x\in\mathcal{X}. \tag{2.17}$$

We will consider an output-stable pair $(C,\mathbf{T})$ where

$$C = \begin{bmatrix} E \\ N \end{bmatrix} \colon \mathcal{X} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \tag{2.18}$$

and $\mathbf{T} = (T_1,\dots,T_d)$ is a commutative d-tuple of operators on $\mathcal{X}$. We let

$$J = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & -I_{\mathcal{U}} \end{bmatrix}. \tag{2.19}$$

In addition, we shall often have use for the operator $J\otimes I_{\mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(k_d)}$ acting on $\mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(k_d)$; we shall abuse notation and write this operator also as simply J. We define the J-gramian $\mathcal{G}_{C,\mathbf{T}}^J$ of the pair $(C,\mathbf{T})$ by

$$\mathcal{G}_{C,\mathbf{T}}^J := \mathcal{O}^*_{\left[\begin{smallmatrix} E \\ N\end{smallmatrix}\right],\mathbf{T}} J \mathcal{O}_{\left[\begin{smallmatrix} E \\ N\end{smallmatrix}\right],\mathbf{T}} = \mathcal{O}^*_{E,\mathbf{T}}\mathcal{O}_{E,\mathbf{T}} - \mathcal{O}^*_{N,\mathbf{T}}\mathcal{O}_{N,\mathbf{T}} = \mathcal{G}_{E,\mathbf{T}} - \mathcal{G}_{N,\mathbf{T}}. \tag{2.20}$$

An important property of $\mathcal{G}_{C,\mathbf{T}}^J$ is that it solves the Stein equation

$$P - \sum_{j=1}^d T_j^* P T_j = C^*JC, \tag{2.21}$$

as follows easily from the fact that $\mathcal{G}_{E,\mathbf{T}}$ and $\mathcal{G}_{N,\mathbf{T}}$ satisfy Stein equations of the type (2.16), or by plugging in the infinite series representations

$$\begin{aligned} \mathcal{G}_{C,\mathbf{T}}^J &= \mathcal{G}_{E,\mathbf{T}} - \mathcal{G}_{N,\mathbf{T}} \\ &= \sum_{v\in\mathcal{F}_d} \mathbf{T}^{v*}\left(E^*E - N^*N\right)\mathbf{T}^v = \sum_{v\in\mathcal{F}_d} \mathbf{T}^{v*}C^*JC\mathbf{T}^v \end{aligned} \tag{2.22}$$

for $\mathcal{G}_{C,\mathbf{T}}^J$.

If $\mathcal{X}$ is a Hilbert space and X is a selfadjoint operator operator on $\mathcal{X}$, we use the notation $(\mathcal{X}, X)$ to denote the space $\mathcal{X}$ with the indefinite inner product induced by X:

$$\langle x, y\rangle_X := \langle Xx, y\rangle_{\mathcal{X}}.$$

Usually it is assumed that X is invertible, so $(\mathcal{X}, X)$ is a Hilbert space if X is positive definite and a Kreĭn space in general. The following result is an indefinite analogue of Theorem 3.14 from [18].

Theorem 2.3. *Let $\mathbf{T} = (T_1, \dots, T_d)$ be a commutative d-tuple on $\mathcal{X}$ and let C of the form* (2.18) *be such that the pair $(C, \mathbf{T})$ is output-stable and the J-gramian $P := \mathcal{G}^J_{C,\mathbf{T}}$ given by* (2.20) *is strictly positive definite on $\mathcal{X}$. Then:*

1. *The operator $\mathcal{O}_{C,\mathbf{T}} : \ (\mathcal{X}, P) \to (\mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(k_d), J)$ is a contraction. This operator is isometric if and only if $\mathbf{T}$ is strongly stable.*
2. *If the space $\mathcal{M} := \operatorname{Ran} \mathcal{O}_{C,\mathbf{T}}$ is given the lifting norm*

$$\|\mathcal{O}_{C,\mathbf{T}}x\|^2 = \langle Px, x\rangle_{\mathcal{X}},$$

then $\mathcal{M}$ is isometrically equal to the reproducing kernel Hilbert space with reproducing kernel $K^P_{C,\mathbf{T}}$ given by

$$K^P_{C,\mathbf{T}}(z,\zeta) = C(I - Z(z)T)^{-1}P^{-1}(I - T^*Z(\zeta)^*)^{-1}C^*. \tag{2.23}$$

3. *If a Hilbert space $\mathcal{F}$ and an operator $\begin{bmatrix} B \\ D \end{bmatrix} : \begin{bmatrix} \mathcal{F} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y}\oplus\mathcal{U} \end{bmatrix}$ are such that the operator*

$$\mathbf{U} = \begin{bmatrix} T & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{F}\oplus\mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y}\oplus\mathcal{U} \end{bmatrix} \tag{2.24}$$

satisfies

$$\mathbf{U}\begin{bmatrix} P^{-1} & 0 \\ 0 & \mathbf{J} \end{bmatrix}\mathbf{U}^* = \begin{bmatrix} P^{-1}\otimes I_d & 0 \\ 0 & J \end{bmatrix}, \quad \text{where} \quad \mathbf{J} = \begin{bmatrix} I_{\mathcal{F}} & 0 \\ 0 & -I_{\mathcal{U}} \end{bmatrix}, \tag{2.25}$$

then the kernel $K^P_{C,\mathbf{T}}(z,\zeta)$ appearing in (2.23) *can be expressed as*

$$K^P_{C,\mathbf{T}}(z,\zeta) = K^{\mathbf{J},J}_{\mathfrak{A}}(z,\zeta) := \frac{J - \mathfrak{A}(z)\mathbf{J}\mathfrak{A}(\zeta)^*}{1 - \langle z,\zeta\rangle} \tag{2.26}$$

where $\mathfrak{A}(z)$ is the characteristic function of the colligation $\mathbf{U}$ in (2.24)*:*

$$\mathfrak{A}(z) = D + C(I - Z(z)T)^{-1}Z(z)B. \tag{2.27}$$

If the operators B and D are such that $\mathbf{U}$ in (2.24) *is subject to*

$$\mathbf{U}\begin{bmatrix} P\otimes I_d & 0 \\ 0 & J \end{bmatrix}\mathbf{U} = \begin{bmatrix} P & 0 \\ 0 & \mathbf{J} \end{bmatrix}, \tag{2.28}$$

then $\mathfrak{A}(z)$ is bi-$(\mathbf{J}, J)$-contractive for each $z \in \mathbb{B}^d$:

$$\mathfrak{A}(z)\mathbf{J}\mathfrak{A}(z)^* \le J, \qquad \mathfrak{A}(z)^*J\mathfrak{A}(z) \le \mathbf{J} \quad (z \in \mathbb{B}^d). \tag{2.29}$$

One such construction of B, D in (2.24) is to take $\mathcal{F} = \mathcal{X}^{d-1} \oplus \mathcal{Y}$ and then to solve the J-Cholesky factorization problem

$$\begin{bmatrix} B \\ D \end{bmatrix} \mathbf{J} \begin{bmatrix} B^* & D^* \end{bmatrix} = \begin{bmatrix} P^{-1} \otimes I_d & 0 \\ 0 & J \end{bmatrix} - \begin{bmatrix} T \\ C \end{bmatrix} P^{-1} \begin{bmatrix} T^* & C^* \end{bmatrix} \tag{2.30}$$

with $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ injective.

Proof. To prove (1), we make subsequent use of (2.20), (2.22) and (2.21) to get

$$\begin{aligned}
\langle J\mathcal{O}_{C,\mathbf{T}}x, \mathcal{O}_{C,\mathbf{T}}\rangle_{\mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(k_d)} &= \left\langle \mathcal{G}^J_{C,\mathbf{T}}x, x\right\rangle_{\mathcal{X}} \\
&= \sum_{v\in\mathcal{F}_d} \langle \mathbf{T}^{v*}C^*JC\mathbf{T}^v x, x\rangle_{\mathcal{X}} \\
&= \sum_{N=0}^{\infty} \sum_{v\in\mathcal{F}_d\colon |v|=N} \left\langle \mathbf{T}^{v*}(P - \sum_{j=1}^{d} T_j^* P T_j)\mathbf{T}^v x, x \right\rangle_{\mathcal{X}} \\
&= \lim_{N\to\infty} \left[\langle Px, x\rangle_{\mathcal{X}} - \sum_{v\in\mathcal{F}_d\colon |v|=N+1} \mathbf{T}^{v*}P\mathbf{T}^v x, x \right\rangle_{\mathcal{X}} \\
&\leq \langle Px, x\rangle
\end{aligned}$$

with equality in the last step for all $x \in \mathcal{X}$ if and only if $\mathbf{T}$ is strongly stable.
Statement (2) follows by standard reproducing kernel Hilbert space considerations; for this we refer the reader to [18].
As for statement (3), assume first that $\mathbf{U}$ as in (2.24) has been constructed so as to satisfy (2.25) and that we set $\mathfrak{A}$ equal to the characteristic function of $\mathbf{U}$ as in (2.27). From the J-coisometry property (2.25) of $\mathbf{U}$ we read off the relations

$$\begin{aligned}
&TP^{-1}T^* + B\mathbf{J}B^* = P^{-1} \otimes I_d, && CP^{-1}T^* + D\mathbf{J}B^* = 0, \\
&TP^{-1}C^* + B\mathbf{J}D^* = 0, && CP^{-1}C^* + D\mathbf{J}D^* = J.
\end{aligned}$$

Then we compute

$$\begin{aligned}
\mathfrak{A}(z)\mathbf{J}\mathfrak{A}(\zeta)^* &= D\mathbf{J}D^* + C(I - Z(z)T)^{-1}Z(z)B\mathbf{J}D^* \\
&\quad + D\mathbf{J}B^*Z(\zeta)^*(I - T^*Z(\zeta)^*)^{-1}C^* \\
&\quad + C(I - Z(z)T)^{-1}Z(z)B\mathbf{J}B^*Z(\zeta)^*(I - TZ(\zeta)^*)^{-1}C^* \\
&= J - CP^{-1}C^* - C(I - Z(z)T)^{-1}Z(z)TP^{-1}C^* \\
&\quad - CP^{-1}T^*Z(\zeta)^*(I - T^*Z(\zeta)^*)^{-1}C^* \\
&\quad + C(I - Z(z)T)^{-1}(P^{-1} \otimes I_d - TP^{-1}T^*)Z(\zeta)^*(I - T^*Z(\zeta)^*)^{-1}C^* \\
&= J - C(I - Z(z)T)^{-1}\Gamma(I - T^*Z(\zeta)^*)^{-1}C^*
\end{aligned}$$

where Γ is given by

$$\begin{aligned}\Gamma &= (I - Z(z)T)P^{-1}(I - T^*Z(\zeta)^*) + Z(z)TP^{-1}(I - T^*Z(\zeta)^*)\\ &\quad + (I - Z(z)T)P^{-1}T^*Z(\zeta)^* + Z(z)(TP^{-1}T^* - P^{-1}\otimes I_d)Z(\zeta)^*\\ &= P^{-1} - Z(z)(P^{-1}\otimes I_d)Z(\zeta)^* = (1 - \langle z,\zeta\rangle)P^{-1}\end{aligned}$$

from which (2.26) follows.

If B and D are constructed so that $\mathbf{U}$ also satisfies (2.28), then we have the additional relations

$$\begin{aligned}&T^*(P\otimes I_d)T + C^*JC = P, &&T^*(P\otimes I_d)B + C^*JD = 0,\\ &B^*(P\otimes I_d)T + D^*JC = 0, &&B^*(P\otimes I_d)B + D^*JD = \mathbf{J}.\end{aligned}$$

Then a computation similar to that used in the previous paragraph shows that

$$\mathbf{J} - \mathfrak{A}(w)^*J\mathfrak{A}(z) = B^*(I - Z(w)^*T^*)^{-1}(P\otimes I_d - Z(w)^*PZ(z))(I - TZ(z))^{-1}B.$$

In particular, taking $w = z$ gives

$$\mathbf{J} - \mathfrak{A}(z)^*J\mathfrak{A}(z) = (I - Z(z)^*T^*)\Pi(z)(I - TZ(z))^{-1}B$$

where

$$\Pi(z) = P\otimes I_d - Z(z)^*PZ(z) = (I_d - z^*z)\otimes P$$

is the tensor product of two positive-definite operators $I_d - z^*z$ (where here we view z as the row matrix $z = \begin{bmatrix} z_1 & \cdots & z_d\end{bmatrix} : \mathbb{C}^d \to \mathbb{C}$) and hence is positive-definite.

To construct B, D so that $\mathbf{U}$ as in (2.24) satisfies (2.25), proceed as follows. From the Stein equation (2.21), we see that $\mathcal{G} := \operatorname{Ran}\begin{bmatrix} T \\ C\end{bmatrix}$ is a uniformly positive subspace of the Kreĭn space $(\mathcal{X}^d\oplus\mathcal{Y}\oplus\mathcal{U}, \left[\begin{smallmatrix} P\otimes I_d & 0 \\ 0 & J\end{smallmatrix}\right])$. Hence the Kreĭn-space orthogonal complement $\mathcal{G}^{[\perp]}$ of $\mathcal{G}$ is also a Kreĭn space in inner product inherited from the ambient space $(\mathcal{X}^d\oplus\mathcal{Y}\oplus\mathcal{U}, \left[\begin{smallmatrix} P\otimes I_d & 0 \\ 0 & J\end{smallmatrix}\right])$ with inertia equal to the complement of the inertia of P with respect to the inertia of $\begin{bmatrix} P\otimes I_d & 0 \\ 0 & J\end{bmatrix}$ on the large space, namely, with inertia equal to that of $\begin{bmatrix} P\otimes I_{d-1} & 0 \\ 0 & J\end{bmatrix}$ on $\mathcal{X}^{d-1}\oplus\mathcal{Y}\oplus\mathcal{U}$. Therefore there is an isometry $\begin{bmatrix} B \\ D\end{bmatrix}$ from $(\mathcal{X}^{d-1}\oplus\mathcal{Y}\oplus\mathcal{U}, \mathbf{J})$ where we set $\mathbf{J} = \left[\begin{smallmatrix} I_{\mathcal{X}^{d-1}\oplus\mathcal{Y}} & 0 \\ 0 & -I_{\mathcal{U}}\end{smallmatrix}\right]$. If $\begin{bmatrix} B \\ D\end{bmatrix}$ is such an isometry, then the orthogonal (with respect to the Kreĭn-space inner product) projection $\mathcal{P}_{\mathcal{G}^{[\perp]}}$ of $\mathcal{X}^d\oplus\mathcal{Y}\oplus\mathcal{U}$ onto $\mathcal{G}^{[\perp]}$ is given by $\mathcal{P}_{\mathcal{G}^{[\perp]}} = \begin{bmatrix} B \\ D\end{bmatrix}\begin{bmatrix} B \\ D\end{bmatrix}^{[*]}$ where $X^{[*]}$ denotes the Kreĭn-space adjoint of the Hilbert space operator X. For the case of

$$\begin{bmatrix} B \\ D\end{bmatrix} : \left(\begin{bmatrix} \mathcal{X}^{d-1}\oplus\mathcal{Y} \\ \mathcal{U}\end{bmatrix}, \mathbf{J}\right) \to \left(\begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y}\oplus\mathcal{U}\end{bmatrix}, \begin{bmatrix} P\otimes I_d & 0 \\ 0 & J\end{bmatrix}\right),$$

the Kreĭn-space adjoint of $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ is given by

$$\begin{bmatrix} B \\ D \end{bmatrix}^{[*]} = \mathbf{J} \begin{bmatrix} B^* & D^* \end{bmatrix} \begin{bmatrix} P \otimes I_d & 0 \\ 0 & J \end{bmatrix}$$

and hence we have

$$\mathcal{P}_{\mathcal{G}^{[\perp]}} = \begin{bmatrix} B \\ D \end{bmatrix} \mathbf{J} \begin{bmatrix} B^* & D^* \end{bmatrix} \begin{bmatrix} P \otimes I_d & 0 \\ 0 & J \end{bmatrix}. \tag{2.31}$$

On the other hand the Kreĭn-space orthogonal projection of $\mathcal{X}^d \oplus \mathcal{Y} \oplus \mathcal{U}$ onto $\mathcal{G} = \operatorname{Ran} \begin{bmatrix} T \\ C \end{bmatrix}$ is given by $\mathcal{P}_{\mathcal{G}} = \begin{bmatrix} T \\ C \end{bmatrix} \begin{bmatrix} T \\ C \end{bmatrix}^{[*]}$ where the Kreĭn-space adjoint of $\left[\begin{smallmatrix} T \\ C \end{smallmatrix}\right]$ is given by

$$\begin{bmatrix} T \\ C \end{bmatrix}^{[*]} = P^{-1} \begin{bmatrix} T^* & C^* \end{bmatrix} \begin{bmatrix} P \otimes I_d & 0 \\ 0 & J \end{bmatrix}.$$

Therefore we get a second expression for the Kreĭn-space orthogonal projection $\mathcal{P}_{\mathcal{G}^{[\perp]}}$, namely

$$\mathcal{P}_{\mathcal{G}^{[\perp]}} = I_{\mathcal{X}^d \oplus \mathcal{Y} \oplus \mathcal{U}} - \mathcal{P}_{\mathcal{G}} = I - \begin{bmatrix} T \\ C \end{bmatrix} P^{-1} \begin{bmatrix} T^* & C^* \end{bmatrix} \begin{bmatrix} P \otimes I_d & 0 \\ 0 & J \end{bmatrix}. \tag{2.32}$$

Equating the two expressions (2.31) and (2.32) for $P_{\mathcal{P}^{[\perp]}}$ gives the identity

$$I - \begin{bmatrix} T \\ C \end{bmatrix} P^{-1} \begin{bmatrix} T^* & C^* \end{bmatrix} \begin{bmatrix} P \otimes I_d & 0 \\ 0 & J \end{bmatrix} = \begin{bmatrix} B \\ D \end{bmatrix} \mathbf{J} \begin{bmatrix} B^* & D^* \end{bmatrix} \begin{bmatrix} P \otimes I_d & 0 \\ 0 & J \end{bmatrix}.$$

Multiplication on the right by $\left[\begin{smallmatrix} P^{-1} \otimes I_d & 0 \\ 0 & J \end{smallmatrix}\right]$ then leaves us with the expression (2.30) as the equation to be solved for B and D to complete the construction. We can always solve (2.30) so that $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ is injective; this then guarantees that the resulting $\mathbf{U}$ of the form (2.24) also satisfies (2.28). □

The hypothesis that $(C, \mathbf{T})$ is output-stable in Theorem 2.3 can be weakened as follows. Rather than assuming that $\mathcal{O}_{C,\mathbf{T}}$ maps the space $\mathcal{X}$ into $\mathcal{H}_{\mathcal{Y} \oplus \mathcal{U}}(k_d)$, we assume only that $\mathcal{O}_{C,\mathbf{T}}$ maps $\mathcal{X}$ into the space $Hol_{\mathcal{Y} \oplus \mathcal{U}}(\mathbb{B}^d)$ of $(\mathcal{Y} \oplus \mathcal{U})$-valued holomorphic functions on $\mathbb{B}^d$. Then there of course is no hope for the validity of statement (1) in Theorem 2.3, but statement (2) holds as stated. For the validity of statement (3) all that is required (for the construction of $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ so that $\mathbf{U}$ as in (2.24) satisfies (2.25) and (2.28)) is that P be a positive definite solution of the Stein equation (2.21). We are led to the following result; as the proof is essentially the same as that of Theorem 2.3, we leave the details to the reader.

