ON DEGENERATE HAMBURGER MOMENT PROBLEM AND EXTENSIONS OF POSITIVE SEMIDEFINITE HANKEL BLOCK MATRICES

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Abstract. In this paper we consider two related objects: singular positive semidefinite Hankel block–matrices and associated degenerate truncated matrix Hamburger moment problems. The description of all solutions of a degenerate matrix Hamburger moment problem is given in terms of a linear fractional transformation. The case of interest is the Hamburger moment problem whose Hankel block–matrix admits a positive semidefinite Hankel extension.

This is the corrected version of the original paper [2]. The work was inspired by V. Dubovoj's paper [4] containing the first systematic study of degenerate matricial interpolation problems. Another sourse of inspiration must have been the paper by R. Curto and L. Fialkow [3] but I was not aware of it then. The original paper contained several erratae and the author is very grateful to A. Ben-Artzi and H. Woerdeman for indicating them. A short proof in Section 5 fixes these incorrectnesses. The remaining four sections are mostly the same as in [2].

1. Introduction

The objective of this article is to describe the solutions of a degenerate truncated matrix Hamburger moment problem **HMP**. We start with a set of Hermitian matrices $s_0, \dots, s_{2n} \in \mathbb{C}^{m\times m}$ and let K_n denote the Hankel block matrix

$$K_n = (s_{i+j})_{i,j=0}^n. \tag{1.1}$$

Let $\mathcal{Z}(K_n)$ denote the set of all solutions of the associated truncated Hamburger moment problem, i.e., the set of nondecreasing right continuous $m \times m$ matrix-valued functions $\sigma(\lambda)$ such that

$$\int_{-\infty}^{\infty} \lambda^k d\sigma(\lambda) = s_k \qquad (k = 0, \dots, 2n-1) \tag{1.2}$$

and

$$\int_{-\infty}^{\infty} \lambda^{2n} d\sigma(\lambda) \le s_{2n}. \tag{1.3}$$

As in the scalar case (see [1: §2.1]) $\mathcal{Z}(K_n)$ is nonempty if and only if K_n is positive semidefinite and, moreover, by a theorem of H. Hamburger and R. Nevanlinna [1: §3.1], the formula

$$w(z) = \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{\lambda - z} \tag{1.4}$$

establishes a one-to-one correspondence between $\mathcal{Z}(K_n)$ and the class $\mathcal{R}(K_n)$ of $\mathbb{C}^{m\times m}$–valued functions $w(z)$ analytic and with positive semidefinite imaginary part in the upper half plane $\mathbb{C}_+$ such that uniformly in the angle $\{z = \rho e^{i\theta} : \varepsilon \le \theta \le \pi - \varepsilon \, , \, \varepsilon > 0\}$,

$$\lim_{z\to\infty} \left\{ z^{2n+1} w(z) + \sum_{k=0}^{2n} s_k z^{2n-k} \right\} \ge 0. \tag{1.5}$$

This correspondence reduces the **HMP** problem to a boundary interpolation problem of finding all $\mathbb{C}^{m\times m}$–valued Pick functions w (which by definition are analytic and with positive semidefinite imaginary part in $\mathbb{C}_+$) with prescribed asymptotic behavour (1.5) at infinity.

In this paper we follow the Potapov's method of the fundamental matrix inequality [9]. The starting point is the following theorem which describes the set $\mathcal{R}(K_n)$ in terms of a matrix inequality (see [9, §1] for the proof).

Theorem 1.1. *Let w be a $\mathbb{C}^{m\times m}$–valued function analytic in $\mathbb{C}_+$. Then w belongs to $R(K_n)$ if and only if it satisfies the inequality*

$$\begin{pmatrix} K_n & (I-zF_{m,n})^{-1}(Uw(z)+M) \\ (w(z)^*U^*+M^*)(I-\bar{z}F_{m,n}^*)^{-1} & \dfrac{w(z)-w(z)^*}{z-\bar{z}} \end{pmatrix} \geq 0 \tag{1.6}$$

for every $z\in\mathbb{C}_+$, where

$$F_{m,n}=\begin{pmatrix} 0_m & & \dots & & 0 \\ I_m & \ddots & & & \\ 0 & I_m & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & I_m & 0_m \end{pmatrix}\in\mathbb{C}^{m(n+1)\times m(n+1)} \tag{1.7}$$

is the matrix of the m-dimensional shift in $\mathbb{C}^{m(n+1)}$ and where $U,M\in\mathbb{C}^{m(n+1)\times m}$ are given by

$$U=\begin{pmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad M=F_{m,n}K_nU=\begin{pmatrix} 0 \\ s_0 \\ \vdots \\ s_{n-1} \end{pmatrix}. \tag{1.8}$$

The matrix K_n (the so–called *Pick matrix* of the **HMP**) satisfies the following Lyapunov identity

$$F_{m,n}K_n-K_nF_{m,n}^*=MU^*-UM^* \tag{1.9}$$

which can be easily verified with help of (1.1), (1.7) and (1.8).