Theorem 2.4. *Suppose that $(C, \mathbf{T})$ is an analytic output-pair, i.e.,*

$$C(I - Z(z)T)^{-1} x \in Hol_{\mathcal{Y} \oplus \mathcal{U}}(\mathbb{B}^d) \text{ for all } x \in \mathcal{X}.$$

Assume also that $P \in \mathcal{L}(\mathcal{X})$ is a positive definite solution of the Stein equation (2.21). *Then statements (2) and (3) in Theorem* 2.3 *hold without change.*

Remark 2.5. In the setting of Theorem 2.3, we have

$$\mathcal{H}(K_{\mathfrak{A}}^{\mathbf{J},J}) \subset \mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(k_d). \tag{2.33}$$

If we assume in addition that $\mathbf{T}$ is strongly stable, then the inclusion (2.33) is isometric. The $\mathcal{L}(\mathcal{F}\oplus\mathcal{U}, \mathcal{Y}\oplus\mathcal{U})$-valued function $\mathfrak{A}$ in general is not in the multiplier class $\mathcal{M}_d(\mathcal{F}\oplus\mathcal{U}, \mathcal{Y}\oplus\mathcal{U})$ but it is true that $\mathfrak{A}$ maps constant functions in $\mathcal{H}_{\mathcal{F}\oplus\mathcal{U}}(k_d)$ (and hence also polynomials) into $\mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(k_d)$. In this case the operator $I - M_{\mathfrak{A}} M_{\mathfrak{A}}^{[*]}$, initially only defined on linear combinations of kernel functions via

$$I - M_{\mathfrak{A}} M_{\mathfrak{A}}^{[*]} = I - M_{\mathfrak{A}} \mathbf{J} M_{\mathfrak{A}}^{*} J \colon k_d(\cdot,\zeta) \begin{bmatrix} y \\ u \end{bmatrix} \mapsto k_d(\cdot,\zeta) \begin{bmatrix} y \\ u \end{bmatrix} - \mathfrak{A}(\cdot) k_d(\cdot,\zeta) \mathbf{J} \mathfrak{A}(\zeta)^* J \begin{bmatrix} y \\ u \end{bmatrix}$$

extends continuously to the J-orthogonal projection operator mapping $\mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(k_d)$ onto $\mathcal{H}(K_{\mathfrak{A}}^{\mathbf{J},J})$. In this situation we say that $\mathfrak{A}$ is a $(\mathbf{J}, J)$-*inner multiplier*. In the single-variable case ($d = 1$), we have $\mathbf{J} = J$ and these functions coincide with the *strongly regular* J-inner functions in the sense of Arov-Dym [9] (see also [24]) which are analytic on the unit disk $\mathbb{D} = \mathbb{B}^1$.

Remark 2.6. Suppose that $\mathcal{M} \subset \mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(k_d)$ is a shift-invariant subspace which is J-regular in the sense that the J-orthogonal complement $\mathcal{M}^{[\perp]_J}$ of $\mathcal{M}$ in $\mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(k_d)$ with respect to the J-inner product together with $\mathcal{M}$ forms a direct-sum decomposition of $\mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(k_d)$:

$$\mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(k_d) = \mathcal{M} \dot{+} \mathcal{M}^{[\perp]_J}.$$

As shown in [20, Theorem 4.8], there exists an output-stable pair $(C, \mathbf{T})$ so that $\mathcal{M} = \operatorname{Ker} \mathcal{O}_{JC,\mathbf{T}}^*$. It then follows that

$$\mathcal{M}^{[\perp]_J} = \operatorname{Ran} \mathcal{O}_{C,\mathbf{T}}.$$

One choice of $C \colon \mathcal{X} \to \mathcal{Y} \oplus \mathcal{U}$ and $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{L}(\mathcal{X})^d$ is

$$\mathcal{X} = \mathcal{M}^{[\perp]_J}, \qquad T_j = M_{z_j}^*|_{\mathcal{M}^{[\perp]_J}} \text{ for } j = 1, \dots, d, \qquad C \colon f \mapsto f(0).$$

(Note that $\mathcal{M}^{[\perp]_J}$ is backward-shift-invariant since $\mathcal{M}$ is shift-invariant.) Since $\mathcal{M}^{[\perp]_J}$ is a Kreĭn subspace of $\mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(k_d)$, it follows that the J-observability gramian $P := \mathcal{G}_{C,\mathbf{T}}^J = \mathcal{O}_{C,\mathbf{T}}^* J \mathcal{O}_{C,\mathbf{T}}$ is invertible. Moreover, $(C, \mathbf{T})$ satisfies the Stein equation (4.4). One can then construct a solution $\mathbf{U}$ of the form (4.10) which satisfies (2.25) and define $\mathfrak{A}(z)$ as in (2.27). Then, as in the proof of Theorem (2.3) we see that $\mathcal{M}^{[\perp]_J}$ is isometrically equal to $\mathcal{H}(K_{\mathfrak{A}}^{\mathbf{J},J})$. If it is the case that $\mathfrak{A}$ is a bounded multiplier between $\mathcal{H}_{\mathcal{F}\oplus\mathcal{U}}(k_k)$ and $\mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(k_d)$, it follows that

$$\mathcal{M} = (\mathcal{M}^{[\perp]_J})^{[\perp]_J} = M_{\mathfrak{A}} \mathcal{H}_{\mathcal{F}\oplus\mathcal{U}}(k_d)$$

where the multiplication operator $M_{\mathfrak{A}}$ in addition is a $(\mathbf{J}, J)$-partial isometry. This representation of the shift-invariant subspace $\mathcal{M}$ is a Kreĭn-spaces version of the Beurling-Lax theorem [12, 48, 18, 20] cited above and goes back to [22] for the single-variable case.

Suppose that we are given a holomorphic $\mathcal{L}(\mathcal{F}\oplus\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$-valued function $\mathfrak{A}(z)$ on $\mathbb{B}^d$ as in Theorem 2.4 so that $\mathfrak{A}(z)$ is bi-$(\mathbf{J},J)$-contractive for each $z\in\mathbb{B}^d$. If we decompose $\mathfrak{A}$ as a block 2×2 matrix

$$\mathfrak{A}(z)=\begin{bmatrix}\mathfrak{A}_{11}(z) & \mathfrak{A}_{12}(z)\\ \mathfrak{A}_{21}(z) & \mathfrak{A}_{22}(z)\end{bmatrix}:\ \begin{bmatrix}\mathcal{F}\\ \mathcal{U}\end{bmatrix}\to\begin{bmatrix}\mathcal{Y}\\ \mathcal{U}\end{bmatrix}$$

and recall the conformal decompositions (2.19) and (2.25) for J and $\mathbf{J}$, then we see from the $(2,2)$-entry in the inequalities (2.29) that

$$\mathfrak{A}_{21}(z)\mathfrak{A}_{21}(z)^*-\mathfrak{A}_{22}(z)\mathfrak{A}_{22}(z)^*\le -I_{\mathcal{U}},$$
$$\mathfrak{A}_{12}(z)^*\mathfrak{A}_{12}(z)-\mathfrak{A}_{22}(z)^*\mathfrak{A}_{22}(z)\le -I_{\mathcal{U}}$$

from which we get

$$\mathfrak{A}_{22}(z)\mathfrak{A}_{22}(z)^*\ge I_{\mathcal{U}}+\mathfrak{A}_{21}(z)\mathfrak{A}_{21}(z)^*,$$
$$\mathfrak{A}_{22}(z)^*\mathfrak{A}_{22}(z)\ge I_{\mathcal{U}}+\mathfrak{A}_{12}(z)^*\mathfrak{A}_{12}(z)$$

and hence also

$$\mathfrak{A}_{22}(z)\text{ is invertible and }\|\mathfrak{A}_{22}(z)^{-1}\mathfrak{A}_{21}(z)\|<1\text{ for each }z\in\mathbb{B}^d. \tag{2.34}$$

We conclude that

$$\mathfrak{A}_{21}(z)\mathcal{E}(z)+\mathfrak{A}_{22}(z)=\mathfrak{A}_{22}(z)(\mathfrak{A}_{22}(z)^{-1}\mathfrak{A}_{21}(z)\mathcal{E}(z)+I)$$

is invertible for all $z\in\mathbb{B}^d$ and $\mathcal{E}\in\mathcal{S}_d(\mathcal{U},\mathcal{F})$ and hence the linear fractional transform of $\mathcal{E}$

$$T_{\mathfrak{A}}[\mathcal{E}](z)=(\mathfrak{A}_{11}(z)\mathcal{E}(z)+\mathfrak{A}_{12}(z))(\mathfrak{A}_{21}(z)\mathcal{E}(z)+\mathfrak{A}_{22}(z))^{-1} \tag{2.35}$$

is a well-defined holomorphic $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function on $\mathbb{B}^d$. We denote by $\mathcal{S}_d^{J,\mathbf{J}}(\mathcal{F}\oplus\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ the indefinite Schur class of $\mathcal{L}(\mathcal{F}\oplus\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$-valued functions $\mathfrak{A}$ analytic and bi-$(\mathbf{J},J)$-contractive on $\mathbb{B}^d$ and such that the kernel $K_{\mathfrak{A}}^{\mathbf{J},J}(z,\zeta)$ given by (3.7) is positive on $\mathbb{B}^d\times\mathbb{B}^d$. Thus, for an $\mathfrak{A}\in\mathcal{S}_d^{J,\mathbf{J}}(\mathcal{F}\oplus\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$, the linear fractional map $T_{\mathfrak{A}}$ given by (2.35) is well-defined on $\mathcal{S}_d(\mathcal{U},\mathcal{F})$. Theorem 2.8 gives a useful characterization of its range. As a first step in this direction, we need the following interpolation result (the Leech theorem for Drury-Arveson-space multipliers) which was established in [3, 14].

Theorem 2.7. *Let $\mathcal{E}$ be a Hilbert space and let*

$$\mathbf{a}\colon\mathbb{B}^d\to\mathcal{L}(\mathcal{Y},\mathcal{E})\quad\text{and}\quad\mathbf{c}\colon\mathbb{B}^d\to\mathcal{L}(\mathcal{U},\mathcal{E})$$

be operator valued functions defined on $\mathbb{B}^d$. The following are equivalent:

1. *There exists a function $S\in\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ such that*

$$\mathbf{a}(z)S(z)=\mathbf{c}(z)\quad\text{for all }z\in\mathbb{B}^d.$$

2. *There exist a Hilbert space $\mathcal{H}$ and a function $R(z):\mathbb{B}^d\to\mathcal{L}(\mathcal{H},\mathcal{E})$, such that*

$$\frac{\mathbf{a}(z)\mathbf{a}(\zeta)^*-\mathbf{c}(z)\mathbf{c}(\zeta)^*}{1-\langle z,\zeta\rangle}=R(z)R(\zeta)^*. \tag{2.36}$$

Theorem 2.8. *Let $\mathfrak{A} \in \mathcal{S}_d^{J,\mathbf{J}}(\mathcal{F} \oplus \mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$. Then a holomorphic $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function S has the form*

$$S = T_{\mathfrak{A}}[\mathcal{E}] \tag{2.37}$$

for some $\mathcal{E} \in \mathcal{S}_d(\mathcal{U},\mathcal{F})$ if and only if $S \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$ and the operator

$$\begin{bmatrix} I & -M_S \end{bmatrix} : \begin{bmatrix} y(z) \\ u(z) \end{bmatrix} \mapsto y(z) - S(z)u(z)$$

maps $\mathcal{H}(K_{\mathfrak{A}}^{\mathbf{J},J})$ contractively into the de Branges-Rovnyak space $\mathcal{H}(K_S)$.

Proof. The result can be found in the proof of Theorem 3.8 in [28] and appears in [33] for the case $d = 1$; we include the short proof for completeness. Suppose that $S \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$ and that $\begin{bmatrix} I & -M_S \end{bmatrix}$ maps $\mathcal{H}(K_{\mathfrak{A}}^{\mathbf{J},J})$ contractively into $\mathcal{H}(K_S)$. Then the kernel K_S is positive and we have the kernel inequality

$$\begin{bmatrix} I & -S(z) \end{bmatrix} K_{\mathfrak{A}}^{\mathbf{J},J}(z,\zeta) \begin{bmatrix} I \\ -S(\zeta)^* \end{bmatrix} \preceq K_S(z,\zeta),$$

or, what is the same,

$$\begin{bmatrix} I & -S(z) \end{bmatrix} \frac{J - \mathfrak{A}(z)\mathbf{J}\mathfrak{A}(\zeta)^*}{1 - \langle z,\zeta \rangle} \begin{bmatrix} I \\ -S(\zeta)^* \end{bmatrix} \preceq \frac{I - S(z)S(\zeta)^*}{1 - \langle z,\zeta \rangle}. \tag{2.38}$$

Note that

$$\begin{bmatrix} I & -S(z) \end{bmatrix} J \begin{bmatrix} I \\ -S(\zeta)^* \end{bmatrix} = I - S(z)S(\zeta)^*.$$

Hence we can rearrange (2.38) to

$$\begin{bmatrix} I & -S(z) \end{bmatrix} \frac{\mathfrak{A}(z)J\mathfrak{A}(\zeta)^*}{1 - \langle z,\zeta \rangle} \begin{bmatrix} I \\ -S(\zeta)^* \end{bmatrix} \succeq 0. \tag{2.39}$$

If we set

$$\begin{bmatrix} u(z) & -v(z) \end{bmatrix} := \begin{bmatrix} I & -S(z) \end{bmatrix} \mathfrak{A}(z),$$

then we get

$$\frac{u(z)u(\zeta)^* - v(z)v(\zeta)^*}{1 - \langle z,\zeta \rangle} \succeq 0, \tag{2.40}$$

where

$$u(z) = \mathfrak{A}_{11}(z) - S(z)\mathfrak{A}_{21}(z), \qquad -v(z) = \mathfrak{A}_{12}(z) - S(z)\mathfrak{A}_{22}(z).$$

By Theorem 2.7, it follows that there exists $\mathcal{E} \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$ so that $v(z) = u(z)\mathcal{E}(z)$, i.e.,

$$-(\mathfrak{A}_{12}(z) - S(z)\mathfrak{A}_{22}(z)) = (\mathfrak{A}_{11}(z) - S(z)\mathfrak{A}_{21}(z))\mathcal{E}(z)$$

which can be rearranged as

$$S(z)(\mathfrak{A}_{21}(z)\mathcal{E}(z) + \mathfrak{A}_{22}(z)) = \mathfrak{A}_{11}(z)\mathcal{E}(z) + \mathfrak{A}_{12}(z).$$

It now follows that we recover S as $S = T_{\mathfrak{A}}[\mathcal{E}]$.
Conversely, suppose that $\mathcal{E} \in \mathcal{S}_d(\mathcal{U},\mathcal{F})$ and $S = T_{\mathfrak{A}}[\mathcal{E}]$. By reversing the steps in the argument above and using that condition (2.40) is necessary as well as sufficient in Theorem 2.36, we arrive at (2.39). We then add $K_S(z,\zeta)$ to both sides of (2.39)

to arrive at (2.38). As $K_{\mathfrak{A}}^{\mathbf{J},J}$ is a positive kernel by assumption, we conclude that K_S is a positive kernel, i.e., that $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. Then the inequality (2.38) is the statement that $\begin{bmatrix} I & -M_S \end{bmatrix}$ maps $\mathcal{H}(K_{\mathfrak{A}}^{\mathbf{J},J})$ contractively into $\mathcal{H}(K_S)$. □

In addition to the linear-fractional transformations of chain-matrix form (2.35) as discussed above we shall also have use of linear-fractional transformations of Redheffer form. To define these, we suppose that we are given a matrix function $\Sigma(z) = \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix}$ holomorphic on the ball $\mathbb{B}^d$ with values in $\mathcal{L}(\mathcal{X} \oplus \widetilde{\Delta}_*, \mathcal{X}' \oplus \widetilde{\Delta})$ for some Hilbert spaces $\mathcal{X}$, $\widetilde{\Delta}_*$, $\mathcal{X}'$, $\widetilde{\Delta}$. We assume that $\|\Sigma_{22}(z)\| < 1$ for all $z \in \mathbb{B}^d$. Suppose that $\mathcal{W}$ is in the Schur-multiplier class $\mathcal{S}_d(\widetilde{\Delta}, \widetilde{\Delta}_*)$. From the positivity of the kernel $K_{\mathcal{W}}(z,\zeta) = [I - \mathcal{W}(z)\mathcal{W}(\zeta)^*]/(1 - \langle z, \zeta \rangle)$ we see in particular that $\|\mathcal{W}(z)\| \leq 1$ for each $z \in \mathbb{B}^d$ and it follows that $(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}$ makes sense as a holomorphic $\mathcal{L}(\widetilde{\Delta}_*)$-valued function on $\mathbb{B}^d$. We then can define the associated *Redheffer linear-fractional map* $\mathfrak{R}_\Sigma$ acting from $\mathcal{S}_d(\widetilde{\Delta}, \widetilde{\Delta}_*)$ to $\mathrm{Hol}_{\mathcal{L}(\mathcal{X},\mathcal{X}')}(\mathbb{B}^d)$ by

$$\mathfrak{R}_\Sigma[\mathcal{W}] := \Sigma_{11}(z) + \Sigma_{12}(z)(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}\mathcal{W}(z)\Sigma_{21}(z). \tag{2.41}$$

The following criterion for a given function S to be in the range of $\mathfrak{R}_\Sigma$, while less explicit than the criterion in Theorem 2.8, nevertheless is useful in some applications (see Theorem 6.4 below). For this purpose we say that a pair of functions

$$\mathbf{a} \in \mathrm{Hol}_{\mathcal{L}(\widetilde{\Delta}_*, \mathcal{X})}(\mathbb{B}^d), \qquad \mathbf{c} \in \mathrm{Hol}_{\mathcal{L}(\widetilde{\Delta}, \mathcal{X}')}(\mathbb{B}^d)$$

is a *Schur-pair* if the associated kernel below is positive:

$$\frac{\mathbf{a}(z)\mathbf{a}(\zeta)^* - \mathbf{c}(z)\mathbf{c}(\zeta)^*}{1 - \langle z, \zeta \rangle} \succeq 0. \tag{2.42}$$

Theorem 2.9. *Suppose that we are given*

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \in \mathrm{Hol}_{\mathcal{L}(\mathcal{X} \oplus \widetilde{\Delta}_*, \mathcal{X}' \oplus \widetilde{\Delta})}(\mathbb{B}^d)$$

with $\|\Sigma_{22}(z)\| < 1$ *for each* $z \in \mathbb{B}^d$. *Suppose that we are also given an operator-valued function* $S \in \mathrm{Hol}_{\mathcal{L}(\mathcal{X},\mathcal{X}')}(\mathbb{B}^d)$. *Then there exists a Schur-class multiplier* $\mathcal{W} \in \mathcal{S}_d(\widetilde{\Delta}, \widetilde{\Delta}_*)$ *such that* $S = \mathfrak{R}_\Sigma[\mathcal{W}]$ *if and only if there exists a Schur-pair* $(\mathbf{a}(z), \mathbf{c}(z))$ *so that*

$$\begin{bmatrix} I & \mathbf{c}(z) \end{bmatrix} \Sigma(z) = \begin{bmatrix} S(z) & \mathbf{a}(z) \end{bmatrix}. \tag{2.43}$$

Proof. Suppose that $(\mathbf{a}(z), \mathbf{c}(z))$ is a Schur-pair satisfying (2.43). By Theorem 2.7, there is a Schur-class multiplier $\mathcal{W} \in \mathcal{S}_d(\widetilde{\Delta}, \widetilde{\Delta}_*)$ so that

$$\mathbf{c}(z) = \mathbf{a}(z)\mathcal{W}(z). \tag{2.44}$$

Then (2.43) can be written as

$$\begin{aligned} \Sigma_{11}(z) + \mathbf{a}(z)\mathcal{W}(z)\Sigma_{21}(z) &= S(z) \\ \Sigma_{12}(z) + \mathbf{a}(z)\mathcal{W}(z)\Sigma_{22}(z) &= \mathbf{a}(z). \end{aligned} \tag{2.45}$$

From the second of equations (2.45) we can solve for $\mathbf{a}(z)$:

$$\mathbf{a}(z) = \Sigma_{12}(z)(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}. \tag{2.46}$$

If we plug this expression into the first of equations (2.45) we get

$$S(z) = \Sigma_{11}(z) + \Sigma_{12}(z)(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}\mathcal{W}(z)\Sigma_{21}(z) = \mathfrak{R}_\Sigma[\mathcal{W}](z)$$

as wanted.

For the converse direction, given that $S = \mathfrak{R}_\Sigma[\mathcal{W}]$, if we define $(\mathbf{a}(z), \mathbf{c}(z))$ by (2.44) and (2.46), then $(\mathbf{a}, \mathbf{c})$ is a Schur-pair meeting the criterion (2.43). □

Remark 2.10. As the solution of the Leech problem appearing in both the proof of Theorem 2.8 and the proof of Theorem 2.9 does not appear to be unique, in general we would expect that S does not uniquely determine $\mathcal{E}$ (or $\mathcal{W}$) in the formula (2.37) (or (2.41)). A similar issue appears in the parametrization of all solutions of the commutant lifting problem for the single-variable case [36, 37] and in connection with the more general *relaxed* commutant lifting problem (see [38, 39]). Remarkably it turns out in this context that one can parametrize the free parameters $\mathcal{E}$ which give rise to a given S and that, in the context of (nonrelaxed or "stiff") commutant lifting, it happens that S does uniquely determine $\mathcal{E}$. It remains to be seen if something similar can be worked out in the multivariable context here.

3. Higher-order left tangential Nevanlinna-Pick interpolation: compact operator-theoretic formulations

Suppose that we are given the data $\{\zeta^{(i)}, a_i^*, c_i^* \colon i = 1, \dots, N\}$ for a **LNPP** (i.e., a left-tangential Nevanlinna-Pick interpolation problem with associated interpolation conditions of the form (1.3)), and suppose that we know one solution $\Psi \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y})$ (with M_Ψ not necessarily contractive) of the set of interpolation conditions (1.3) (with Ψ in place of S). We note that the subspace $\mathcal{N} \subset \mathcal{H}_\mathcal{Y}(k_d)$ defined by

$$\mathcal{N} = \{h \in \mathcal{H}_\mathcal{Y}(k_d) \colon a_i^* h(\zeta^{(i)}) = 0 \ \text{ for } \ i = 1, \dots, N\}$$

is closed and shift-invariant. Hence, by the Beurling-Lax theorem for the Drury-Arveson space mentioned at the end of Section 2 above, we know that there is an inner multiplier $\Theta \in \mathcal{S}_d(\mathcal{F}, \mathcal{Y})$ (for some auxiliary Hilbert space $\mathcal{F}$) so that $\mathcal{N} = \Theta\mathcal{H}_\mathcal{F}(k_d)$. If $S \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y})$ is another solution of the interpolation conditions (1.3), then we see that

$$M_{S-\Psi} \colon \mathcal{H}_\mathcal{U}(k_d) \to \Theta \cdot \mathcal{H}_\mathcal{F}(k_d).$$

It then follows as a consequence of the Leech theorem for multipliers on the Drury-Arveson space (see [3]) that there is a $F \in \mathcal{M}_d(\mathcal{U}, \mathcal{F})$ so that S has the form $S = \Psi + \Theta F$. With the new data set $\{\Psi, \Theta\}$ (where F is a multiplier in $\mathcal{M}_d(\mathcal{U}, \mathcal{Y})$ and where Θ is an inner multiplier in $\mathcal{S}_d(\mathcal{F}, \mathcal{U})$, we see that the **LNPP** can be reformulated as a *Sarason interpolation problem*:

Sarason interpolation problem (SIP): *Given* $\Psi \in \mathcal{M}_d(\mathcal{U},\mathcal{Y})$ *and an inner* $\Theta \in \mathcal{S}_d(\mathcal{F},\mathcal{U})$, *find all* $S \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$ *such that*

$$S(z) = \Psi(z) + \Theta(z)F(z) \quad \text{for some} \quad F \in \mathcal{M}_d(\mathcal{U},\mathcal{F}).$$

As was first done for the single-variable case in the classical paper [55] of Sarason and then in [50, 53] for both the noncommutative and commutative ball setting, the **SIP** can be put in more operator-theoretic form as follows. Given the data set $\{\Psi,\Theta\}$ for a Sarason interpolation problem, introduce the subspace $\mathcal{M}$ by

$$\mathcal{M} = \mathcal{H}_{\mathcal{Y}}(k_d) \ominus \Theta\mathcal{H}_{\mathcal{Y}}(k_d)$$

and define the operator $\Phi\colon \mathcal{H}_{\mathcal{U}}(k_d) \to \mathcal{M}$ by

$$\Phi = \mathcal{P}_{\mathcal{M}} M_{\Psi}.$$

Note that $\mathcal{M}$ is backward-shift-invariant and that Φ satisfies the intertwining property:

$$\mathcal{P}_{\mathcal{M}} M_{z_j}\Phi = \Phi M_{z_j} \quad (j = 1,\dots,d). \tag{3.1}$$

Furthermore, one easily checks that a multiplier $S \in \mathcal{M}_d(\mathcal{U},\mathcal{Y})$ solves the **SIP**(Ψ,Θ) if and only if M_S satisfies the conditions

$$\mathcal{P}_{\mathcal{M}} M_S = \Phi \quad \text{and} \quad \|M_S\| \le 1.$$

From these conditions we read off that necessarily

$$\|\Phi\| \le 1 \tag{3.2}$$

if **SIP**(Ψ,Θ) has a solution. As mentioned previously, multipliers M_S are characterized as those operators between $\mathcal{H}_{\mathcal{U}}(k_d)$ and $\mathcal{H}_{\mathcal{Y}}(k_d)$ which intertwine the respective shift operators $M_{z_j} \otimes I_{\mathcal{U}}$ and $M_{z_j} \otimes I_{\mathcal{Y}}$ for $j = 1,\dots,d$. It now follows that the **SIP** can be reformulated as the following *Commutant Lifting Problem*:

Commutant Lifting Problem (CLP): *Given an* $\mathbf{M}_{\boldsymbol{z}}^*$*-invariant subspace* $\mathcal{M}$ *of* $\mathcal{H}_{\mathcal{Y}}(k_d)$ *and an operator* $\Phi : \mathcal{H}_{\mathcal{U}}(k_d) \to \mathcal{M}$ *subject to* (3.1), *find an operator* $R : \mathcal{H}_{\mathcal{U}}(k_d) \to \mathcal{H}_{\mathcal{Y}}(k_d)$ *such that*

$$\|R\| \le 1, \quad \mathcal{P}_{\mathcal{M}} R = \Phi \quad \text{and} \quad M_{z_j}R = RM_{z_j} \quad (j = 1,\dots,d), \tag{3.3}$$

or equivalently, find a contractive multiplier $S \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$ *such that*

$$\mathcal{P}_{\mathcal{M}} M_S = \Phi. \tag{3.4}$$

If R is a solution of **CLP**, then it follows from (3.3) that $\|\Phi\| = \|\mathcal{P}_{\mathcal{M}}R\| \le \|R\| \le 1$ and also

$$\Phi M_{z_j} = \mathcal{P}_{\mathcal{M}} R M_{z_j} = \mathcal{P}_{\mathcal{M}} M_{z_j} R = \mathcal{P}_{\mathcal{M}} M_{z_j} \mathcal{P}_{\mathcal{M}} R = \mathcal{P}_{\mathcal{M}} M_{z_j} \Phi,$$

where the third equality holds due to the backward shift invariance of $\mathcal{M}$. Hence the conditions (3.1) and (3.2) are certainly necessary for the existence of a solution to **CLP**. That the converse holds is the assertion of the commutant lifting theorem

(see [55, 36] for the single-variable case and [26] as well as [49] combined with [7, 30] for the present Drury-Arveson space setting).
Given a Sarason interpolation problem $\mathbf{SIP}(\Psi, \Theta)$, we have seen how to pass to a $\mathbf{CLP}(\mathcal{M}, \Phi)$. Conversely, it is possible to pass from a $\mathbf{CLP}(\mathcal{M}, \Phi)$ to a $\mathbf{SIP}(\Psi, \Theta)$ as follows. Take any Beurling-Lax representer $\Theta \in \mathcal{S}_d(\mathcal{F}, \mathcal{Y})$ for $\mathcal{M} \subset \mathcal{H}_{\mathcal{Y}}(k_d)$ and choose any $\Psi \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y})$ (not necessarily contractive) so that $\Phi = P_{\mathcal{M}} M_\Psi$. The fact that such a multiplier Ψ always exists is of course a consequence of the commutant lifting theorem.
We next formulate the left-tangential interpolation problem with operator argument and show that the **CLP** is a particular case of this problem.
For an output-stable pair $(E, \mathbf{T})$ (in the sense defined in Section 2) with a *commutative* d-tuple $\mathbf{T}$ we define a *left-tangential functional calculus* $f \to (E^* f)^{\wedge L}(\mathbf{T}^*)$ on $\mathcal{H}_{\mathcal{Y}}(k_d)$ by

$$(E^* f)^{\wedge L}(\mathbf{T}^*) = \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \mathbf{T}^{*\mathbf{n}} E^* f_{\mathbf{n}} \quad \text{if} \quad f = \sum_{\mathbf{n} \in \mathbb{Z}_+^d} f_{\mathbf{n}} z^{\mathbf{n}} \in \mathcal{H}_{\mathcal{Y}}(k_d). \tag{3.5}$$

The computation

$$\begin{aligned} \left\langle \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \mathbf{T}^{*\mathbf{n}} E^* f_{\mathbf{n}},\ x \right\rangle_{\mathcal{X}} &= \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \langle f_{\mathbf{n}},\ E\mathbf{T}^{\mathbf{n}} x \rangle_{\mathcal{Y}} \\ &= \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \frac{\mathbf{n}!}{|\mathbf{n}|!} \left\langle f_{\mathbf{n}}, \frac{|\mathbf{n}|!}{\mathbf{n}!} E\mathbf{T}^{\mathbf{n}} x \right\rangle_{\mathcal{Y}} = \langle f,\ \mathcal{O}_{E,\mathbf{T}} x \rangle_{\mathcal{H}_{\mathcal{Y}}(k_d)} \end{aligned}$$

shows that the output-stability of the pair $(E, \mathbf{T})$ is exactly what is needed to verify that the infinite series in the definition (3.5) of $(E^* f)^{\wedge L}(\mathbf{T}^*)$ converges in the weak topology on $\mathcal{X}$. In fact the left-tangential evaluation with operator argument $f \to (E^* f)^{\wedge L}(\mathbf{T}^*)$ amounts to the adjoint of the observability operator:

$$(E^* f)^{\wedge L}(\mathbf{T}^*) = \mathcal{O}_{E,\mathbf{T}}^* f \quad \text{for} \quad f \in \mathcal{H}_{\mathcal{Y}}(k_d). \tag{3.6}$$

The evaluation map (3.6) extends to multipliers $S \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y})$ by

$$(E^* S)^{\wedge L}(\mathbf{T}^*) = \mathcal{O}_{E,\mathbf{T}}^* M_S : \mathcal{U} \to \mathcal{X} \tag{3.7}$$

and suggests the interpolation problem with operator argument $\mathbf{OAP}(\mathbf{T}, E, N)$ whose data set consists of a commutative d-tuple $\mathbf{T} = (T_1, \dots, T_d)$ and operators $E \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $N \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ such that the pair $(E, \mathbf{T})$ is output stable.

Operator Argument interpolation Problem ($\mathbf{OAP}(\mathbf{T}, E, N)$): *Given the data set* $\{\mathbf{T}, E, N\}$ *as above, find all* $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ *such that*

$$(E^* S)^{\wedge L}(\mathbf{T}^*) := \mathcal{O}_{E,\mathbf{T}}^* M_S|_{\mathcal{U}} = N^*. \tag{3.8}$$

Such problems have been considered in [52, 16, 53, 17] for the commutative and noncommutative setting. In case the d-tuple $\mathbf{T} = (T_1, \dots, T_d)$ is *strongly stable* (see

(2.17)), we shall refer to $\mathbf{OAP}(\mathbf{T}, E, N)$ as a *strongly stable Operator Argument interpolation Problem* ($\mathbf{ssOAP}(\mathbf{T}, E, N)$).

If $(\zeta^{(i)}, a_i^*, c_i^*)$ $(i = 1, \dots, N)$ is the data set for a left Nevanlinna-Pick problem **LNPP** and if we set

$$T_j^* = \begin{bmatrix} \zeta_j^{(1)} & & \\ & \ddots & \\ & & \zeta_j^{(N)} \end{bmatrix} \text{ for } j = 1, \dots, d, \qquad E^* = \begin{bmatrix} a_1^* \\ \vdots \\ a_N^* \end{bmatrix}, \qquad N^* = \begin{bmatrix} c_1^* \\ \vdots \\ c_N^* \end{bmatrix}$$

and if we set $\mathbf{T} = (T_1, \dots, T_d)$, then it is easily checked that condition (3.8) amounts to the **LNPP** interpolation conditions (1.3). Furthermore, as by assumption $\zeta^{(k)} \in \mathbb{B}^d$ for $k = 1, \dots, N$, one can show that $\mathbf{T}$ is strongly stable. Thus we see that any **LNPP** can be encoded as a strongly stable Operator Argument interpolation Problem **ssOAP**. More generally, Carathéodory-Fejér-type interpolation problems can be embedded into **ssOAP** problems—see [16].

We shall see in Theorem 3.3 below that, more generally, **ssOAP** is exactly equivalent to commutant lifting and Sarason interpolation problems after appropriate transformations of the data sets. First we observe a simple necessary conditions for a problem $\mathbf{OAP}(\mathbf{T}, E, N)$ (as well as for a problem $\mathbf{ssOAP}(\mathbf{T}, E, N)$) to have a solution.

Proposition 3.1. *Let $(E, \mathbf{T})$ be an output-stable pair with $E \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, let $S \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y})$ and let N be defined as in (3.8). Then*

1. *The pair $(N, \mathbf{T})$ is output stable and*

$$\mathcal{O}_{E,\mathbf{T}}^* M_S = \mathcal{O}_{N,\mathbf{T}}^* : \mathcal{H}_{\mathcal{U}}(k_d) \to \mathcal{X}. \tag{3.9}$$

2. *If $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, then $\mathcal{O}_{N,\mathbf{T}}^* \mathcal{O}_{N,\mathbf{T}} \leq \mathcal{O}_{E,\mathbf{T}}^* \mathcal{O}_{E,\mathbf{T}}$.*

Hence, if the problem $\mathbf{OAP}(\mathbf{T}, E, N)$ has a solution $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, then $(N, \mathbf{T})$ is also output-stable and

$$P := \mathcal{O}_{E,\mathbf{T}}^* \mathcal{O}_{E,\mathbf{T}} - \mathcal{O}_{N,\mathbf{T}}^* \mathcal{O}_{N,\mathbf{T}} \geq 0. \tag{3.10}$$

Proof. Let $h(z) = \sum_{\mathbf{n} \in \mathbb{Z}_d^+} h_{\mathbf{n}} z^{\mathbf{n}} \in \mathcal{H}_{\mathcal{U}}(k_d)$. By (3.6) and (3.8),

$$\mathcal{O}_{E,\mathbf{T}}^* M_S h = (E^* S h)^{\wedge L}(\mathbf{T}^*) = \sum_{\mathbf{n}, \mathbf{m} \in \mathbb{Z}_d^+} (\mathbf{T}^*)^{\mathbf{n}+\mathbf{m}} E^* S_{\mathbf{m}} h_{\mathbf{n}} \tag{3.11}$$

where the latter series converges weakly since the pair $(E, \mathbf{T})$ is output stable and since $Sh \in \mathcal{H}_{\mathcal{Y}}(k_d)$. On the other hand,

$$\begin{aligned} \mathcal{O}_{N,\mathbf{T}}^* h &= (N^*h)^{\wedge L}(\mathbf{T}^*) = \sum_{\mathbf{n}\in\mathbb{Z}_d^+} \mathbf{T}^{*\mathbf{n}} N^* h_{\mathbf{n}} \\ &= \sum_{\mathbf{n}\in\mathbb{Z}_d^+} \mathbf{T}^{*\mathbf{n}} \left(\sum_{\mathbf{m}\in\mathbb{Z}_d^+} \mathbf{T}^{*\mathbf{m}} E^* S_{\mathbf{m}} \right) h_{\mathbf{n}} = \sum_{\mathbf{n},\mathbf{m}\in\mathbb{Z}_d^+} (\mathbf{T}^*)^{\mathbf{n}+\mathbf{m}} E^* S_{\mathbf{m}} h_{\mathbf{n}} \end{aligned}$$

where all the series converge weakly, since that in (3.11) does. Since h was picked arbitrarily in $\mathcal{H}_{\mathcal{U}}(k_d)$, we get (3.9). The operator $\mathcal{O}_{N,\mathbf{T}}^*$ is bounded and therefore the pair $(N, \mathbf{T})$ is output stable.
If $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, then $I - M_S M_S^* \geq 0$ and by (3.9) we have for every $x \in \mathcal{X}$,

$$\begin{aligned} 0 &\leq \langle (I - M_S M_S^*)\mathcal{O}_{E,\mathbf{T}} x,\ \mathcal{O}_{E,\mathbf{T}} x \rangle_{\mathcal{H}_{\mathcal{Y}}(k_d)} \\ &= \|\mathcal{O}_{E,\mathbf{T}} x\|^2_{\mathcal{H}_{\mathcal{Y}}(k_d)} - \|M_S^* \mathcal{O}_{E,\mathbf{T}} x\|^2_{\mathcal{H}_{\mathcal{U}}(k_d)} \\ &= \|\mathcal{O}_{E,\mathbf{T}} x\|^2_{\mathcal{H}_{\mathcal{Y}}(k_d)} - \|\mathcal{O}_{N,\mathbf{T}} x\|^2_{\mathcal{H}_{\mathcal{U}}(k_d)} \end{aligned}$$

which proves the second statement and completes the proof of the proposition. □

Corollary 3.2. *Conditions* (3.8) *and* (3.9) *are equivalent.*

Proof. Indeed, Proposition 3.1 shows that (3.8) implies (3.9). The converse implication follows upon restricting equality (3.9) to constant functions from $\mathcal{H}_{\mathcal{U}}(k_d)$:

$$\mathcal{O}_{E,\mathbf{T}}^* M_S u = \mathcal{O}_{N,\mathbf{T}}^* u$$

and taking into account that $\mathcal{O}_{E,\mathbf{T}}^* M_S u = (E^*S)^{\wedge L}(\mathbf{T}^*)u$ and $\mathcal{O}_{N,\mathbf{T}}^*|_{\mathcal{U}} = N^*$. □

A convenient way to impose various hypotheses in the formulation of a particular class of interpolation problems is to demand that the data set for the interpolation problem of the class satisfy certain *admissibility* criteria. In what follows, we call the data set $\{\mathcal{M}, \Phi\}$ to be admissible for a **CLP** if $\mathcal{M}$ is a backward shift invariant subspace of $\mathcal{H}_{\mathcal{Y}}(k_d)$ and $\Phi : \mathcal{H}_{\mathcal{U}}(k_d) \to \mathcal{M}$ satisfies relations (3.1), Furthermore, the collection $\{\mathbf{T} = (T_1, \ldots, T_d), E, N\}$ is admissible for an **OAP** if $\mathbf{T}$ is a commutative d-tuple of operators on the Hilbert space $\mathcal{X}$ and $E\colon \mathcal{X} \to \mathcal{Y}$ and $N\colon \mathcal{X} \to \mathcal{U}$ are such that both $(E, \mathbf{T})$ and $(N, \mathbf{T})$ are output-stable. The same collection will be called admissible for a **ssOAP**, if in addition, the d-tuple $\mathbf{T}$ is strongly stable.

Theorem 3.3. *Let $\mathcal{M}$ be a backward shift invariant subspace of $\mathcal{H}_{\mathcal{Y}}(k_d)$, let $\Phi \in \mathcal{H}_{\mathcal{U}}(k_d) \to \mathcal{M}$ satisfy conditions* (3.1) *and let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{L}(\mathcal{M})^d$, $E \in \mathcal{L}(\mathcal{M}, \mathcal{Y})$ and $N \in \mathcal{L}(\mathcal{M}, \mathcal{U})$ be defined by*

$$T_j = M_{z_j}^*|_{\mathcal{M}} \quad (j = 1, \ldots, d), \quad E\colon f \mapsto f(0) \quad \text{and} \quad N\colon h \mapsto (\Phi^* h)(0). \tag{3.12}$$

Then $\{\mathbf{T}, E, N\}$ is an admissible data set for a problem $\mathbf{ssOAP}(\mathbf{T}, E, N)$ *and a contractive multiplier S solves* $\mathbf{CLP}(\mathcal{M}, \Phi)$ *if and only if S solves* $\mathbf{ssOAP}(\mathbf{T}, E, N)$.

Conversely, suppose that $\{\mathbf{T}, E, N\}$ *is an admissible data set for a problem* **ssOAP**. *Set* $\mathcal{M} = \operatorname{Ran} \mathcal{O}_{E,\mathbf{T}}$ *and define* $\Phi\colon \mathcal{H}_{\mathcal{U}}(k_d) \to \mathcal{M}$ *via its adjoint* Φ^**:*

$$\Phi^*\colon \mathcal{O}_{E,\mathbf{T}}x \mapsto \mathcal{O}_{N,\mathbf{T}}x. \tag{3.13}$$

Then $\{\mathcal{M}, \Phi\}$ *is the admissible data set for a Commutant Lifting Problem and a contractive multiplier* $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ *solves* $\mathbf{ssOAP}(\mathbf{T}, E, N)$ *if and only if* S *solves* $\mathbf{CLP}(\mathcal{M}, \Phi)$.