The **HMP** is called *nondegenerate* if its Pick matrix K_n is strictly positive and it is termed *degenerate* if K_n is singular and positive semidefinite. The parametrization of all solutions to the inequality (1.6) for the case $K_n>0$ was obtained in [9] and will be recalled in Theorem 1.3 below. To formulate this theorem we first introduce some needed definitions and notations. We will denote bt $\mathbf{W}$ the class of $\mathbb{C}^{2m\times 2m}$–valued meromorphic functions Θ which are J–unitary on $\mathbb{R}$ and J–expansive in $\mathbb{C}_+$:

$$\Theta(z)J\Theta(z)^*=J \quad (z\in\mathbb{R}), \qquad \Theta(z)J\Theta(z)^*\geq J \quad (z\in\mathbb{C}_+) \tag{1.10}$$

where

$$J=\begin{pmatrix} 0 & iI_m \\ -iI_m & 0 \end{pmatrix}. \tag{1.11}$$

Definition 1.2. *A pair $\{p,\ q\}$ of $\mathbb{C}^{m\times m}$-valued functions meromorphic in $\mathbb{C}\backslash\mathbb{R}$ is called a Nevanlinna pair if*

$$\begin{array}{ll} (i) & \det\,(p(z)^*p(z)+q(z)^*q(z))\not\equiv 0 \qquad \textit{(the nondegeneracy of the pair)} \\ (ii) & \dfrac{q(z)^*p(z)-p(z)^*q(z)}{z-\bar{z}}=(p(z)^*,\ q(z)^*)\,\dfrac{J}{i(\bar{z}-z)}\begin{pmatrix} p(z) \\ q(z) \end{pmatrix}\geq 0 \qquad (\Im z\neq 0). \end{array} \tag{1.12}$$

A pair $\{p,\ q\}$ is said to be *equivalent* to the pair $\{p_1,\ q_1\}$ if there exists a $\mathbb{C}^{m\times m}$-valued function Ω ($\det\,\Omega(z)\not\equiv 0$) meromorphic in $\mathbb{C}\backslash\mathbb{R}$ such that $p_1=p\,\Omega$ and $q_1=q\Omega$. The set of all $m\times m$ matrix valued Nevanlinna pairs will be denoted by $\mathbf{N}_m$.

Theorem 1.3. *Let K_n be a strictly positive matrix given by* (1.1) *and let $F_{m,n}$, U and M be defined by* (1.7), (1.8). *Then*

(1) *The function*

$$\Theta(z) = \begin{pmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{pmatrix} = I_{2m} + z \begin{pmatrix} M^* \\ -U^* \end{pmatrix} (I - zF_{m,n}^*)^{-1} K_n^{-1}(U, M) \tag{1.13}$$

belongs to the class $\mathbf{W}$.

(2) *The formula*

$$w(z) = (\theta_{11}(z)p(z) + \theta_{12}(z)q(z))(\theta_{21}(z)p(z) + \theta_{22}(z)q(z))^{-1} \tag{1.14}$$

gives all the solutions w to the inequality (1.6) *when $\{p,\ q\}$ varies in $\mathbf{N}_m$.*

(3) *Two pairs $\{p(z),\ q(z)\}$ and $\{p_1(z),\ q_1(z)\}$ lead by* (1.14) *to the same function $w(z)$ if and only if these pairs are equivalent.*