Proof. Let $\mathbf{T}, E, N$ be defined as in (3.12). From the fact that the backward-shift d-tuple $\mathbf{M}_{\boldsymbol{z}}^* = (M_{z_1}^*, \dots, M_{z_d}^*)$ is strongly stable on $\mathcal{H}_{\mathcal{Y}}(k_d)$ we see that $\mathbf{T} = \mathbf{M}_{\boldsymbol{z}}^*|_{\mathcal{M}}$ is strongly stable.
Now note that $(\mathcal{O}_{E,\mathbf{T}}h)(z) = h(z)$ for every $h \in \mathcal{M}$, i.e., that the observability operator $\mathcal{O}_{E,\mathbf{T}}$ acting on an element $h \in \mathcal{M}$ simply reproduces h and hence can be viewed as the operator of inclusion of $\mathcal{M}$ in $\mathcal{H}_{\mathcal{Y}}(k_d)$; in particular, $(E, \mathbf{T})$ is also output-stable. Therefore we have

$$\mathcal{O}_{E,\mathbf{T}}^*\mathcal{O}_{E,\mathbf{T}} = I_{\mathcal{M}}, \quad \mathcal{O}_{E,\mathbf{T}}^*|_{\mathcal{M}} = I_{\mathcal{M}} \tag{3.14}$$

and furthermore,

$$\mathcal{O}_{E,\mathbf{T}}^*|_{\mathcal{M}^\perp} = 0 \quad \text{and} \quad \mathcal{P}_{\mathcal{M}} = \mathcal{O}_{E,\mathbf{T}}^*. \tag{3.15}$$

We refer to [18, Section 3] for more details.
Next we show that, for operators T_j and N given by (3.12), we have

$$\mathcal{O}_{N,\mathbf{T}} = \Phi^* \tag{3.16}$$

from which it will follow immediately that the pair $(N, \mathbf{T})$ is also output-stable. To this end, pick up an arbitrary $h(z) = \sum_{\mathbf{n} \in \mathbb{Z}_d^+} h_{\mathbf{n}} z^{\mathbf{n}} \in \mathcal{M}$ and note that

$$(\mathbf{M}_{\boldsymbol{z}}^{*\mathbf{n}}\Phi^* h)(0) = \frac{\mathbf{n}!}{|\mathbf{n}|!}(\Phi^* h)_{\mathbf{n}} \quad \text{for every} \quad \mathbf{n} \in \mathbb{Z}_+^d.$$

By (3.1), $\Phi^* M_{z_j}^*|_{\mathcal{M}} = M_{z_j}^*\Phi^*$ for $j = 1, \dots, d$ and therefore, $\Phi^*\mathbf{M}_{\boldsymbol{z}}^{*\mathbf{n}}|_{\mathcal{M}} = \mathbf{M}_{\boldsymbol{z}}^{*\mathbf{n}}\Phi^*$ for every $\mathbf{n} \in \mathbb{Z}_+^d$. Now we have

$$\begin{aligned}
N(I - \sum_{j=1}^d z_j T_j)^{-1} h &= \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} z^{\mathbf{n}} N \mathbf{T}^{\mathbf{n}} h \\
&= \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} z^{\mathbf{n}} (\Phi^* \mathbf{M}_{\boldsymbol{z}}^{*\mathbf{n}} h)(0) \\
&= \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} z^{\mathbf{n}} (\mathbf{M}_{\boldsymbol{z}}^{*\mathbf{n}} \Phi^* h)(0) \\
&= \sum_{\mathbf{n} \in \mathbb{Z}_+^d} z^{\mathbf{n}} (\Phi^* h)_{\mathbf{n}} = (\Phi^* h)(z)
\end{aligned}$$

and since h is arbitrary, (3.16) follows. It follows immediately from (3.16) that the pair $(N, \mathbf{T})$ is output-stable and therefore $\{\mathbf{T}, E, N\}$ is an admissible data set for a problem **ssOAP**. By Corollary 3.2, condition (3.8) is equivalent to

$$\mathcal{O}_{E,\mathbf{T}}^{*} M_S = \mathcal{O}_{N,\mathbf{T}}^{*} = \Phi$$

which coincides with (3.4) due to (3.15).

Conversely, let $\{\mathbf{T}, E, N\}$ be an admissible data set for a problem **ssOAP**. Since $\mathbf{T}$ is strongly stale, $\mathcal{M} = \operatorname{Ran} \mathcal{O}_{E,\mathbf{T}}$ is a closed backward-shift-invariant subspace of $\mathcal{H}_{\mathcal{Y}}(k_d)$ and $\mathcal{O}_{E,\mathbf{T}}$ is an isomorphism of $\mathcal{X}$ onto $\mathcal{M}$ (see [18]). We define $\Phi \colon \mathcal{H}_{\mathcal{U}}(k_d) \to \mathcal{M}$ via its adjoint Φ^* given by (3.13). Then, for $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, from Proposition 3.1 we see that S solves $\mathbf{ssOAP}(\mathbf{T}, E, N)$ is equivalent to condition (3.9), or, after taking adjoints, to

$$M_S^* \mathcal{O}_{E,\mathbf{T}} = \mathcal{O}_{N,\mathbf{T}}. \tag{3.17}$$

By definition, this in turn is equivalent to $M_S^*|_{\mathcal{M}} = \Phi^*$, i.e., to S solving the problem $\mathbf{CLP}(\mathcal{M}, \Phi)$. □

Remark 3.4. We have seen in Proposition 3.1 that a necessary condition for a problem $\mathbf{OAP}(\mathbf{T}, E, N)$ to have a solution is that $P \geq 0$ where $P \colon \mathcal{X} \to \mathcal{X}$ is given by (3.10). In the context of a problem $\mathbf{ssOAP}(\mathbf{T}, E, N)$, note that the operator Φ given by (3.13) satisfies $\|\Phi\| \leq 1$ if and only if $P \geq 0$. It thus follows from the Commutant Lifting Theorem that the condition $P \geq 0$ is also sufficient for a problem $\mathbf{ssOAP}(\mathbf{T}, E, N)$ to have a solution. In the next section we shall see that the same statement holds for the more general problem $\mathbf{OAP}(\mathbf{T}, E, N)$.

4. The general Operator Argument interpolation Problem

In this section we present our solution of the general Operator Argument Interpolation Problem, including the parametrization of the set of all solutions for the case where the operator P (3.10) is invertible. As a first step we present several useful reformulations of the problem. The main tool for this analysis is the following well-known Hilbert space result.

Proposition 4.1. *A Hilbert space operator*

$$\begin{bmatrix} P & B^* \\ B & A \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{H} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{H} \end{bmatrix}$$

is positive semidefinite if and only if A is positive semidefinite and for every $x \in \mathcal{X}$, there exists a vector $h_x \in \mathcal{H} \ominus \operatorname{Ker} A$ such that

$$A^{\frac{1}{2}} h_x = Bx \quad \text{and} \quad \|h_x\|_{\mathcal{H}} \leq \|P^{\frac{1}{2}} x\|_{\mathcal{X}}.$$

Theorem 4.2. *Let $\{\mathbf{T}, E, N\}$ be an admissible data set for the* $\mathbf{OAP}(\mathbf{T}, E, N)$*. Let $P\colon \mathcal{X} \to \mathcal{X}$ be defined as in (3.10), let S be an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function defined on $\mathbb{B}^d$, and let $F^S : \mathcal{X} \to \mathcal{H}_{\mathcal{Y}}(k_d)$ be the linear map given by*

$$F^S : x \to (\mathcal{O}_{E,\mathbf{T}} - M_S \mathcal{O}_{N,\mathbf{T}})\, x. \tag{4.1}$$

The following are equivalent:

1. *S is a solution of the* $\mathbf{OAP}(\mathbf{T}, E, N)$.
2. *The operator*
$$\mathbf{P} := \begin{bmatrix} P & (F^S)^* \\ F^S & I - M_S M_S^* \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{H}_{\mathcal{Y}}(k_d) \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{H}_{\mathcal{Y}}(k_d) \end{bmatrix} \tag{4.2}$$
is positive semidefinite.
3. *The following kernel is positive on $\mathbb{B}^d \times \mathbb{B}^d$:*
$$\mathbf{K}(z, \zeta) = \begin{bmatrix} P & G(\zeta)^* \left(E^* - N^* S(\zeta)^*\right) \\ (E - S(z)N)\, G(z) & \dfrac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1 - \langle z, \zeta \rangle} \end{bmatrix} \succeq 0 \tag{4.3}$$
where
$$G(z) = (I - z_1 T_1 - \cdots - z_d T_d)^{-1}. \tag{4.4}$$
4. *$S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and the function $F^S x$ belongs to the de Branges-Rovnyak space $\mathcal{H}(K_S)$ and satisfies*
$$\|F^S x\|_{\mathcal{H}(K_S)} \le \|P^{\frac{1}{2}} x\|_{\mathcal{X}} \quad \text{for every} \quad x \in \mathcal{X}. \tag{4.5}$$
5. *$S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and the function $F^S x$ belongs $\mathcal{H}(K_S)$ and satisfies*
$$\|F^S x\|_{\mathcal{H}(K_S)} = \|P^{\frac{1}{2}} x\|_{\mathcal{X}} \quad \text{for every} \quad x \in \mathcal{X}. \tag{4.6}$$

Proof. First note, as was observed in (3.17), that $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ solves the problem $\mathbf{OAP}(\mathbf{T}, E, N)$ if and only

$$M_S^* \mathcal{O}_{E,\mathbf{T}} = \mathcal{O}_{N,\mathbf{T}} : \ \mathcal{X} \to \mathcal{H}_{\mathcal{U}}(k_d). \tag{4.7}$$

To prove Theorem 4.2, we shall show that (2) $\Longleftrightarrow$ (3) and that (1) $\Longrightarrow$ (5) $\Longrightarrow$ (4) $\Longrightarrow$ (2) $\Longleftrightarrow$ (1).

(2) $\Longleftrightarrow$ (3): Simply note that that the identity

$$\langle \mathbf{P} f, \ f \rangle_{\mathcal{X} \oplus \mathcal{H}_{\mathcal{Y}}(k_d)} = \sum_{j,\ell=1}^{r} \left\langle \mathbf{K}(z^{(j)}, z^{(\ell)}) \begin{bmatrix} x_\ell \\ y_\ell \end{bmatrix}, \begin{bmatrix} x_j \\ y_j \end{bmatrix} \right\rangle_{\mathcal{X} \oplus \mathcal{Y}}$$

holds for every vector $f \in \mathcal{X} \oplus \mathcal{H}_{\mathcal{Y}}(k_d)$ of the form

$$f = \sum_{j=1}^{r} \begin{bmatrix} x_j \\ k_d(\cdot\,, z^{(j)}) y_j \end{bmatrix} \qquad (x_j \in \mathcal{X}, \ y_j \in \mathcal{Y}, \ z^{(j)} \in \mathbb{B}^d)$$

(see [28, Theorem 2.4] for details).

(1) $\Longrightarrow$ (5): Assume that $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ solves $\mathbf{OAP}(\mathbf{T}, E, N)$. Then from (4.7) we see that

$$F^S = \mathcal{O}_{E,\mathbf{T}} - M_S \mathcal{O}_{N,\mathbf{T}} = \mathcal{O}_{E,\mathbf{T}} - M_S M_S^* \mathcal{O}_{E,\mathbf{T}} = (I - M_S M_S^*) \mathcal{O}_{E,T}.$$

Hence

$$\|F^S x\|^2_{\mathcal{H}(K_S)} = \langle (I - M_S M_S^*)\mathcal{O}_{E,\mathbf{T}}x, \mathcal{O}_{E,\mathbf{T}}x\rangle_{\mathcal{H}_{\mathcal{Y}}(k_d)}$$
$$= \langle (\mathcal{O}_{E,\mathbf{T}}^*\mathcal{O}_{E,\mathbf{T}} - \mathcal{O}_{N,\mathbf{T}}^*\mathcal{O}_{N,\mathbf{T}})x, x\rangle_{\mathcal{X}} = \langle Px, x\rangle_{\mathcal{X}} = \|P^{\frac{1}{2}}x\|^2_{\mathcal{X}}$$

for all $x \in \mathcal{X}$ and (5) follows.

(5) $\Longrightarrow$ (4): This is trivial.

(4) $\Longrightarrow$ (2): Since S is in $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, the operator $A := I - M_S M_S^*$ is positive semidefinite on $\mathcal{H}_{\mathcal{Y}}(k_d)$. Furthermore, $F^S x$ belongs to $\mathcal{H}(K_S)$ for every $x \in \mathcal{X}$ which means, due to (2.9) that $F^S x = (I - M_S M_S^*)^{\frac{1}{2}} h_x$ for some element $h_x \in \mathcal{H}_{\mathcal{Y}}(k_d)$ which can be chosen to be orthogonal to the $\operatorname{Ker}(I - M_S M_S^*)$. Then the norm constraint (4.5) implies

$$\left\|(I - M_S M_S^*)^{\frac{1}{2}} h_x\right\|_{\mathcal{H}(K_S)} = \|h_x\|_{\mathcal{H}_{\mathcal{Y}}(k_d)} \le \|P^{\frac{1}{2}}x\|_{\mathcal{X}}$$

and positivity of the operator (4.2) follows by Proposition 4.1.

(2) $\Longleftrightarrow$ (1): Let the operator (4.2) be positive semidefinite. Then the operator $I - M_S M_S^*$ is positive semidefinite (equivalently, M_S is a contraction) which implies $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. By definitions (3.10) and (4.1) we have

$$\mathbf{P} = \begin{bmatrix} \mathcal{O}_{E,\mathbf{T}}^*\mathcal{O}_{E,\mathbf{T}} - \mathcal{O}_{N,\mathbf{T}}^*\mathcal{O}_{N,\mathbf{T}} & \mathcal{O}_{E,\mathbf{T}}^* - \mathcal{O}_{N,\mathbf{T}}^* M_S^* \\ \mathcal{O}_{E,\mathbf{T}} - M_S\mathcal{O}_{N,\mathbf{T}} & I - M_S M_S^* \end{bmatrix} \ge 0.$$

By the standard Schur complement argument, the latter inequality is equivalent to

$$\widehat{\mathbf{P}} := \begin{bmatrix} I_{\mathcal{H}_{\mathcal{U}}(k_d)} & \mathcal{O}_{N,\mathbf{T}} & M_S^* \\ \mathcal{O}_{N,\mathbf{T}}^* & \mathcal{O}_{E,\mathbf{T}}^*\mathcal{O}_{E,\mathbf{T}} & \mathcal{O}_{E,\mathbf{T}}^* \\ M_S & \mathcal{O}_{E,\mathbf{T}} & I_{\mathcal{H}_{\mathcal{Y}}(k_d)} \end{bmatrix} \ge 0,$$

since $\mathbf{P}$ is the Schur complement of the block $I_{\mathcal{H}_{\mathcal{U}}(k_d)}$ in $\widehat{\mathbf{P}}$. On the other hand, the latter inequality holds if and only if the Schur complement of the block $I_{\mathcal{H}_{\mathcal{Y}}(k_d)}$ in $\widehat{\mathbf{P}}$ is positive semidefinite:

$$\begin{bmatrix} I_{\mathcal{H}_{\mathcal{U}}(k_d)} & \mathcal{O}_{N,\mathbf{T}} \\ \mathcal{O}_{N,\mathbf{T}}^* & \mathcal{O}_{E,\mathbf{T}}^*\mathcal{O}_{E,\mathbf{T}} \end{bmatrix} - \begin{bmatrix} M_S^* \\ \mathcal{O}_{E,\mathbf{T}}^* \end{bmatrix} \begin{bmatrix} M_S & \mathcal{O}_{E,\mathbf{T}} \end{bmatrix} \ge 0. \tag{4.8}$$

Now we write (4.8) as

$$\begin{bmatrix} I_{\mathcal{H}_{\mathcal{U}}(k_d)} - M_S^* M_S & \mathcal{O}_{N,\mathbf{T}} - M_S^*\mathcal{O}_{E,\mathbf{T}} \\ \mathcal{O}_{N,\mathbf{T}}^* - \mathcal{O}_{E,\mathbf{T}}^* M_S & 0 \end{bmatrix} \ge 0$$

and arrive at $\mathcal{O}_{E,\mathbf{T}}^* M_S = \mathcal{O}_{N,\mathbf{T}}$ which means that S is a solution of $\mathbf{OAP}(\mathbf{T}, E, N)$. Since we have already proved (1) $\Longrightarrow$ (5) $\Longrightarrow$ (4) $\Longrightarrow$ (2), it follows that (1) $\Longrightarrow$ (2). However we think it instructive to include the following direct path:

(1) $\Longrightarrow$ (2): If $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is a solution of $\mathbf{OAP}(\mathbf{T}, E, N)$, then we know that $I - M_S M_S^* \ge 0$ and we have the identity (4.7). Recalling also the definition (3.10) of P, we then have

$$\mathbf{P} = \begin{bmatrix} \mathcal{O}_{E,\mathbf{T}}^*\mathcal{O}_{E,\mathbf{T}} - \mathcal{O}_{E,\mathbf{T}}^* M_S M_S^* \mathcal{O}_{E,\mathbf{T}} & \mathcal{O}_{E,\mathbf{T}}^*(I - M_S M_S^*) \\ (I - M_S M_S^*)\mathcal{O}_{E,\mathbf{T}} & I - M_S M_S^* \end{bmatrix}$$
$$= \begin{bmatrix} \mathcal{O}_{E,\mathbf{T}}^* \\ I \end{bmatrix} (I - M_S M_S^*) \begin{bmatrix} \mathcal{O}_{E,\mathbf{T}}^* & I \end{bmatrix} \geq 0$$

and (2) follows.

Similarly, since we have already proved (2) $\Longrightarrow$ (1) $\Longrightarrow$ (5) $\Longrightarrow$ (4), we know that (2) $\Longrightarrow$ (4). However, for purposes of more general considerations to come in the next section, we shall need the following direct path:

(2) $\Longrightarrow$ (4): If the operator (4.2) is positive semidefinite, then by Proposition 4.1 the operator $I - M_S M_S^*$ is positive semidefinite (i.e., M_S is a contraction which implies $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$) and for every $x \in \mathcal{X}$, there exists a function $h_x \in \mathcal{H}_{\mathcal{Y}}(k_d)$ which is orthogonal to the Ker $(I - M_S M_S^*)$ such that

$$(I - M_S M_S^*)^{\frac{1}{2}} h_x = F^S x \quad \text{and} \quad \|h_x\|_{\mathcal{H}_{\mathcal{Y}}(k_d)} \leq \|P^{\frac{1}{2}} x\|_{\mathcal{X}}. \tag{4.9}$$

The first relation in (4.9) implies in particular that $F^S x$ belongs to Ran$(I - M_S M_S^*)^{\frac{1}{2}}$ or equivalently, to $\mathcal{H}(K_S)$, due to characterization (2.9). Furthermore, since h_x is orthogonal to Ker $(I - M_S M_S^*)$, we conclude from (2.10) and (4.9) that

$$\begin{aligned} \|F^S x\|_{\mathcal{H}(K_S)} &= \|(I - M_S M_S^*)^{\frac{1}{2}} h_x\|_{\mathcal{H}(K_S)} \\ &= \|(I - \pi) h_x\|_{\mathcal{H}_{\mathcal{Y}}(k_d)} = \|h_x\|_{\mathcal{H}_{\mathcal{Y}}(k_d)} \leq \|P^{\frac{1}{2}} x\|_{\mathcal{X}}, \end{aligned}$$

which proves (4.5). □

As a corollary, we get the following curious reformulation of the $\mathbf{CLP}(\mathcal{M}, \Phi)$.

Theorem 4.3. *Assume that we are given an admissible data set $\{\mathcal{M}, \Phi\}$ for a commutant lifting problem and that the necessary conditions* (3.1) *and* (3.2) *are satisfied so that the operator $P := I_{\mathcal{M}} - \Phi\Phi^*$ is positive semidefinite. Then $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is a solution of the $\mathbf{CLP}(\mathcal{M}, \Phi)$ if and only if for every $h \in \mathcal{M}$, the function*

$$F_h(z) := h(z) - S(z)(\Phi^* h)(z) \tag{4.10}$$

belongs to the de Branges-Rovnyak space $\mathcal{H}(K_S)$ and $\|F_h\|_{\mathcal{H}(K_S)} \leq \|P^{\frac{1}{2}} h\|_{\mathcal{M}}$.

Proof. By Theorem 3.3, a function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is a solution of the $\mathbf{CLP}(\mathcal{M}, \Phi)$ if and only if it is a solution of the $\mathbf{OAP}(\mathbf{T}, E, N)$ with the data given by (3.12) which is equivalent (by Theorem 4.2) to the function

$$(F^S h)(z) := (E - S(z)N)\,(I_{\mathcal{M}} - \sum_{j=1}^d z_j T_j)^{-1} h \tag{4.11}$$

being an element of $\mathcal{H}(K_S)$ and satisfying $\|F^S h\|_{\mathcal{H}(K_S)} \leq \|P^{\frac{1}{2}} h\|_{\mathcal{M}}$ for every $h \in \mathcal{X} = \mathcal{M}$. It remains to show that the right hand side expressions in formulas

(4.11) and (4.10) coincide, i.e., that

$$(E - S(z)N)\,(I_{\mathcal{M}} - \sum_{j=1}^{d} z_j T_j)^{-1} h = h(z) - S(z)(\Phi^* h)(z) \quad \text{for every} \quad h \in \mathcal{M}.$$

But this follows from (3.16) and the second equality in (3.14). □

Reformulation of the problem $\mathbf{OAP}(\mathbf{T}, E, N)$ in terms of the operator

$$F^S = \begin{bmatrix} I & -M_S \end{bmatrix} \mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], \mathbf{T}}$$

mapping $(\mathcal{X}, P)$ contractively into the de Branges-Rovnyak space $\mathcal{H}(K_S)$ (condition (4) in Theorem 4.2), when combined with Theorems 2.3 and 2.8, leads immediately the a linear-fractional description of the set of all solutions in case the Pick operator $P = \mathcal{G}^J_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], \mathbf{T}}$ (see (2.20) and (3.10)) is strictly positive definite.

Theorem 4.4. *Let $\{\mathbf{T}, E, N\}$ be an admissible data set for the* **OAP** *and let the operator P be defined as in* (3.10) *be strictly positive. Also let*

$$\mathfrak{A}(z) = \begin{bmatrix} \mathfrak{A}_{11}(z) & \mathfrak{A}_{12}(z) \\ \mathfrak{A}_{21}(z) & \mathfrak{A}_{22}(z) \end{bmatrix} = D + \begin{bmatrix} E \\ N \end{bmatrix} (I - Z(z)T)^{-1} Z(z) B$$

be the $(\mathbf{J}, J)$-inner operator-valued function constructed according to the recipe in Theorem 2.3. *Then an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function S is a solution of the problem* $\mathbf{OAP}(\mathbf{T}, E, N)$ *if and only if S can be written in the form*

$$S(z) = (\mathfrak{A}_{11}(z)\mathcal{E}(z) + \mathfrak{A}_{12}(z))\,(\mathfrak{A}_{21}(z)\mathcal{E}(z) + \mathfrak{A}_{22}(z))^{-1}, \tag{4.12}$$

for some $\mathcal{E} \in \mathcal{S}_d(\mathcal{U},\, \mathcal{Y} \oplus \mathcal{X}^{d-1})$.
Moreover the condition $P \geq 0$ is both necessary and sufficient for the problem $\mathbf{OAP}(\mathbf{T}, E, N)$ *to have solutions.*

Proof. By condition (4) in Theorem 4.2 we know that S solves $\mathbf{OAP}(\mathbf{T}, E, N)$ if and only if $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and the operator

$$F^S := \begin{bmatrix} I & -M_S \end{bmatrix} \begin{bmatrix} \mathcal{O}_{E,\mathbf{T}} \\ \mathcal{O}_{N,\mathbf{T}} \end{bmatrix}$$

maps $(\mathcal{X}, P)$ contractively into the de Branges-Rovnyak space $\mathcal{H}(K_S)$. By Theorem 2.3, we know that $\mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], \mathbf{T}}$ is a unitary identification between $(\mathcal{X}, P)$ and $\mathcal{H}(K^P_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], \mathbf{T}}) = \mathcal{H}(K^{\mathbf{J},J}_{\mathfrak{A}})$. Hence the condition for S to solve $\mathbf{OAP}(\mathbf{T}, E, N)$ translates to: $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ *and the operator* $\begin{bmatrix} I & -M_S \end{bmatrix}$ *maps* $\mathcal{H}(K^{\mathbf{J},J}_{\mathfrak{A}})$ *contractively into* $\mathcal{H}(K_S)$. By Theorem 2.8, this last condition is equivalent to $S = T_{\mathfrak{A}}[\mathcal{E}]$ for some $\mathcal{E} \in \mathcal{S}_d(\mathcal{U}, \mathcal{F})$.
If P is strictly positive definite, it follows in particular that $\mathbf{OAP}(\mathbf{T}, E, N)$ has solutions. If we only have $P \geq 0$, then via a rescaling the result for the strictly positive definite case implies that, for each $\delta > 0$ there exists solutions $S_\delta \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ of the interpolation conditions (3.8) with $\|M_{S_\delta}\| \leq 1 + \delta$. The existence of a solution

S of (3.8) with $\|M_S\| \leq 1$ then follows by a standard weak-$*$ compactness argument which makes use of the fact that the operators $\mathcal{O}_{E,\mathbf{T}}$ and $\mathcal{O}_{N,\mathbf{T}}$ have range inside the Drury-Arveson spaces $\mathcal{H}_{\mathcal{Y}}(k_d)$ and $\mathcal{H}_{\mathcal{U}}(k_d)$ respectively. The necessity of the condition $P \geq 0$ for the existence of solutions is the content of part (2) of Proposition 3.1. □

5. The analytic Abstract Interpolation Problem

The very formulation of the problem $\mathbf{OAP}(\mathbf{T}, E, N)$ appears to require that the operators $\mathcal{O}_{E,\mathbf{T}}$ and $\mathcal{O}_{N,\mathbf{T}}$ be bounded operators from $\mathcal{X}$ into $H_{\mathcal{Y}}(k_d)$ and $\mathcal{H}_{\mathcal{U}}(k_d)$ respectively. However, upon close inspection, one can see that conditions (2), (3), (4) in Theorem 4.2 make sense if we take P to be any positive semidefinite operator on $\mathcal{X}$ and if we only assume that

(a) $\mathbf{T} = (T_1, \dots, T_d)$ is a not necessarily commutative d-tuple of operators on $\mathcal{X}$ and $E\colon \mathcal{X} \to \mathcal{Y}$ and $N\colon \mathcal{X} \to \mathcal{U}$ are such that

$$\mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right],\mathbf{T}} : x \mapsto \begin{bmatrix} E \\ N \end{bmatrix} (I - Z(z)T)^{-1}x$$

maps $\mathcal{X}$ into the space $Hol_{\mathcal{Y}\oplus\mathcal{U}}(\mathbb{B}^d)$ of holomorphic $(\mathcal{Y}\oplus\mathcal{U})$-valued functions on $\mathbb{B}^d$.

A careful inspection of the proof of Theorem 4.2 (specifically, of steps (2) $\iff$ (3), (4) $\implies$ (2) and (2) $\implies$ (4) (direct path)) shows that the mutual equivalence of conditions (2), (3), (4) continues to hold in this more general situation. This suggests that we use any of these conditions as the *definition* of a more general interpolation problem. In order to apply Theorems 2.4 and 2.8 to this more general situation, we must also require:

(b) P is a positive semidefinite solution of the Stein equation (2.21).

This leads to the formulation of the *analytic Abstract Interpolation Problem*:

The analytic Abstract Interpolation Problem ($\mathbf{aAIP}(\mathbf{T}, E, N, P)$): *Given the data* $\{E, N, \mathbf{T}, P\}$ *subject to assumptions* (a), (b)*, find all* $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ *such that the function* $F^S x$ *defined as in* (4.1) *belongs to the de Branges-Rovnyak space* $\mathcal{H}(K_S)$ *and satisfies the norm constraint* (4.5)*.*

The next Theorem summarizes the observations made above. We note that the proof of the linear-fractional parametrization goes through Theorem 2.4 in place of Theorem 2.3.

Theorem 5.1. *Let* P*,* $\mathbf{T}$*,* E *and* N *satisfy assumptions* (a), (b)*. The following are equivalent:*

1. *S is a solution of the* $\mathbf{aAIP}(E, N, \mathbf{T}, P)$*.*
2. *The operator* $\mathbf{P} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{H}_{\mathcal{Y}}(k_d))$ *of the form* (4.2) *is positive semidefinite.*
3. *The kernel* $\mathbf{K}(z, \zeta)$ *of the form* (4.3) *is positive on* $\mathbb{B}^d \times \mathbb{B}^d$*.*

Moreover, if P is strictly positive definite and if the function $\mathfrak{A} \in \mathcal{S}_d^{J,\mathbf{J}}(\mathcal{F}\oplus\mathcal{U}, \mathcal{Y}\oplus\mathcal{U})$ is constructed as in Theorem 2.4, then S solves $\mathbf{aAIP}(E, N, \mathbf{T}, P)$ if and only if S can be realized in the form (4.12) for a Schur-class function $\mathcal{E} \in \mathcal{S}_d(\mathcal{U}, \mathcal{F})$.

Remark 5.2. It is tempting to use a weak-$*$ compactness argument as in the proof of Theorem 4.4 to conclude from Theorem 5.1 that the problem **aAIP** always has a solution (even when P is only positive semidefinite rather than strictly positive definite). However the details of such an argument are not so clear since the observability operators $\mathcal{O}_{E,\mathbf{T}}$ and $\mathcal{O}_{N,\mathbf{T}}$ no longer have range in the Drury-Arveson space. We will see that the problem $\mathbf{aAIP}(\mathbf{T}, E, N, P)$ always has solutions based on a different approach whereby we get a description, even in the degenerate case, of the set of all solutions in terms of a Redheffer-type linear-fractional map (see Corollary 6.6 below).

By Theorem 4.2, the $\mathbf{OAP}(\mathbf{T}, E, N)$ is a particular case of the $\mathbf{aAIP}(E, N, \mathbf{T}, P)$ (corresponding to a commutative $\mathbf{T}$, output stable $(E, \mathbf{T})$, $(N, \mathbf{T})$ and the Pick operator P defined in (3.10). One of the special features of this case is expressed by the equivalence (**4**) $\Leftrightarrow$ (**5**) in Theorem 4.2: for every solution S of the problem, inequality (4.5) implies equality (4.6) (in [41] such problems were called *possessing the Parseval equality*). We next present another interesting particular case of the $\mathbf{aAIP}(E, N, \mathbf{T}, P)$ for which this phenomenon does not take place.

The boundary Nevanlinna-Pick problem: *Given n points $t^{(i)} = (t_1^{(i)}, \dots, t_d^{(i)})$ ($i = 1, \dots, n$) on the unit sphere $\mathbb{S}^d = \{t = (t_1, \dots, t_d) : \sum_{j=1}^d |t_j|^2 = 1\}$, given vectors $\xi_i \in \mathcal{Y}$ and $\eta_i \in \mathcal{U}$ and given numbers $\gamma_i \geq 0$, ($i = 1, \dots, n$), find all $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ such that*

$$\lim_{r\to 1} S(rt^{(i)})^*\xi_j = \eta_j \quad \text{and} \quad \lim_{r\to 1} \left\langle \frac{I_{\mathcal{Y}} - S(rt^{(i)})S(rt^{(i)})^*}{1 - r^2}\, \xi_j,\ \xi_j \right\rangle_{\mathcal{Y}} \leq \gamma_i \tag{5.1}$$

for $i = 1, \dots, n$.

This problem has been studied in [4, 27, 14]. From [27, 14] it is known that the problem has a solution if and only if

$$\|\xi_i\|_{\mathcal{Y}} = \|\eta_i\|_{\mathcal{U}} \quad \text{for} \quad i = 1, \dots, n \tag{5.2}$$

and the matrix $P = [P_{ij}]_{i,j=1}^n$ with the entries

$$P_{ij} = \frac{\langle \xi_j, \xi_i\rangle_{\mathcal{Y}} - \langle \eta_j, \eta_i\rangle_{\mathcal{U}}}{1 - \langle t^{(i)}, t^{(j)}\rangle} \quad (i \neq j) \quad \text{and} \quad P_{ii} = \gamma_i$$

is positive semidefinite. It is easily seen that P defined as above satisfies the Stein identity

$$P - \sum_{j=1}^d T_j^* P T_j = E^*E - N^*N \tag{5.3}$$

where T_j's are the diagonal $n \times n$ matrices defined by

$$T_j = \operatorname{diag}\left\{t_1^{(j)}, t_2^{(j)}, \dots, t_n^{(j)}\right\} \quad j = 1, \dots, d.$$

and where

$$E = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \end{bmatrix}, \quad N = \begin{bmatrix} \eta_1 & \eta_2 & \dots & \eta_n \end{bmatrix}.$$

Note that the pairs $(E, \mathbf{T})$ and $(N, \mathbf{T})$ are not output stable, but the functions $E(I - Z(z)T)^{-1}$ and $N(I - Z(z)T)^{-1}$ are analytic on $\mathbb{B}^d$. Thus the problem $\mathbf{aAIP}(E, N, \mathbf{T}, P)$ is well defined. As it was shown in [27], this latter problem is equivalent to the boundary Nevanlinna-Pick problem (5.1). Note that equalities (5.2) are necessary for the Stein equation (5.3) to have a solution. Note also that P is not completely determined by the Stein equation (5.1); its diagonal entries form a piece of information independent of that contained in $\mathbf{T}$, E and N. Finally, note that equality (4.6) for a solution S of the $\mathbf{aAIP}(E, N, \mathbf{T}, P)$ corresponds to equalities in the second series of interpolation conditions (5.1). When one considers the boundary Nevanlinna-Pick problem with the inequality in (5.1) replaced by equality, then a necessary and sufficient condition for existence of solutions as well as a description for the set of all solutions is unknown; it is known that $P \geq 0$ is necessary and that $P > 0$ is sufficient for existence of solutions.

Remark 5.3. If the tuple $\mathbf{T} = (T_1, \dots, T_d)$ is not commutative, then condition (3.8) is not equivalent to (3.9) in general. For example, let

$$T_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}.$$

Then

$$(I - z_1 T_1 - z_2 T_2)^{-1} = \begin{bmatrix} 1 & z_1 & z_1 z_2 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix},$$

so that

$$\begin{aligned} \mathcal{O}_{E,\mathbf{T}} x &= x_1 + x_2 z_1 + x_3 z_1 z_2, \\ \mathcal{O}_{N,\mathbf{T}} x &= x_1 \alpha + x_2(\alpha z_1 + \beta) + x_3(\alpha z_1 z_2 + \beta z_2 + \gamma) \quad \text{if} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{C}^3. \end{aligned}$$

Using the definition of inner product in $\mathcal{H}(K_d)$ it is readily seen that

$$\mathcal{O}^*_{E,\mathbf{T}} f = \begin{bmatrix} f_{00} \\ f_{01} \\ \frac{1}{2} f_{11} \end{bmatrix} \quad \text{and} \quad \mathcal{O}^*_{N,\mathbf{T}} f = \begin{bmatrix} \overline{\alpha} f_{00} \\ \overline{\alpha} f_{01} + \overline{\beta} f_{00} \\ \frac{\overline{\alpha}}{2} f_{11} + \overline{\beta} f_{01} + \overline{\gamma} f_{00} \end{bmatrix}$$

if $f(z_1, z_2) = \sum_{i,j=0}^{\infty} f_{ij} z_1^i z_2^j \in \mathcal{H}(k_2)$. Condition (3.8) is equivalent to

$$S_{00} = \overline{\alpha}, \quad S_{01} = \overline{\beta}, \quad S_{11} = 2\overline{\gamma}. \tag{5.4}$$

On the other hand, condition (3.9) is equivalent to the conditions

$$(Sh)_{00} = \overline{\alpha} h_{00}, \quad (Sh)_{01} = \overline{\alpha} h_{10} + \overline{\beta} h_{00}, \quad \frac{1}{2}(Sh)_{11} = \frac{\overline{\alpha}}{2} h_{11} + \overline{\beta} h_{01} + \overline{\gamma} h_{00},$$

or, in more detail,

$$\begin{aligned}
&S_{00}h_{00} = \overline{\alpha}h_{00},\\
&S_{00}h_{01} + S_{01}h_{00} = \overline{h_{01}} + \overline{\beta}h_{00},\\
&\frac{1}{2}\left(S_{00}h_{11} + S_{10}h_{01} + S_{01}h_{10} + S_{11}h_{00}\right) = \frac{\overline{\alpha}}{2}h_{11} + \overline{\beta}h_{01} + \overline{\gamma}h_{00}
\end{aligned}$$

holding for every $h(z_1, z_2) = \sum_{i,j=0}^{\infty} h_{ij} z_1^i z_2^j \in \mathcal{H}(k_2)$. It is easily checked that this is the case if and only if

$$S_{00} = \overline{\alpha}, \quad S_{10} = S_{01} = \overline{\beta} = 0, \quad S_{11} = 2\overline{\gamma}. \tag{5.5}$$

We conclude that conditions (5.4) and (5.5) are not equivalent.

6. The Abstract Interpolation Problem

We are now ready to formulate the Abstract Interpolation Problem **AIP** based on a data set $\{D, \mathfrak{T}, \mathbf{T}, E, N\}$ described as follows. We are given a linear space $\mathcal{X}$, a positive semidefinite Hermitian form D on $\mathcal{X}$, Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, linear operators $\mathfrak{T}$, $\mathbf{T} = (T_1, \dots, T_d)$ on $\mathcal{X}$, and linear operators $N\colon \mathcal{X} \to \mathcal{U}$ and $E\colon \mathcal{X} \to \mathcal{Y}$. In addition we assume that

$$D(\mathfrak{T}x, \mathfrak{T}x) + \|Nx\|_{\mathcal{U}}^2 = \sum_{j=1}^{d} D(T_j x, T_j x) + \|Ex\|_{\mathcal{Y}}^2 \quad \text{for every } x \in \mathcal{X}. \tag{6.1}$$

Definition 6.1. *Suppose that we are given the data set* $\{D, \mathfrak{T}, \mathbf{T}, E, N\}$ *for an* **AIP** *as in* (6.1). *We say that the pair* (F, S) *is a solution of the* **AIP** *if* S *is a Schur-class function* $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ *and* F *is a linear mapping from* $\mathcal{X}$ *into* $\mathcal{H}(K_S)$ *such that the two conditions hold:*

$$\|Fx\|_{\mathcal{H}(K_S)}^2 \le D(x, x) \quad \textit{for all} \quad x \in \mathcal{X}, \tag{6.2}$$

$$(F\mathfrak{T}x)(z) - \sum_{j=1}^{d} z_j (FT_j x)(z) = (E - S(z)N)\,x \quad \textit{for all} \quad z \in \mathbb{B}^d. \tag{6.3}$$

Denote by $\mathcal{X}_0$ the Hilbert space equal to the completion of the space of equivalence classes of elements of $\mathcal{X}$ (where the zero equivalence class consists of elements x with $D(x, x) = 0$) in the D-inner product. Note that if (S, F) solves **AIP**, then condition (6.2) implies that F has a factorization $F_0 \circ \pi$ where π is the canonical projection operator $\pi\colon \mathcal{X} \to \mathcal{X}_0$ and where $F_0\colon \mathcal{X}_0 \to \mathcal{H}(K_S)$ has $\|F_0\| \le 1$. We abuse notation and denote also by $\mathfrak{T}$ and T_k the operators $\mathfrak{T}$ and T_k followed by the canonical projection into the equivalence class in $\mathcal{X}_0$; so $\mathfrak{T}$, $T_k\colon \mathcal{X} \to \mathcal{X}_0$. Let for short

$$T = \begin{bmatrix} T_1 \\ \vdots \\ T_d \end{bmatrix} \colon \mathcal{X} \to \mathcal{X}_0^d, \qquad Z(z) = \begin{bmatrix} z_1 I_{\mathcal{X}_0} & \cdots & z_d I_{\mathcal{X}_0} \end{bmatrix}.$$

If we further identify F_0 with the operator-valued holomorphic function $z \mapsto F_0(z) \in \mathcal{L}(\mathcal{X}_0, \mathcal{Y})$ defined by

$$F_0(z)x_0 = (F_0 x_0)(z)$$

then we can rewrite (6.3) in the form

$$F_0(z)\mathfrak{T} - F_0(z)Z(z)T = E - S(z)N \quad \text{for all} \quad z \in \mathbb{B}^d. \tag{6.4}$$

Note that the import of the hypothesis (6.1) is that there is a well-defined isometry $\mathbf{V}$ from

$$\mathcal{D}_{\mathbf{V}} = \overline{\operatorname{Ran}} \begin{bmatrix} \mathfrak{T} \\ N \end{bmatrix} \subset \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \end{bmatrix} \quad \text{onto} \quad \mathcal{R}_{\mathbf{V}} = \overline{\operatorname{Ran}} \begin{bmatrix} T \\ E \end{bmatrix} \subset \begin{bmatrix} \mathcal{X}_0^d \\ \mathcal{Y} \end{bmatrix} \tag{6.5}$$

such that

$$\mathbf{V}\colon \begin{bmatrix} \mathfrak{T} \\ N \end{bmatrix} x \to \begin{bmatrix} T \\ E \end{bmatrix} x \quad \text{for all } x \in \mathcal{X}. \tag{6.6}$$

Note also that the definition (6.5) and (6.6) of $\mathbf{V}$ is completely determined by the problem data $\{D, \mathfrak{T}, \mathbf{T}, E, N\}$.