The degenerate scalar **HMP** is simple: $\mathcal{R}(K_n)$ consists of the unique rational function $w(z)$ (this follows immediately from (1.6)). In the degenerate *matrix* case, the description of $\mathcal{R}(K_n)$ depends on the degeneracy of K_n, but we still have a parametrization of all the solutions as a linear fractional transformation (1.14) with the coefficient matrix Θ from the class $\mathbf{W}$ and for a suitable choice of parameters $\{p,\ q\}$ (see Theorem 4.6 below). To construct the coefficient matrix of the degenerate **HMP**, we follow the method of V. Dubovoj which was applied in [4] to the degenerate Schur problem. Note that if $\det \theta_{22} \not\equiv 0$, the transformation (1.14) can be written as

$$w(z) = \psi_{11}(z) + \psi_{12}(z)p(z)(\psi_{22}(z)p(z) + q(z))^{-1}\psi_{21} \tag{1.15}$$

where

$$\psi_{11} = \theta_{11}\theta_{22}^{-1}, \quad \psi_{12} = \theta_{11} - \theta_{12}\theta_{22}^{-1}\theta_{22}, \quad \psi_{21} = \theta_{22}^{-1}, \quad \psi_{22} = \theta_{22}^{-1}\theta_{21} \tag{1.16}$$

and it turns out that the function $\Psi(z) = \begin{pmatrix} \psi_{11}(z) & \psi_{12}(z) \\ \psi_{21}(z) & \psi_{22}(z) \end{pmatrix}$ is a Pick function (i.e. analytic and with positive semidefinite imaginary part in $\mathbb{C}_+$). If $\det \theta_{22} \equiv 0$, formulas (1.16) make no sense, but nevertheless the set $\mathcal{R}(K_n)$ can be parametrized by the transformation (1.15) with a coefficient matrix Ψ from the Pick class. This Ψ can be constructed as a characteristic function of certain unitary colligation associated with the initial data $\{s_j\}$ of the problem. This approach (see [8]) is much more stable with respect to a possible degeneracy of the Pick matrix K_n. The degenerate **HMP** will be discussed in some more detail in Section 2.

2. Positive semidefinite Hankel extensions of Hankel block matrices

Let $\mathcal{H}_{m,n}$ be the set of all positive semidefinite Hankel block matrices of the form (1.1). We say that a matrix $K_n \in \mathcal{H}_{m,n}$ *admits a positive semidefinite Hankel extension* if there exist Hermitian matrices $s_{2n+1}, s_{2n+2} \in \mathbb{C}^{m\times m}$ such that the block matrix $K_{n+1} = (s_{i+j})_{i,j=0}^{n+1}$ is still positive semidefinite. The class of such matrices will be denoted by $\mathcal{H}_{m,n}^+$:

$$\mathcal{H}_{m,n}^+ = \left\{ K_n \in \mathcal{H}_{m,n} : (s_{i+j})_{i,j=0}^{n+1} \geq 0 \text{ for some } s_1 = s_1^* \text{ and } s_2 = s_2^* \right\}. \tag{2.1}$$

In the scalar case ($m = 1$) every positive semidefinite Hankel matrix admits a positive semidefinite Hankel extension and therefore, $\mathcal{H}_{1,n}^+ = \mathcal{H}_{1,n}$. For $n \geq 2$, $\mathcal{H}_{m,n}^+$ is a proper subset of $\mathcal{H}_{m,n}$ as can be seen from the example

$$K_2 = \begin{pmatrix} s_0 & s_1 \\ s_1 & s_2 \end{pmatrix}, \qquad s_0 = s_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We introduce two more subsets of $\mathcal{H}_{m,n}$:

$$\widetilde{\mathcal{H}}_{m,n} := \left\{ K_n \in \mathcal{H}_{m,n} : \mathbf{P}_{Ker K_{n-1}} \begin{pmatrix} s_{n+1} \\ \vdots \\ s_{2n} \end{pmatrix} = 0 \right\} \tag{2.2}$$

and

$$\widehat{\mathcal{H}}_{m,n} := \left\{ K_n \in \mathcal{H}_{m,n} : \, s_{2n} = \int_{-\infty}^{\infty} \lambda^{2n} d\sigma(\lambda) \text{ for some } \sigma \in \mathcal{Z}(K_n) \right\}. \tag{2.3}$$

Thus, $\widehat{\mathcal{H}}_{m,n}$ consists of all matrices $K_n \in \mathcal{H}_{m,n}$, the associated truncated Hamburger moment problem admits an "exact" solution σ such that

$$\int_{-\infty}^{\infty} \lambda^k d\sigma(\lambda) = s_k \qquad (k = 0, \ldots, 2n), \tag{2.4}$$

that is, with equality for the last assigned moment s_{2n} rather than inequality (1.3). In (2.2) and in what follows, $\mathbf{P}_{KerK}$ denotes the orthogonal projection onto the kernel of K. We will show below that

$$\mathcal{H}^{+}_{m,n} = \widetilde{\mathcal{H}}_{m,n} = \widehat{\mathcal{H}}_{m,n} \tag{2.5}$$

which will provide therefore, several equivalent characterizations of Hankel block matrices admitting positive semidefinite Hankel extensions. The following two propositions can be easily verified.