If $\mathcal{X}$ is already a Hilbert space and there exists a bounded positive semidefinite operator $P \geq 0$ such that $D(x, y) = \langle Px, \ y\rangle_{\mathcal{X}}$ for every x, $y \in \mathcal{X}$, then identity (6.1) can be written as

$$\mathfrak{T}^* P \mathfrak{T} - \sum_{k=1}^{d} T_j^* P T_j = E^*E - N^*N.$$

Furthermore, equality (6.4) can be written as

$$F_0(z)(\mathfrak{T} - Z(z)T)x = (E - S(z)N)\, x$$

and if the pencil $(\mathfrak{T} - Z(z)T)$ is invertible for every $z \in \mathbb{B}^d$, then the latter equation defines F_0 uniquely by

$$F_0(z)x = (E - S(z)N)\,(\mathfrak{T} - Z(z)T)^{-1}x.$$

If furthermore, $\mathfrak{T} = I_{\mathcal{X}}$, then it is readily seen that the $\mathbf{AIP}(D, I_{\mathcal{X}}, \mathbf{T}, E, N)$ collapses to the $\mathbf{aAIP}(E, N, \mathbf{T}, P)$.

However, it can happen that $S(z)$ does not uniquely determine $F_0(z)$ and therefore the problem **AIP** cannot be reduced to a problem **aAIP**. The following example illustrates this point.

Example 6.2. Consider the following single-variable example. Choose operators $\mathcal{T}$ and T on a Hilbert space $\mathcal{X}$ so that

1. the pencil $\mathcal{T} - zT$ is singular,
2. $T^*T - \mathcal{T}^*\mathcal{T}$ is positive semidefinite, and
3. $\operatorname{Ran}\mathcal{T} + \operatorname{Ran} T$ is dense in $\mathcal{X}$

One such choice is

$$\mathcal{X} = \mathbb{C}^2, \qquad \mathcal{T} = \begin{bmatrix} 0 & 0 \\ \delta & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

for some $\delta > 0$. Then choose the operator $N\colon \mathcal{X} \to \mathcal{U}$ so that

$$N^*N = T^*T - \mathcal{T}^*\mathcal{T}.$$

For the example at hand, we take

$$\mathcal{U} = \mathbb{C}, \qquad N = \begin{bmatrix} \sqrt{1-\delta^2} & 0 \end{bmatrix}.$$

We also take the output space $\mathcal{Y} = \mathbb{C}$ and we set $E = 0\colon \mathcal{X} \to \mathcal{Y}$. We take the Hermitian form D on $\mathcal{X}$ to be $D(x,x) = \langle x, x\rangle_{\mathcal{X}}$. Then $\{D, \mathcal{T}, T, E, N\}$ is an admissible data set for an **AIP** in the sense above with $d = 1$. If we define $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ by $S(z) = 0$, then $\mathcal{H}(K_S) = H^2$. We seek solutions of the **AIP** associated with this data set such that $S(z) = 0$. Thus we seek $F\colon \mathcal{X} = \mathbb{C}^2 \to \mathcal{H}(K_S) = H^2$ so that

$$(F\mathcal{T}x)(z) - z(FTx)(z) = Ex - S(z)Nx = 0.$$

The associated operator-valued function $z \mapsto F_0(z)$ given by $F_0(z)x = (Fx)(z)$ then must satisfy

$$F_0(z)\mathcal{T}x - zF_0(z)Tx = 0.$$

Expressing $F_0(z)$ as a row matrix $F_0(z) = \begin{bmatrix} F_1(z) & F_2(z) \end{bmatrix}$, we arrive at

$$\begin{bmatrix} F_1(z) & F_2(z) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \delta & 0 \end{bmatrix} - z \begin{bmatrix} F_1(z) & F_2(z) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

which can be rewritten as

$$\begin{bmatrix} \delta F_2(z) - zF_1(z) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Solving gives

$$\begin{bmatrix} F_1(z) & F_2(z) \end{bmatrix} = F_0(z) \begin{bmatrix} \delta & z \end{bmatrix}$$

where F_0 is a free-parameter H^2-function. Choosing F_0 of sufficiently small 2-norm then guarantees that the resulting operator $M_F\colon \mathcal{X} \to H^2$ is contractive. In particular, such F's are not uniquely determined. Thus there are many distinct solution-pair solutions to **AIP** of the form $(F, 0)$.

Our next goal is to show that solutions of a problem $\mathbf{AIP}(D, \mathfrak{T}, \mathbf{T}, E, N)$ correspond to minimal unitary-colligation extensions of the partially defined isometric colligation $\mathbf{V}$ in (6.5), (6.6). Here we say that the unitary colligation $\mathcal{C}$ with connecting operator $\mathbf{U}\colon \mathcal{H} \oplus \mathcal{U} \to \mathcal{H} \oplus \mathcal{Y}$ is a *minimal unitary-colligation extension of* $\mathbf{V}$ if

1. $\mathcal{X}_0$ is a subspace of $\mathcal{H}$,
2. $\mathbf{U}|_{\mathcal{D}_{\mathbf{V}}} = \mathbf{V}\colon \mathcal{D}_{\mathbf{V}} \to \mathcal{R}_{\mathbf{V}}$, and
3. the smallest reducing subspace for U contained in $\mathcal{H}$ and containing $\mathcal{X}$ is the whole space $\mathcal{H}$.

Theorem 6.3. *Let* $\mathbf{V}$ *be the isometry defined by* (6.6) *associated with the data of a problem* **AIP** *and let*

$$\mathcal{C} = \left\{ \mathcal{X}_0 \oplus \mathcal{X}_1, \mathcal{U}, \mathcal{Y}, \mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\}$$

be a minimal unitary-colligation extension of $\mathbf{V}$*. Define* $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ *and* $F_0 \colon \mathcal{X}_0 \oplus \mathcal{X}_1 \to \mathcal{H}(K_S)$ *by*

$$S(z) = D + C(I - Z(z)A)^{-1} Z(z)B, \tag{6.7}$$

$$F_0(z) = C(I - Z(z)A)^{-1}|_{\mathcal{X}_0}. \tag{6.8}$$

Then the pair (S, F_0) *is a solution of* **AIP***.*
Conversely, every solution of **AIP** *arises in this way.*

Proof. Let $\mathcal{H} = \mathcal{X}_0 \oplus \mathcal{X}_1$, let

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}^d \\ \mathcal{Y} \end{bmatrix}$$

be the connecting operator for a minimal unitary-colligation extension of $\mathbf{V}$ and let

$$S(z) = D + C\,(I - Z(z)A)^{-1}\,Z(z)B \tag{6.9}$$

be the characteristic function of the colligation $\mathcal{C} = \{\mathcal{H},\ \mathcal{U},\ \mathcal{Y},\ \mathbf{U}\}$. Then $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ by Theorem 2.1. Furthermore, let $\mathcal{H}(K_S)$ be the associated de Branges-Rovnyak space and define $F_0 \colon \mathcal{X} \to \mathcal{H}(K_S)$ by

$$(F_0 x_0)(z) = F_0(z)x_0 = C(I - Z(z)A)^{-1} \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \ \text{ for } x_0 \in \mathcal{X}_0. \tag{6.10}$$

Then F_0 is a contraction (see e.g. part (2) of Theorem 2.1 in [19]). It remains to check the identity (6.4) which, due to (6.10) is the same as

$$H(z)\mathfrak{T}x = H(z)Z(z)Tx + Ex - S(z)Nx \tag{6.11}$$

(with $H(z)$ as in (2.5)). Using the unitary realization (6.9) of S written as

$$S(z) = D + H(z)Z(z)B,$$

we rewrite (6.11) as

$$H(z)\mathfrak{T} = H(z)Z(z)T + E - [D + H(z)Z(z)B]N \tag{6.12}$$

To establish (6.12), we use the identity

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathfrak{T} \\ N \end{bmatrix} = \begin{bmatrix} T \\ E \end{bmatrix},$$

or, in more detail,

$$A\mathfrak{T} + BN = T, \qquad C\mathfrak{T} + DN = E,$$

which is true by the hypothesis that $\mathbf{U}$ extends $\mathbf{V}$, to see that the right hand side of (6.12) is equal to

$$\begin{aligned} &H(z)Z(z)T + E - DN - H(z)Z(z)BN \\ &= H(z)Z(z)T + C\mathfrak{T} - H(z)Z(z)(T - A\mathfrak{T}) \\ &= C\mathfrak{T} + H(z)Z(z)A\mathfrak{T} = H(z)\mathfrak{T} \end{aligned}$$

as wanted.

We postpone the proof of the converse direction to the proof of Theorem 6.4 where a more general result is proved. We leave as an open question the problem of finding a direct proof of the converse direction. □

We next introduce the defect spaces

$$\Delta = \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \end{bmatrix} \ominus \mathcal{D}_{\mathbf{V}} \quad \text{and} \quad \Delta_* = \begin{bmatrix} \mathcal{X}_0^d \\ \mathcal{Y} \end{bmatrix} \ominus \mathcal{R}_{\mathbf{V}}$$

and let $\widetilde{\Delta}$ be another copy of Δ and $\widetilde{\Delta}_*$ another copy of Δ_* with unitary identification maps

$$i\colon \Delta \to \widetilde{\Delta} \quad \text{and} \quad i_*:\ \Delta_* \to \widetilde{\Delta}_*.$$

Define a unitary operator $\mathbf{U}_0$ from $\mathcal{D}_{\mathbf{V}} \oplus \Delta \oplus \widetilde{\Delta}_*$ onto $\mathcal{R}_{\mathbf{V}} \oplus \Delta_* \oplus \widetilde{\Delta}$ by the rule

$$\mathbf{U}_0 x = \begin{cases} \mathbf{V}x, & \text{if } x \in \mathcal{D}_{\mathbf{V}}, \\ i(x) & \text{if } x \in \Delta, \\ i_*^{-1}(x) & \text{if } x \in \widetilde{\Delta}_*. \end{cases} \tag{6.13}$$

Identifying $\begin{bmatrix} \mathcal{D}_{\mathbf{V}} \\ \Delta \end{bmatrix}$ with $\begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \end{bmatrix}$ and $\begin{bmatrix} \mathcal{R}_{\mathbf{V}} \\ \Delta_* \end{bmatrix}$ with $\begin{bmatrix} \mathcal{X}_0^d \\ \mathcal{Y} \end{bmatrix}$, we decompose $\mathbf{U}_0$ defined by (6.13) according to

$$\mathbf{U}_0 = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & 0 \end{bmatrix} : \quad \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \\ \widetilde{\Delta}_* \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_0^d \\ \mathcal{Y} \\ \widetilde{\Delta} \end{bmatrix}. \tag{6.14}$$

The $(3,3)$ block in this decomposition is zero, since (by definition (6.13)), for every $\widetilde{\delta}_* \in \widetilde{\Delta}_*$, the vector $\mathbf{U}_0\widetilde{\delta}_*$ belongs to Δ, which is a subspace of $\begin{bmatrix} \mathcal{X}_0^d \\ \mathcal{Y} \end{bmatrix}$ and therefore, is orthogonal to $\widetilde{\Delta}$; in other words $\mathcal{P}_{\widetilde{\Delta}}\mathbf{U}_0|_{\widetilde{\Delta}_*} = 0$ where $\mathcal{P}_{\widetilde{\Delta}}$ stands for the orthogonal projection of $\mathcal{R}_{\mathbf{V}} \oplus \Delta_* \oplus \widetilde{\Delta}$ onto $\widetilde{\Delta}$.
The unitary operator $\mathbf{U}_0$ is the connecting operator of the unitary colligation

$$\mathcal{C}_0 = \left\{ \mathcal{X}_0,\ \begin{bmatrix} \mathcal{U} \\ \widetilde{\Delta}_* \end{bmatrix},\ \begin{bmatrix} \mathcal{Y} \\ \widetilde{\Delta} \end{bmatrix},\ \mathbf{U}_0 \right\}, \tag{6.15}$$

which is called *the universal unitary colligation* associated with the **AIP**.
According to (2.6), the characteristic function of the colligation $\mathcal{C}_0$ defined in (6.15) is given by

$$\begin{aligned} \Sigma(z) &= \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} \\ &= \begin{bmatrix} U_{22} & U_{23} \\ U_{32} & 0 \end{bmatrix} + \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} (I - Z(z)U_{11})^{-1} Z(z) \begin{bmatrix} U_{12} & U_{13} \end{bmatrix} \\ &= \mathcal{P}_{\mathcal{Y}\oplus\widetilde{\Delta}}\mathbf{U}_0(I - \mathcal{P}_{\mathcal{X}_0}^* Z(z)\mathcal{P}_{\mathcal{X}_0}^* Z(z)\mathcal{P}_{\mathcal{X}_0^d}\mathbf{U}_0)^{-1}|_{\mathcal{U}\oplus\widetilde{\Delta}_*} \end{aligned} \tag{6.16}$$

and belongs to the class $\mathcal{S}_d(\mathcal{U} \oplus \widetilde{\Delta}_*,\ \mathcal{Y} \oplus \widetilde{\Delta})$ by Theorem 2.1. The associated observability operator is given by

$$\begin{aligned} H_\Sigma(z) &= \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} (I - Z(z)U_{11})^{-1} \\ &= \mathcal{P}_{\mathcal{Y}\oplus\widetilde{\Delta}} \mathbf{U}_0 (I - \mathcal{P}_{\mathcal{X}_0}^* Z(z) \mathcal{P}_{\mathcal{X}_0^d} \mathbf{U}_0)^{-1}|_{\mathcal{X}_0}. \end{aligned} \tag{6.17}$$

By another application of Theorem 2.1 we see that

$$K_\Sigma(z,\zeta) := \frac{I - \Sigma(z)\Sigma(\zeta)^*}{1 - \langle z, \zeta \rangle} = H_\Sigma(z) H_\Sigma(\zeta)^*. \tag{6.18}$$

We shall also need an enlarged colligation

$$\mathcal{C}_{0,e} = \left\{ \mathcal{X}_0, \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \\ \widetilde{\Delta}_* \end{bmatrix}, \begin{bmatrix} \mathcal{Y} \\ \widetilde{\Delta} \end{bmatrix}, \mathbf{U}_{0,e} = \begin{bmatrix} U_{11} & U_{11} & U_{12} & U_{13} \\ U_{21} & U_{21} & U_{22} & U_{23} \\ U_{31} & U_{31} & U_{32} & 0 \end{bmatrix} \right\} \tag{6.19}$$

with associated characteristic function

$$\begin{aligned} \Sigma_e(z) &= \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{31} & U_{32} & 0 \end{bmatrix} + \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} (I - Z(z)U_{11})^{-1} Z(z) \begin{bmatrix} U_{11} & U_{12} & U_{13} \end{bmatrix} \\ &= \begin{bmatrix} U_{21}(I - Z(z)U_{11})^{-1} & \Sigma_{11}(z) & \Sigma_{12}(z) \\ U_{31}(I - Z(z)U_{11})^{-1} & \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} \\ &= \mathcal{P}_{\mathcal{Y}\oplus\widetilde{\Delta}} \mathbf{U}_0 (I - \mathcal{P}_{\mathcal{X}_0}^* Z(z) \mathcal{P}_{\mathcal{X}_0^d} \mathbf{U}_0)^{-1}. \end{aligned} \tag{6.20}$$

These are the ingredients for the following parametrization for the set of all solutions of **AIP**. In particular, solutions of **AIP** exist for any data set $(D, \mathfrak{T}, \mathbf{T}, E, N)$ which is admissible (i.e., condition (6.1) is satisfied).

Theorem 6.4. *Suppose that* $(D, \mathfrak{T}, \mathbf{T}, E, N\}$ *is an admissible data set for a problem* **AIP**. *Let* $\mathbf{U}_0$ *be the universal unitary-colligation extension of* $\mathbf{V}$ *given by* (6.13) *with characteristic function* (6.16) *and let* $\mathbf{U}_{0,e}$ *be the connecting operator for the enlarged universal unitary colligation* $\mathcal{C}_{0,e}$ *given by* (6.19)*. Then the pair* $(S(z),\ F_0(z))$ *solves the problem* **AIP** *if and only if there is a Schur-class multiplier* $\mathcal{W} \in \mathcal{S}_d(\widetilde{\Delta}, \widetilde{\Delta}_*)$ *such that*

$$\begin{bmatrix} F_0(z) & S(z) \end{bmatrix} = \mathfrak{R}_{\Sigma_e}[\mathcal{W}](z), \tag{6.21}$$

i.e., such that

$$\begin{aligned} S(z) &= \Sigma_{11}(z) + \Sigma_{12}(z)(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}\mathcal{W}(z)\Sigma_{21}(z), \\ F_0(z) &= U_{21}(I - Z(z)U_{11})^{-1} \\ &\quad + \Sigma_{12}(z)(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}\mathcal{W}(z)U_{31}(I - Z(z)U_{11})^{-1}. \end{aligned} \tag{6.22}$$

Proof. We have already seen in Theorem 6.3 that $(F_0(z), S(z))$ is a solution of **AIP** whenever

$$\begin{aligned} S(z) &= D + C(I - Z(z)A)^{-1}B, \\ F_0(z) &= C(I - Z(z)A)^{-1}|_{\mathcal{X}_0} \end{aligned} \tag{6.23}$$

where

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \oplus \mathcal{X}_1 \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_0^d \oplus \mathcal{X}_1^d \\ \mathcal{Y} \end{bmatrix}$$

is the connecting operator for a unitary-colligation extension of the partially defined isometry $\mathbf{V}$ ((6.5) and (6.6)). It is also known (see [10, 11]) that such unitary extensions $\mathbf{U}$ are parametrized via a free-parameter closely-connected unitary colligation matrix

$$\mathbf{U}_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \widetilde{\Delta} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_1 \\ \widetilde{\Delta}_* \end{bmatrix}$$

in *feedback connection* with the universal colligation $\mathbf{U}_0$:

$$\mathbf{U}\colon \begin{bmatrix} x_0 \\ x_1 \\ u \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{x}_0 \\ \widetilde{x}_1 \\ y \end{bmatrix}$$

if and only if there exist $\widetilde{\delta} \in \widetilde{\Delta}$, $\widetilde{\delta}_* \in \widetilde{\Delta}_*$ so that

$$\mathbf{U}_0\colon \begin{bmatrix} x_0 \\ u \\ \widetilde{\delta}_* \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{x}_0 \\ y \\ \widetilde{\delta} \end{bmatrix} \quad \text{and} \quad \mathbf{U}_1\colon \begin{bmatrix} x_1 \\ \widetilde{\delta} \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{\delta}_* \end{bmatrix}.$$

One can solve explicitly for $\mathbf{U} := \mathcal{F}_{\mathbf{U}_0}[\mathbf{U}_1]$ and arrive at

$$\begin{aligned} \mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \begin{bmatrix} \begin{bmatrix} \mathbf{A}_{00} & \mathbf{A}_{01} \\ \mathbf{A}_{10} & \mathbf{A}_{11} \end{bmatrix} & \begin{bmatrix} \mathbf{B}_0 \\ \mathbf{B}_1 \end{bmatrix} \\ \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 \end{bmatrix} & \mathbf{D} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} U_{11} + U_{12} D_1 U_{31} & U_{13} C_1 \\ B_1 U_{31} & A_1 \end{bmatrix} & \begin{bmatrix} U_{12} + U_{13} D_1 U_{32} \\ B_1 U_{32} \end{bmatrix} \\ \begin{bmatrix} U_{21} + U_{23} D_1 U_{31} & U_{23} C_1 \end{bmatrix} & U_{22} + U_{23} D_1 U_{32} \end{bmatrix}. \end{aligned} \tag{6.24}$$

To plug this formula into (6.23) we need to be able to compute the resolvent term $(I - Z(z)A)^{-1}$. The following computation is an adaptation of the ideas in [43] where the same result for the case $d = 1$ is proved (for the closely related **AIP** problem where the de Branges-Rovnyak space is taken to have two components). We introduce the associated multidimensional system of Fornasini-Marchesini type (see [18])

$$\Sigma(\mathbf{U})\colon \begin{cases} x(n) &= \mathbf{A}_1 x(n - \mathbf{e}_1) + \cdots + \mathbf{A}_d x(n - \mathbf{e}_d) \\ & \qquad + \mathbf{B}_1 u(n - \mathbf{e}_1) + \cdots + b B_d u(n - \mathbf{e}_d) \\ y(n) &= \mathbf{C} x(n) + \mathbf{D} u(n) \end{cases}$$

where we write

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_d \end{bmatrix} : \mathcal{X}_0 \oplus \mathcal{X}_1 \to \begin{bmatrix} \mathcal{X}_0 \oplus \mathcal{X}_1 \\ \vdots \\ \mathcal{X}_0 \oplus \mathcal{X}_1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \vdots \\ \mathbf{B}_d \end{bmatrix} : \mathcal{U} \to \begin{bmatrix} \mathcal{X}_0 \oplus \mathcal{X}_1 \\ \vdots \\ \mathcal{X}_0 \oplus \mathcal{X}_1 \end{bmatrix}.$$

We assume an initial condition $x(0) = x^{(0)}$ and we assume zero boundary conditions: $x(n) = 0$ for $n \in \mathbb{Z}^d_+ \setminus \{0\}$ with $n_k = 0$ for some $k \in \{1, \dots, d\}$. Then we define the initial condition/input–output map

$$T_{\Sigma(\mathbf{U})} \colon \begin{bmatrix} \mathcal{X}_0 \oplus \mathcal{X}_1 \\ \mathcal{H}_{\mathcal{U}}(k_d) \end{bmatrix} \to \begin{bmatrix} \mathrm{Hol}_{\mathcal{X}_0 \oplus \mathcal{X}_1}(\mathbb{B}^d) \\ \mathcal{H}_{\mathcal{Y}}(k_d) \end{bmatrix} \tag{6.25}$$

by

$$T_{\Sigma(\mathbf{U})} \colon \begin{bmatrix} x^{(0)} \\ \sum_{n \in \mathbb{Z}^d_+} u(n) \end{bmatrix} \mapsto \begin{bmatrix} \sum_{n \in \mathbb{Z}^d_+} x(n) z^n \\ \sum_{n \in \mathbb{Z}^d} y(n) \end{bmatrix}.$$

Then it can be shown (see e.g. [18]) that

$$T_{\Sigma(\mathbf{U})} = \begin{bmatrix} (I - Z(z)\mathbf{A})^{-1} & (I - Z(z)\mathbf{A})^{-1}\mathbf{B} \\ \mathbf{C}(I - Z(z)\mathbf{A})^{-1} & S(z) \end{bmatrix}.$$

This object in turn can be computed as the feedback connection $\mathcal{F}_{T_{\Sigma(\mathbf{U}_0)}}[T_{\Sigma(\mathbf{U}_1)}]$ of the state/input—state-trajectory/output map

$$T_{\Sigma(\mathbf{U}_0)} = \begin{bmatrix} (I - Z(z)U_{11})^{-1} & (I - Z(z)U_{11})^{-1} Z(z) \begin{bmatrix} U_{12} & U_{13} \end{bmatrix} \\ \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} (I - Z(z)U_{11})^{-1} & \Sigma(z) \end{bmatrix}$$

with the the state/input–state-trajectory/output map as the free-parameter load

$$T_{\Sigma(\mathbf{U}_1)} = \begin{bmatrix} (I - Z(z)A_1)^{-1} & (I - Z(z)A_1)^{-1} Z(z) B_1 \\ C_1 (I - Z(z)A_1)^{-1} & \mathcal{W}(z) \end{bmatrix}$$

where

$$\mathcal{W}(z) = D_1 + C_1 (I - Z(z)A_1)^{-1} Z(z) B_1$$

is the characteristic function of the free-parameter unitary colligation $\mathcal{C}_1$, and therefore is itself a free-parameter Schur-class function in $\mathcal{S}_d(\widetilde{\Delta}, \widetilde{\Delta}_*)$. We note that $(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}$ makes sense for all $z \in \mathbb{B}^d$ since both $\mathcal{W}$ and Σ_{22} are Schur multipliers and hence have contractive values on $\mathbb{B}^d$ and the value of $\Sigma_{22}(z)$ is actually strictly contractive for $z \in \mathbb{B}^d$ since $\Sigma_{22}(0) = 0$. Hence the feedback connection $\mathcal{F}_{T_{\Sigma(\mathbf{U}_0)}}[T_{\Sigma(\mathbf{U}_1)}]$ is well-defined. Moreover, one can see that

$$T_{\Sigma(\mathbf{U})} = \mathcal{F}_{T_{\Sigma(\mathbf{U}_0)}}[T_{\Sigma(\mathbf{U}_1)}]$$

and hence the various matrix entries on the right-hand side of (6.25) can be computed explicitly in terms of matrix entries of $\mathbf{U}_0$ and $\mathbf{U}_1$. In particular, one can show that

$$(I - Z(z)\mathbf{A})^{-1} = \begin{bmatrix} X_{00}(z) & X_{01}(z) \\ X_{10}(z) & X_{11}(z) \end{bmatrix}$$

where

$$\begin{aligned}X_{00}(z) &= (I-Z(z)U_{11})^{-1} + (I-Z(z)U_1)^{-1}Z(z)U_{13}(I-\mathcal{W}(z)\Sigma_{22}(z))^{-1}\mathcal{W}(z)\\ &\qquad \times U_{31}(I-Z(z)U_{11})^{-1},\\ X_{01}(z) &= (I-Z(z)U_{11})^{-1}Z(z)U_{13}(I-\mathcal{W}(z)\Sigma_{22}(z))^{-1}C_1(I-Z(z)A_1)^{-1},\\ X_{10}(z) &= (I-Z(z)A_1)^{-1}B_1(I-\Sigma_{22}(z)\mathcal{W}(z))^{-1}U_{31}(I-Z9z)U_{11})^{-1},\\ X_{11}(z) &= (I-Z(z)A_1)^{-1} + (I-Z(z)A_1)^{-1}Z(z)B_1(I-\Sigma_{22}(z)\mathcal{W}(z))^{-1}\Sigma_{22}(z)\\ &\qquad \times C_1(I-Z(z)A_1)^{-1}. \end{aligned} \tag{6.26}$$

Using (6.24) and (6.26), we compute from (6.23) that

$$\begin{aligned}S(z) &= U_{22} + U_{23}D_1U_{32}\\ &\qquad + \begin{bmatrix}U_{21}+U_{23}D_1U_{33} & U_{23}C_1\end{bmatrix}\begin{bmatrix}X_{00}(z) & X_{01}(z)\\ X_{10}(z) & X_{11}(z)\end{bmatrix}\begin{bmatrix}U_{12}+U_{13}D_1U_{32}\\ B_1U_{32}\end{bmatrix},\\ F_0(z) &= \begin{bmatrix}U_{21}+U_{23}D_1U_{31} & U_{23}C_1\end{bmatrix}\begin{bmatrix}X_{00}(z)\\ X_{10}(z)\end{bmatrix}\\ &= (U_{21}+U_{23}D_1U_{31})X_{00}(z) + U_{23}C_1X_{10}(z).\end{aligned}$$

After a lengthy but elementary calculation, one can see that these formulas collapse to (6.22) as asserted. Furthermore, a closer look at these formulas reveals that the two equations in (6.22) can be combined into a single matrix equation

$$\begin{bmatrix}F_0(z) & S(z)\end{bmatrix} = \Sigma_{e,11}(z) + \Sigma_{e,12}(z)(I-\mathcal{W}(z)\Sigma_{22,e}(z))^{-1}\mathcal{W}(z)\Sigma_{e,21}(z)$$

where

$$\begin{aligned}\Sigma_e(z) &= \begin{bmatrix}\Sigma_{e,11}(z) & \Sigma_{e,12}(z)\\ \Sigma_{e,21}(z) & \Sigma_{e,22}(z)\end{bmatrix}\\ &= \begin{bmatrix}\begin{bmatrix}U_{21}(I-Z(z)U_{11})^{-1} & \Sigma_{11}(z)\end{bmatrix} & \Sigma_{12}(z)\\ \\ \begin{bmatrix}U_{31}(I-Z(z)U_{11})^{-1} & \Sigma_{21}(z)\end{bmatrix} & \Sigma_{22}(z)\end{bmatrix}\\ &= \begin{bmatrix}U_{21} & U_{22} & U_{23}\\ U_{31} & U_{32} & 0\end{bmatrix} + \begin{bmatrix}U_{21}\\ U_{31}\end{bmatrix}(I-Z(z)U_{11})^{-1}Z(z)\begin{bmatrix}U_{11} & U_{12} & U_{13}\end{bmatrix}\end{aligned}$$

from which we see that $\Sigma_e(z)$ can be viewed as the characteristic function of the colligation $\mathcal{C}_{0,e}$ in (6.19). We conclude that the solution $(F_0(z), S(z))$ of the problem **AIP** has the compact representation given by (6.21).

Conversely, let us suppose that the pair $(F_0(z), S(z))$ is a solution of **AIP**. The problem is to show that necessarily $\begin{bmatrix}F_0(z) & S(z)\end{bmatrix}$ is in the range of the linear-fractional map $\mathfrak{R}_{\Sigma_e}$ acting on the Schur class $\mathcal{S}_d(\widetilde{\Delta}, \widetilde{\Delta}_*)$. To show that there is a $\mathcal{W} \in \mathcal{S}_d(\widetilde{\Delta}, \widetilde{\Delta}_*)$ so that (6.21) holds, by Theorem 2.9 it suffices to produce a Schur pair $(\mathbf{a}, \mathbf{c})$ so that

$$\begin{bmatrix}I & \mathbf{c}(z)\end{bmatrix}\Sigma_e(z) = \begin{bmatrix}F_0(z) & S(z) & \mathbf{a}(z)\end{bmatrix}.$$

Using the last expression for $\Sigma_e(z)$ in (6.20), we may rewrite this condition as

$$\begin{bmatrix} I & \mathbf{c}(z) \end{bmatrix} P_{\mathcal{Y}\oplus\widetilde{\Delta}} \mathbf{U}_0 (I - \mathcal{P}_{\mathcal{X}_0^*} Z(z) \mathcal{P}_{\mathcal{X}_0^d} \mathbf{U}_0)^{-1} = \begin{bmatrix} F_0(z) & S(z) & \mathbf{a}(z) \end{bmatrix}$$

which in turn can be converted to the more linear form

$$\begin{bmatrix} I & \mathbf{c}(z) \end{bmatrix} \mathcal{P}_{\mathcal{Y}\oplus\widetilde{\Delta}} \mathbf{U}_0 = \begin{bmatrix} F_0(z) & S(z) & \mathbf{a}(z) \end{bmatrix} (I - \mathcal{P}_{\mathcal{X}_0}^* Z(z) \mathcal{P}_{\mathcal{X}_0^d} \mathbf{U}_0). \tag{6.27}$$

Let us define analytic operator-valued functions

$$\mathbf{a} \colon \mathbb{B}^d \to \mathcal{L}(\widetilde{\Delta}_*, \mathcal{Y}), \qquad \mathbf{c} \colon \mathbb{B}^d \to \mathcal{L}(\widetilde{\Delta}, \mathcal{Y})$$

by the formulas

$$\mathbf{a} = F_0(z) \left(Z(z) \mathcal{P}_{\mathcal{X}_0^d} \mathbf{U}_0 \right)\Big|_{\widetilde{\Delta}_*} + \mathcal{P}_{\mathcal{Y}} \mathbf{U}_0|_{\widetilde{\Delta}_*}, \tag{6.28}$$

$$\mathbf{c} = F_0(z) \mathcal{P}_{\mathcal{X}_0} \mathbf{U}_0^*|_{\widetilde{\Delta}} + S(z)\, \mathcal{P}_{\mathcal{U}} \mathbf{U}_0^*|_{\widetilde{\Delta}}. \tag{6.29}$$

Our goal is to show that $(\mathbf{a}, \mathbf{c})$ is a Schur-pair satisfying the condition (6.27). This will then complete the proof of Theorem 6.4.

Note that the condition (6.27) must be verified on vectors from the space $\mathcal{X}_0 \oplus \mathcal{U} \oplus \widetilde{\Delta}_*$. Recall that $\mathcal{X}_0 \oplus \mathcal{U}$ has the alternative decomposition

$$\mathcal{X}_0 \oplus \mathcal{U} = \mathcal{D}_{\mathbf{V}} \oplus \Delta.$$

To verify the validity of (6.27), it suffices to consider the three cases: (1) $y \in \mathcal{D}_{\mathbf{V}}$, (2) $y \in \Delta$, and (3) $y \in \widetilde{\Delta}_*$.

Case 1: $y \in \mathcal{D}_{\mathbf{V}}$. By construction a dense subset of $\mathcal{D}_{\mathbf{V}}$ consists of vectors of the form

$$y = \begin{bmatrix} \mathfrak{T}x \\ Nx \\ 0 \end{bmatrix} \text{ where } x \in \mathcal{X}.$$

By definition we then have

$$\mathbf{U}_0 y = \begin{bmatrix} Tx \\ Ex \\ 0 \end{bmatrix}.$$

Then condition (6.27) applied to the vector y for this case becomes simply

$$Ex = F_0(z)\mathfrak{T}x + S(z)Ex - F_0(z)Z(z)\mathbf{T}x$$

which holds true due the data-admissibility condition (6.4). Note that this case holds automatically independently of the definition of $(\mathbf{a}, \mathbf{c})$.

Step 2: $y = \begin{bmatrix} \delta \\ 0 \end{bmatrix}$ with $\delta \in \Delta$. Note that in this case

$$\mathbf{U}_0 \begin{bmatrix} \delta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ i(\delta) \end{bmatrix}.$$

and hence the left-hand side of (6.27) is simply

$$\mathbf{c}(z) i(\delta). \tag{6.30}$$

On the other hand, the right-hand side is

$$F_0(z)\mathcal{P}_{\mathcal{X}_0}\delta + S(z)\mathcal{P}_{\mathcal{U}}\delta. \tag{6.31}$$

The equality of (6.30) with (6.31) amounts to the definition of $\mathbf{c}(z)$ in (6.29).

Step 3: $y = 0 \oplus 0 \oplus \widetilde{\delta}_*$ **with** $\widetilde{\delta}_* \in \widetilde{\Delta}_*$**.** In this case we know that

$$\mathbf{U}_0 y = \begin{bmatrix} i_*^{-1}(\delta_*) \\ 0 \end{bmatrix}.$$

Then the left-hand side of (6.27) applied to a vector y of this form gives us

$$\begin{bmatrix} I & \mathbf{c}(z) \end{bmatrix} \begin{bmatrix} P_{\mathcal{Y}} i_*^{-1}(\widetilde{\delta}_*) \\ 0 \end{bmatrix} = P_{\mathcal{Y}} i_*^{-1}(\widetilde{\delta}_*) \tag{6.32}$$

while the right-hand side gives us

$$\begin{bmatrix} F_0(z) & S(z) & \mathbf{a}(z) \end{bmatrix} \begin{bmatrix} -Z(z)\mathcal{P}_{\mathcal{X}_0^d} i_*^{-1}(\widetilde{\delta}_*) \\ 0 \\ \widetilde{\delta}_* \end{bmatrix} = -F_0(z)Z(z)\mathcal{P}_{\mathcal{X}_0^d} i_*^{-1}(\widetilde{\delta}_*) + \mathbf{a}(z)\widetilde{\delta}_*. \tag{6.33}$$

Equality of (6.32) with (6.33) collapses to the definition (6.28) of $\mathbf{a}(z)$.

It remains only to verify that $(\mathbf{a}, \mathbf{c})$ defined via (6.28) and (6.29) is a Schur-pair. We use the notation $H_\Sigma(z)$ for the observability operator (6.17) associated with the universal colligation $\mathcal{C}_0$ and $H(z)$ for any function giving rise to a factorization of the kernel $K_S(z,\zeta)$ as in (2.1). We note that a particular consequence of (6.27) is that

$$\begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z) \end{bmatrix} H_\Sigma(z)x_0 = F_0(z)x_0 \in \mathcal{H}(K_S) \tag{6.34}$$

for each $x_0 \in \mathcal{X}_0$. Furthermore, for every $x \in \mathcal{X}_0$, there is a unique $g_x \in \mathcal{H}$ which is orthogonal to $\operatorname{Ker} H(z)$ for every $z \in \mathbb{B}^d$ and such that

$$\begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z) \end{bmatrix} H_\Sigma(z)x = H(z)g_x.$$

Therefore we can define a linear operator $\Gamma:\ \mathcal{X} \to \mathcal{H}$ by the rule $\Gamma x = g_x$. Thus,

$$\begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z) \end{bmatrix} H_\Sigma(z) = H(z)\Gamma. \tag{6.35}$$

By the definition of the norm in $\mathcal{H}(K_S)$,

$$\|Fx\|_{\mathcal{H}(K_S)} = \|g_x\|_{\mathcal{H}} = \|\Gamma x\|_{\mathcal{H}}.$$

On the other hand, the operator $F\colon\ \mathcal{X}_0 \to \mathcal{H}(K_S)$ is contractive by assumption; hence $\|Fx\|_{\mathcal{H}(K_S)} \le \|x\|_{\mathcal{X}_0}$ and Γ is a contraction:

$$\|\Gamma x\|_{\mathcal{H}} = \|Fx\|_{\mathcal{H}(K_S)} \le \|x\|_{\mathcal{X}_0}.$$

The next step is to show that the functions $\mathbf{a}$ and $\mathbf{c}$ defined in (6.28) and (6.29) satisfy

$$\frac{\mathbf{a}(z)\mathbf{a}(\zeta)^* - \mathbf{c}(z)\mathbf{c}(\zeta)^*}{1 - \langle z, \zeta \rangle} = R(z)R(\zeta)^*$$

with

$$R(z) = H(z)(I - \Gamma\Gamma^*)^{\frac{1}{2}} \tag{6.36}$$

from which it will follow that $(\mathbf{a},\mathbf{c})$ is a Schur-pair. Indeed, by (6.34), (2.1), (6.18) and (6.35),

$$\begin{aligned}
&\frac{\mathbf{a}(z)\mathbf{a}(\zeta)^* - \mathbf{c}(z)\mathbf{c}(\zeta)^*}{1-\langle z,\,\zeta\rangle}\\
&= \frac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1-\langle z,\,\zeta\rangle} + \frac{\begin{bmatrix} S(z) & \mathbf{a}(z)\end{bmatrix}\begin{bmatrix} S(\zeta)^* \\ \mathbf{a}(\zeta)^*\end{bmatrix} - \begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z)\end{bmatrix}\begin{bmatrix} I_{\mathcal{Y}} \\ \mathbf{c}(\zeta)^*\end{bmatrix}}{1-\langle z,\,\zeta\rangle}\\
&= \frac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1-\langle z,\,\zeta\rangle} - \begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z)\end{bmatrix}\frac{I-\Sigma(z)\Sigma(\zeta)^*}{1-\langle z,\,\zeta\rangle}\begin{bmatrix} I_{\mathcal{Y}} \\ \mathbf{c}(\zeta)^*\end{bmatrix}\\
&= H(z)H(\zeta)^* - \begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z)\end{bmatrix} H_{\Sigma}(z)H_{\Sigma}(\zeta)^*\begin{bmatrix} I_{\mathcal{Y}} \\ \mathbf{c}(\zeta)^*\end{bmatrix}\\
&= H(z)\left(I-\Gamma\Gamma^*\right)H(\zeta)^* = R(z)R(\zeta)^*,
\end{aligned}$$

where R is defined in (6.36) It follows that $(\mathbf{a},\mathbf{c})$ is a Schur-pair and the proof of Theorem 6.4 is complete. □

We are now in position to complete the proof of Theorem 6.3

Proof of the converse direction in Theorem 6.3. Suppose that $(F_0(z), S_0(z))$ is a solution of **AIP**. Then Theorem 6.4 tells us that there is a Schur-class multiplier $\mathcal{W}\in\mathcal{S}_d(\widetilde{\Delta},\widetilde{\Delta}_*)$ so that $S(z)=\mathfrak{R}_{\Sigma}[\mathcal{W}](z)$. If we plug in a closely-connected unitary colligation

$$\mathcal{C}_1 = \{\mathcal{X}_1, \widetilde{\Delta}, \widetilde{\Delta}_*, \mathbf{U}_1\}$$

having $\mathcal{W}$ as its characteristic function together with the realization $\mathcal{C}_0$ for Σ, we arrive at a realization

$$\mathcal{C} = \{\mathcal{X}_0\oplus\mathcal{X}_1, \mathcal{U}, \mathcal{Y}, \mathbf{U}\}$$

for $S(z)$ having connecting operator $\mathbf{U}$ which is a minimal unitary-colligation extension of $\mathbf{V}$. Moreover, the associated mapping $F_0(z)$ is the restriction of the associated observability operator $x\mapsto \mathcal{P}_{\mathcal{Y}}\mathbf{U}(I-P^*_{\mathcal{X}_0\oplus\mathcal{X}_1}Z(z)P_{(\mathcal{X}_0\oplus\mathcal{X}_1)^d}\mathbf{U})^{-1}|_{\mathcal{X}_0\oplus\mathcal{X}_1}$ to $\mathcal{X}_0$. Thus every solution of the **AIP** arises from the procedure given in the statement of Theorem 6.3. □

Corollary 6.5. *Suppose that* $\{\mathbf{T},E,N\}$ *is an admissible data set for a problem* $\mathbf{OAP}(\mathbf{T},E,N)$ *and we set*

$$P = \mathcal{O}^*_{E,\mathbf{T}}\mathcal{O}_{E,\mathbf{T}} - \mathcal{O}^*_{N,\mathbf{T}}\mathcal{O}_{N,\mathbf{T}}.$$

Then P is the minimal solution of the Stein equation (2.21)*, i.e., if $\widetilde{P}$ is a solution of* (2.21) *with $\widetilde{P}\le P$, then $\widetilde{P}=P$.*

Proof. Let $\widetilde{P}$ be a positive semidefinite solution of the Stein equation (2.21) and let us assume that $\widetilde{P}\le P := \mathcal{O}^*_{E,\mathbf{T}}\mathcal{O}_{E,\mathbf{T}} - \mathcal{O}^*_{N,\mathbf{T}}\mathcal{O}_{N,\mathbf{T}}$. Then any solution S of the

$\mathbf{aAIP}(E, N, \mathbf{T}, \widetilde{P})$ is also a solution of the $\mathbf{aAIP}(E, N, \mathbf{T}, P)$. In other words, for every $x \in \mathcal{X}$, the function $F^S x$ belongs to $\mathcal{H}(K_S)$ and

$$\|F^S x\|^2_{\mathcal{H}(K_S)} \le \langle \widetilde{P}x,\, x\rangle_{\mathcal{X}} \le \langle Px,\, x\rangle_{\mathcal{X}}.$$

But by Theorem 4.2, $\|F^S x\|^2_{\mathcal{H}(K_S)} = \langle Px,\, x\rangle_{\mathcal{X}}$ which implies therefore, that $P = \widetilde{P}$. □

Corollary 6.6. *For any* **aAIP***-admissible data set* $\{\mathbf{T}, E, N, P\}$*,* **aAIP** *has solutions.*

Proof. We have already observed that the **aAIP** is a special form of **AIP**. Hence the result of Theorem 2.7 that any admissible problem of the type **AIP** has solutions implies the same for **aAIP**. □

We conclude by continuing Example 6.2 to illustrate the general theory.