Lemma 2.1. *The block matrix* $T = (t_{ij})_{i,j=0}^n$ $(t_{ij} \in \mathbb{C}^{r \times l})$ *is Hankel if and only if*

$$F^*_{l,n}(F_{l,n}T - TF^*_{r,n})F_{r,n} = 0 \tag{2.6}$$

where F *is a shift matrix defined via* (1.7).

Lemma 2.2. *Let* $K, V \in \mathbb{C}^{N \times N}$ *and* $A \in \mathbb{C}^{N \times r}$ *be matrices such that* $K = K^*$ *and* $\det V \neq 0$. *Then,* $\mathbf{P}_{KerK}A = \mathbf{P}_{KerVKV^*}VA$.

Given a $K \geq 0$, let Q be a matrix such that

$$QKQ^* > 0 \quad \text{ and } \quad \text{rank } QKQ^* = \text{rank } K. \tag{2.7}$$

We define the pseudoinverse matrix $K^{[-1]}$ by

$$K^{[-1]} = Q^* \left(QKQ^*\right)^{-1} Q. \tag{2.8}$$

Since the pseudoinverse matrix depends on the choice of Q, it is not uniquely defined.

Lemma 2.3. *For every choice of* $K^{[-1]}$,

$$I - KK^{[-1]} = \left(I - KK^{[-1]}\right) \mathbf{P}_{KerK}. \tag{2.9}$$

Proof: By (2.7), every vector f can be decomposed as $f = g + hQ$ for some $g \in Ker\ K$ and $h \in \mathbb{C}^{1 \times \text{rank}K}$. Therefore,

$$f\left(I - KK^{[-1]}\right) = (g + hQ)\left(I - KQ^*(QKQ^*)^{-1}Q\right) = g$$

which implies (2.9). □

Lemma 2.4. *The block matrix* $\begin{pmatrix} K & B \\ B^* & C \end{pmatrix}$ *is positive semidefinite if and only if*

$$K \geq 0, \quad \mathbf{P}_{kerK}B = 0 \quad \text{and} \quad R = C - B^*K^{[-1]}B \geq 0.$$

Moreover, if $\begin{pmatrix} K & B \\ B^* & C \end{pmatrix} \geq 0$, *then the matrix* R *does not depend on the choice of* $K^{[-1]}$.

Proof: The first assertion of lemma follows from the factorization

$$\begin{pmatrix} K & B \\ B^* & C \end{pmatrix} = \begin{pmatrix} I & 0 \\ B^*K^{[-1]} & I \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} I & K^{[-1]}B \\ 0 & I \end{pmatrix}$$

which in view of (2.9), is valid if and only if $\mathbf{P}_{kerK}B = 0$.
Furthermore, let C admit two different representations $C = R_i + B^*K_i^{[-1]}B$ $(i = 1, 2)$. Then

$$R_1 - R_2 = B^*\left(K_2^{[-1]} - K_1^{[-1]}\right)B. \tag{2.10}$$

In view of (2.9),

$$K\left(K_2^{[-1]}-K_1^{[-1]}\right)B=\left\{\left(I-KK_1^{[-1]}\right)-\left(I-KK_2^{[-1]}\right)\right\}\mathbf{P}_{KerK}B=0.$$

Since $\begin{pmatrix} K & B \\ B^* & C \end{pmatrix}\geq 0$, then also $B^*\left(K_2^{[-1]}-K_1^{[-1]}\right)B=0$ which both with (2.10) implies $R_1=R_2$. ☐

Lemma 2.5. *Let $K_n\in\mathcal{H}_{m,n}$ and let $\mathcal{L}$ be the subspace of $\mathbb{C}^{1\times m}$ given by*

$$\mathcal{L}=\{f\in\mathbb{C}^{1\times m}:(f_0,\dots,f_{n-2},f)\in KerK_{n-1}\ \ for\ \ some\ \ f_0,\dots,f_{n-2}\in\mathbb{C}^{1\times m}\}. \tag{2.11}$$