Example 6.2 **continued:** Note that the partial unitary colligation associated with the **AIP** given in Example 6.2 is $V : \mathcal{D}_V \mapsto \mathcal{R}_V$ given by

$$\mathbf{V} : \begin{bmatrix} 0 \\ \delta \\ \sqrt{1-\delta^2} \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

with

$$\mathcal{D}_{\mathbf{V}} = \operatorname{Ran} \begin{bmatrix} 0 \\ \delta \\ \sqrt{1-\delta^2} \end{bmatrix} \quad \text{and} \quad \mathcal{R}_{\mathbf{V}} = \operatorname{Ran} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore,

$$\Delta = \operatorname{Ran} \begin{bmatrix} 0 & 1 \\ \sqrt{1-\delta^2} & 0 \\ -\delta & 0 \end{bmatrix} \quad \text{and} \quad \Delta_* = \operatorname{Ran} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The universal colligation $\mathbf{U}_0$ takes the form

$$\mathbf{U}_0 = \begin{bmatrix} 0 & \delta & \sqrt{1-\delta^2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1-\delta^2} & -\delta & 0 & 0 \end{bmatrix}$$

that is,

$$U_{11} = \begin{bmatrix} 0 & \delta \\ 0 & 0 \end{bmatrix}, \quad U_{31} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\delta^2} \end{bmatrix}, \quad U_{13} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad U_{33} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$U_{21} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad U_{12} = \begin{bmatrix} \sqrt{1-\delta^2} \\ 0 \end{bmatrix}, \quad U_{22} = 0, \quad U_{32} = \begin{bmatrix} 0 \\ -\delta \end{bmatrix}, \quad U_{23} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned}
\Sigma(z) &= \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} \\
&= \begin{bmatrix} U_{22} & U_{23} \\ U_{32} & 0 \end{bmatrix} + z \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} (I - zU_{11})^{-1} \begin{bmatrix} U_{12} & U_{13} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -\delta & 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \sqrt{1-\delta^2} \end{bmatrix} \begin{bmatrix} 1 & z\delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{1-\delta^2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ z\sqrt{1-\delta^2} & z^2\delta & 0 \\ -\delta & z\sqrt{1-\delta^2} & 0 \end{bmatrix}.
\end{aligned}$$

Thus,

$$\Sigma_{11}(z) = 0,\ \Sigma_{12}(z) = \begin{bmatrix} 0 & 1 \end{bmatrix},\ \Sigma_{21}(z) = \begin{bmatrix} z\sqrt{1-\delta^2} \\ -\delta \end{bmatrix},\ \Sigma_{22}(z) = \begin{bmatrix} z^2\delta & 0 \\ z\sqrt{1-\delta^2} & 0 \end{bmatrix}.$$

Now we apply Theorem 6.4 to get the linear fractional parametrizations:

$$\begin{aligned}
S(z) &= \Sigma_{11}(z) + \Sigma_{12}(z)\mathcal{W}(z)(I - \Sigma_{22}(z)\mathcal{W}(z))^{-1}\Sigma_{21}(z), \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{W}(z) \left(I - \begin{bmatrix} z^2\delta & 0 \\ z\sqrt{1-\delta^2} & 0 \end{bmatrix} \mathcal{W}(z) \right)^{-1} \begin{bmatrix} z\sqrt{1-\delta^2} \\ -\delta \end{bmatrix}
\end{aligned} \tag{6.37}$$

and

$$\begin{aligned}
F_0(z) &= U_{21}(I - zU_{11})^{-1} \\
&\quad + \Sigma_{12}(z)(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}\mathcal{W}(z)U_{31}(I - zU_{11})^{-1} \\
&= \Sigma_{12}(z)\mathcal{W}(z)(I - \Sigma_{22}(z)\mathcal{W}(z))^{-1}U_{31}(I - zU_{11})^{-1} \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{W}(z) \left(I - \begin{bmatrix} z^2\delta & 0 \\ z\sqrt{1-\delta^2} & 0 \end{bmatrix} \mathcal{W}(z) \right)^{-1} \begin{bmatrix} 1 & z\delta \\ 0 & \sqrt{1-\delta^2} \end{bmatrix}
\end{aligned} \tag{6.38}$$

where $\mathcal{W}$ is a free-parameter 2×2 matrix valued Schur function. The function

$$G(z) = \begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix} := \mathcal{W}(z) \left(I - \begin{bmatrix} z^2\delta & 0 \\ z\sqrt{1-\delta^2} & 0 \end{bmatrix} \mathcal{W}(z) \right)^{-1} \tag{6.39}$$

belongs to H^2 and in turn,

$$\mathcal{W}(z) = G(z) \left(I - \begin{bmatrix} z^2\delta & 0 \\ z\sqrt{1-\delta^2} & 0 \end{bmatrix} G(z) \right)^{-1}. \tag{6.40}$$

The formulas (6.37) and (6.38) can be written in terms of G as follows:

$$S(z) = z\sqrt{1-\delta^2}G_{21}(z) - \delta G_{22}(z), \tag{6.41}$$

$$F_0(z) = \begin{bmatrix} G_{21}(z) & z\delta G_{21}(z) + \sqrt{1-\delta^2}G_{22}(z) \end{bmatrix}. \tag{6.42}$$

We conclude that *different parameters $\mathcal{W}$ may lead via formula* (6.37) *to the same S and to different F_0's via formula* (6.38). This phenomenon exhibits explicitly the nonuniqueness of F_0 corresponding to the same S.
For example, the function $S(z) \equiv 0$ is a solution corresponding to the parameter $\mathcal{W} \equiv 0$; then formula (6.38) gives $F_0 = 0 : \mathcal{X} \to \mathcal{H}(K_S) = H^2$. But also we have we have $S(z) \equiv 0$ via (6.41) whenever

$$G_{22}(z) = \frac{\sqrt{1-\delta^2}}{\delta} z G_{21}(z) \tag{6.43}$$

and for this relation in force we have from (6.42)

$$F_0(z) = \begin{bmatrix} G_{21}(z) & \frac{z}{\delta} G_{21}(z) \end{bmatrix} = G_{21}(z) \begin{bmatrix} 1 & \frac{z}{\delta} \end{bmatrix}. \tag{6.44}$$

To show that there are many F_0's corresponding to $S(z) \equiv 0$, take G_{21} in H^∞ (rather than in H^2) with $\|G_{21}\|_{H^\infty}$ small enough. Then define G_{22} as in (6.43) and choose $G_{11}(z)$ and $G_{12}(z)$ so that $G(z)$ in (6.39) has $\|G\|_\infty$ still small (much less than one). Then formula (6.40) gives a Schur function $\mathcal{W}$ which produces via formulas (6.37) and (6.38) $S(z) \equiv 0$ and the corresponding F_0 of the form (6.44) with the prescribed G_{21}.

References

[1] J. Agler, *Some interpolation theorems of Nevanlinna–Pick type*, 1988, Preprint.

[2] J. Agler and J. E. McCarthy, *Complete Nevanlinna-Pick kernels*, J. Funct. Anal., **175** (2000), no. 1, 111–124.

[3] D. Alpay, V. Bolotnikov and H.T. Kaptanoğlu, *The Schur algorithm and reproducing kernel Hilbert spaces in the ball*, Linear Algebra Appl. **343** (2002), 163-186.

[4] D. Alpay and C. Dubi, *Boundary interpolation in the ball*, Linear Algebra Appl. **340** (2002), 33-54.

[5] D. Alpay and C. Dubi, *Carathéodory-Fejér interpolation in the ball*, Linear Algebra Appl. **359** (2003), 1–19.

[6] D. Alpay and C. Dubi, *Carathéodory-Fejér interpolation in the ball with mixed derivatives*, Linear Algebra Appl. **382** (2004), 117–123.

[7] A. Arias and G. Popescu, *Noncommutative interpolation and Poisson transforms*, Israel J. Math. **115** (2000), 205–234.

[8] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc., **68** (1950), 337–404.

[9] D.Z. Arov and H. Dym, *Criteria for the strong regularity of J-inner functions and γ-generating matrices*, J. Math. Anal. Appl. **280** (2003) no. 2, 387–399.

[10] D.Z. Arov and L.Z. Grossman, *Scattering matrices in the theory of unitary extensions of isometric operators*, Soviet Math. Dokl. **270** (1983), 17–20.

[11] D.Z. Arov and L.Z. Grossman, *Scattering matrices in the theory of unitary extensions of isometric operators*, Math. Nachr. **157** (1992), 105–123.

[12] W. Arveson, *Subalgebras of C^*-algebras. III. Multivariable operator theory*, Acta Math. **181** (1998), no. 2, 159–228.

[13] J.A. Ball and V. Bolotnikov, *On a bitangential interpolation problem for contractive valued functions on the unit ball*, Linear Algebra Appl. **353** (2002), 107–147.

[14] J.A. Ball and V. Bolotnikov, *A bitangential interpolation problem on the closed unit ball for multipliers of the Arveson space*, Integral Equations and Operator Theory **46** (2003), 125-164.

[15] J.A. Ball and V. Bolotnikov, *Boundary interpolation for contractive-valued functions on circular domains in C^n*, in *Current Trends in Operator Theory and its Applications* (Ed. J.A. Ball, J.W. Helton, M. Klaus and L. Rodman) pp. 107-132, **OT 149** Birkhäuser, Basel, 2004.

[16] J.A. Ball and V. Bolotnikov, *Nevanlinna-Pick interpolation for Schur-Agler-class functions on domains with matrix polynomial defining function in C^n*, New York J. Math. **11** (2005), 1-44.

[17] J.A. Ball and V. Bolotnikov, *Interpolation in the noncommutative Schur-Agler class*, J. Operator Theory **58** (2007) no. 1, 83–126.

[18] J.A. Ball, V. Bolotnikov and Q. Fang, *Multivariable backward-shift invariant subspaces and observability operators*, Multidimens. Syst. Signal Process. **18** (2007), no. 4, 191–248.

[19] J.A. Ball, V. Bolotnikov and Q. Fang, *Transfer-function realization for multipliers of the Arveson space*, J. Math. Anal. Appl. **333** (2007), no. 1, 68–92

[20] J.A. Ball, V. Bolotnikov and Q. Fang, *Schur-class multipliers on the Arveson space: de Branges-Rovnyak reproducing kernel spaces and commutative transfer-function realizations*, J. Math. Anal. Appl. **341** (2008), 519–539.

[21] J.A. Ball, I. Gohberg, and L. Rodman. *Interpolation of Rational Matrix Functions*, **OT 45** Birkhäuser, Basel, 1990.

[22] J.A. Ball and J.W. Helton, *A Beurling-Lax theorem for the Lie group $U(m, n)$ which contains most classical interpolation*, J. Operator Theory **9** (1983), no.1, 107-142.

[23] J.A. Ball and S. ter Horst, *Multivariable operator-valued Nevanlinna-Pick interpolation: a survey*, Proceedings IWOTA 2007, Potchefstroom, South Africa, Birkhäuser volume to appear.

[24] J.A. Ball and M.W. Raney, *Discrete-time dichotomous well-posed linear systems and generalized Schur-Nevanlinna-Pick interpolation*, Complex Anal. Oper. Theory **1** (2007), no. 1, 1–54.

[25] J.A. Ball and T.T. Trent, *The abstract interpolation problem and commutant lifting: a coordinate-free approach*, in: *Operator theory and interpolation* (Ed. H. Bercovici and C.I. Foias), Oper. Theory Adv. Appl. **115**, Birkhäuser, Basel, 2000, pp. 51–83.

[26] J.A. Ball, T.T. Trent and V. Vinnikov, *Interpolation and commutant lifting for multipliers on reproducing kernels Hilbert spaces*, in: Operator Theory and Analysis (Ed. H. Bart, I. Gohberg and A.C.M. Ran), Oper. Theory Adv. Appl. **122**, Birkhäuser, Basel, 2001, pp. 89-138.

[27] V. Bolotnikov, *A boundary Nevanlinna–Pick problem for a class of analytic matrix-valued functions in the unit ball*, Linear Algebra Appl. **346** (2002), 239–260.

[28] V. Bolotnikov, *Interpolation for multipliers on reproducing kernel Hilbert spaces*, Proc. Amer. Math. Soc., **131** (2003), 1373-1383.

[29] L. de Branges and J. Rovnyak, *S*quare summable power series, Holt, Rinehart and Winston, New–York, 1966.

[30] K.R. Davidson and D. R. Pitts, *Nevanlinna–Pick interpolation for non-commutative analytic Toeplitz algebras*, Integral Equations Operator Theory **31** (1998), no. 3, 321–337.

[31] K.R. Davidson and D. R. Pitts, *Invariant subspaces and hyper-reflexivity for free semigroup algebras*, Proc. London Math. Soc. **78** (1999) no. 2, 401–430.

[32] S.W. Drury, *A generalization of von Neumann's inequality to the complex ball*, Proc. Amer. Math. Soc., **68** (1978), 300-304.

[33] H. Dym, *J-Contractive Matrix Functions, Reproducing Kernel Spaces and Interpolation*, CBMS Lecture Notes **71**, Amer. Math. Soc., Providence, 1989.

[34] J. Eschmeier and M. Putinar, *Spherical contractions and interpolation problems on the unit ball*, J. Reine Angew. Math. **542** (2002), 219–236.

[35] Q. Fang, *Multivariable Interpolation Problems*, Ph.D. dissertation, Virginia Tech, 2008.

[36] C. Foias and A.E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, **OT 44** Birkhäuser, Basel, 1990.

[37] C. Foias, A.E. Frazho, I. Gohberg and M.A. Kaashoek, *Metric Constrained Interpolation, Commutant Lifting and Systems*, **OT 100**, Birkhäuser, Basel, 1998.

[38] A.E. Frazho, S. ter Horst and M.A. Kaashoek, *Coupling and relaxed commutant lifting*, Integral Equations and Operator Theory **54** (2006), no.1, 33-67.

[39] A.E. Frazho, S. ter Horst and M.A. Kaashoek, *All solutions to the relaxed commutant lifting problem*, Acta Sci. Math. (Szeged), **72** (2006), 299–318.

[40] V. Katsnelson, A. Kheifets and P. Yuditskii, *An abstract interpolation problem and extension theory of isometric operators*, in: *Operators in Spaces of Functions and Problems in Function Theory* (V.A. Marchenko, ed.), **146**, Naukova Dumka, Kiev, 1987, pp. 83–96. English transl. in: *Topics in Interpolation Theory* (H. Dym, B. Fritzsche, V. Katsnelson and B. Kirstein, eds.), Oper. Theory Adv. Appl. **95**, Birkhäuser, Basel, 1997, pp. 283–298.

[41] A. Kheifets, *The Parseval equality in an abstract problem of interpolation, and the union of open systems*, Teor. Funktsii Funktsional. Anal. i Prilozhen., **49**, 1988, 112–120, and **50**, 1988, 98–103. English transl. in: *J. Soviet Math.*, **49(4)**, 1990, 114–1120 and **49(6)**, 1990, 1307–1310.

[42] A. Kheifets *Scattering matrices and Parseval Equality in Abstract Interpolation Problem*, Ph.D. Thesis, 1990, Kharkov State University.

[43] A. Kheifets, *The abstract interpolation problem and applications*, in: *Holomorphic spaces* (Ed. D. Sarason, S. Axler, J. McCarthy), pp. 351–379, Cambridge Univ. Press, Cambridge, 1998.

[44] A. Kheifets and P. Yuditskii, *An analysis and extension of V.P. Potapov's approach to interpolation problems with applications to the generalized bitangential Schur-Nevanlinna-Pick problem and j-inner-outer factorization*, in: *Matrix and Operator Valued Functions* (Ed. I. Gohberg and L.A. Sakhnovich), Oper. Theory Adv. Appl. **72**, Birkhäuser, Basel, 1994, pp. 133-161.

[45] I.V. Kovalishina and V.P. Potapov, *Seven Papers Translated from the Russian*, Amer. Math. Soc. Transl. (2) **138**, Providence, R.I., 1988.

[46] S. Kupin, *Lifting theorem as a special case of abstract interpolation problem*, Z. Anal. Anwendungen **15** (1996), no. 4, 789–798.

[47] S. McCullough, *The local de Branges-Rovnyak construction and complete Nevanlinna-Pick kernels*, in *Algebraic methods in operator theory* (Ed. R. Curto and P. E. T. Jorgensen), Birkhäuser, Boston, 1994, pp. 15–24.

[48] S. McCullough and T.T. Trent, *Invariant subspaces and Nevanlinna-Pick kernels*, J. Funct. Anal. **178** (2000), no. 1, 226–249.

[49] G. Popescu, *Isometric dilations for infinite sequences of noncommuting operators*, Trans. Amer. Math. Soc. **316** (1989), no. 2, 523–536.

[50] G. Popescu, *Characteristic functions for infinite sequences of noncommuting operators*, J. Operator Theory **22** (1989), no. 1, 51–71.

[51] G. Popescu, *Interpolation problems in several variables*, J. Math. Anal. Appl., **227** (1998), 227–250.

[52] G. Popescu, *Multivariable Nehari problem and interpolation*, J. Funct. Anal. **200** (2003), no. 2, 536–581.

[53] G. Popescu, *Entropy and multivariable interpolation*, Mem. Amer. Math. Soc. **184** (2006), no. 868,

[54] P. Quiggin, *For which reproducing kernel Hilbert spaces is Pick's theorem true?* Integral Equations and Operator Theory **16** (1993), no. 2, 244–266.

[55] D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 179–203.

[56] D. Sarason, *Sub-Hardy Hilbert spaces in the unit disk*, John Wiley and Sons Inc., New York, 1994.

Joseph A. Ball
Department of Mathematics
Virginia Polytechnic Institute
Blacksburg, VA 24061-0123
USA
e-mail: `ball@calvin.math.vt.edu`

Vladimir Bolotnikov
Department of Mathematics
The College of William and Mary
Williamsburg, VA 23187-8795
USA
e-mail: `vladi@math.wm.edu`

Submitted: October 9, 2007.
Revised: June 19, 2008.