Then K_n belongs to $\widetilde{\mathcal{H}}_{m,n}$, that is

$$\mathbf{P}_{KerK_{n-1}}\begin{pmatrix} s_{n+1}\\ \vdots \\ s_{2n}\end{pmatrix}=0, \tag{2.12}$$

if and only if the block s_{2n} is of the form

$$s_{2n}=(s_n,\dots,s_{2n-1})K_{n-1}^{[-1]}(s_n,\dots,s_{2n-1})^*+R \tag{2.13}$$

for some positive semidefinite matrix $R\in\mathbb{C}^{m\times m}$ which vanishes on the subspace $\mathcal{L}$ and does not depend on the choice of $K_{n-1}^{[-1]}$.

Proof: Since $K_n\geq 0$, then by Lemma 2.4,

$$s_{2n}-(s_n,\dots,s_{2n-1})K_{n-1}^{[-1]}(s_n,\dots,s_{2n-1})^*\geq 0$$

and therefore, s_{2n} admits a representation (2.13) for some $R\geq 0$. Moreover, since $K_n\geq 0$, then for every vector $(f_0,\dots,f_{n-1})$ from $KerK_{n-1}$

$$(f_0,\dots,f_{n-1})\begin{pmatrix} s_1 & \dots & s_n\\ \vdots & & \vdots\\ s_n & \dots & s_{2n-1}\end{pmatrix}=0$$

and therefore,

$$\begin{aligned} f_{n-1}(s_n,\dots,s_{2n-1}) &= -(f_0,\dots,f_{n-2})\begin{pmatrix} s_1 & \dots & s_n\\ \vdots & & \vdots\\ s_{n-1} & \dots & s_{2n-2}\end{pmatrix}\\ &= -(0,f_0,\dots,f_{n-2})K_{n-1}. \end{aligned} \tag{2.14}$$

Thus,

$$\begin{aligned} &f_0s_{n+1}+\dots+f_{n-2}s_{2n-1}+f_{n-1}(s_n,\dots,s_{2n-1})K_{n-1}^{[-1]}(s_n,\dots,s_{2n-1})^*\\ &=(0,f_0,\dots,f_{n-2})\left\{I-K_{n-1}K_{n-1}^{[-1]}\right\}(s_n,\dots,s_{2n-1})^*\\ &=(0,f_0,\dots,f_{n-2})\left\{I-K_{n-1}K_{n-1}^{[-1]}\right\}\mathbf{P}_{KerK_{n-1}}(s_n,\dots,s_{2n-1})^*=0 \end{aligned} \tag{2.15}$$

where the first equality holds due to (2.14), the second follows by (2.9) and the last one holds since $K_n\geq 0$ and therefore, $\mathbf{P}_{KerK_{n-1}}(s_n,\dots,s_{2n-1})^*=0$. Comparing (2.15) with (2.13) gives

$$f_0s_{n+1}+\dots+f_{n-1}s_{2n}=f_{n-1}R. \tag{2.16}$$

It remains to show that R vanishes on the subspace $\mathcal{L}$ if and only if (2.12) holds. To this end, let us observe that condition (2.12) means that $f_0s_{n+1}+\dots+f_{n-1}s_{2n}=0$ for every vector $(f_0,\dots,f_{n-1})\in KerK_{n-1}$. The latter is equivalent, in view of (2.16) and (2.11), to $f_{n-1}R=0$ for all $f_{n-1}\in\mathcal{L}$. By Lemma 2.4, the matrix $R=s_{2n}-(s_n,\dots,s_{2n-1})K_{n-1}^{[-1]}(s_n,\dots,s_{2n-1})^*$ does not depend on the choice of $K_{n-1}^{[-1]}$. ☐

Lemma 2.6. *Let $\mathcal{H}^+_{m,n}$, $\widetilde{\mathcal{H}}_{m,n}$ and $\widehat{\mathcal{H}}_{m,n}$ be the classes defined in* (2.1)–(2.3)*. Then*

$$\mathcal{H}^+_{m,n} \subseteq \widehat{\mathcal{H}}_{m,n} \subseteq \widetilde{\mathcal{H}}_{m,n}. \tag{2.17}$$

Proof: Let K_{n+1} be a positive semidefinite Hankel extension of K_n. Since $K_{n+1} \geq 0$, by the solvability criterion for the associated Hamburger moment problem, the set $\mathcal{Z}(K_{n+1})$ is nonempty. Furthermore, for every $\sigma \in \mathcal{Z}(K_{n+1})$

$$\int_{-\infty}^{\infty} \lambda^k d\sigma(\lambda) = s_k \quad (k = 0, \dots, 2n+1) \text{and} \quad \int_{-\infty}^{\infty} \lambda^{2n+2} d\sigma(\lambda) \leq s_{2n+2}$$

and therefore, $K_n \in \widehat{\mathcal{H}}_{m,n}$ which proves the first containment in (2.17).

Now let us assume that K_n belongs to $\widehat{\mathcal{H}}_{m,n}$ and let $d\sigma$ be the measure satisfying conditions (2.4). Then

$$K_n = \int_{-\infty}^{\infty} (I_m, \dots, \lambda^n I_m)^* \, d\sigma(\lambda) \, (I_m, \dots, \lambda^n I_m). \tag{2.18}$$

Let $\mathbf{f} = (f_0, \dots, f_{n-1}) \in \mathbb{C}^{1\times mn}$ be a vector from $Ker \ K_{n-1}$. Then $\int_{-\infty}^{\infty} f(\lambda) d\sigma(\lambda) f(\lambda)^* = 0$, where

$$f(\lambda) = f_0 + \lambda f_1 + \ldots + \lambda^{n-1} f_{n-1} = \mathbf{f} \left(I_m, \dots, \lambda^{n-1} I_m\right)^*. \tag{2.19}$$

In particular, for every choice of $-\infty < a < b < +\infty$,

$$\int_a^b f(\lambda) d\sigma(\lambda) f(\lambda)^* = 0. \tag{2.20}$$

Let $g \in \mathbb{C}^{1\times m}$ be an arbitrary nonzero vector. By the Cauchy inequality,

$$\int_a^b f(\lambda) d\sigma(\lambda) \lambda^{n+1} g^* \leq \left(\int_a^b f(\lambda) d\sigma(\lambda) f(\lambda)^* \int_a^b \lambda^{2n+2} g d\sigma(\lambda) g^* \right)^{\frac{1}{2}}$$

which in view of (2.20) implies $\int_a^b f(\lambda) d\sigma(\lambda) \lambda^{n+1} g^* = 0$. Since $a, b \in \mathbb{R}$ and $g \in \mathbb{C}^{1\times m}$ are arbitrary, then

$$\int_{-\infty}^{\infty} f(\lambda) d\sigma(\lambda) \lambda^{n+1} I_m = 0$$

which on account of (2.4)–(2.19) can be rewritten as

$$\mathbf{f}(s_{n+1}, \dots, s_{2n})^* = 0. \tag{2.21}$$

Thus, every vector $\mathbf{f} \in Ker \ K_{n-1}$ satisfies (2.21) or in other words, $\mathbf{P}_{Ker K_{n-1}} (s_{n+1}, \dots, s_{2n})^* = 0$ and therefore, $K_n \in \widetilde{\mathcal{H}}_{m,n}$, which completes the proof of the second inclusion in (2.17). □

In connection with the last lemma we consider the following question: *to describe all matrices $s \in \mathbb{C}^{m\times m}$ such that $s = \int_{-\infty}^{\infty} \lambda^{2n} d\sigma(\lambda)$ for some $\sigma \in \mathcal{Z}(K_n)$.*

Lemma 2.7. *Let $K_n \geq 0$ be a block matrix of the form* (1.1) *with the block s_{2n} of the form*

$$s_{2n} = (s_n, \dots, s_{2n-1}) \, K_{n-1}^{[-1]} \, (s_n, \dots, s_{2n-1})^* + R \tag{2.22}$$

for some matrix $R \geq 0$ (which does not depend on the choice of $K_{n-1}^{[-1]}$) and let $s \in \mathbb{C}^{m\times m}$ be defined by

$$s = \int_{-\infty}^{\infty} \lambda^{2n} d\sigma(\lambda) \tag{2.23}$$

for some $\sigma \in \mathcal{Z}(K_n)$. Then there exists a positive semidefinite matrix $R_0 \leq R$ which vanishes on the subspace $\mathcal{L}$ defined by (2.11) *and such that*

$$s = (s_n, \dots, s_{2n-1}) K_{n-1}^{[-1]} \, (s_n, \dots, s_{2n-1})^* + R_0 \quad (0 \leq R_0 \leq R \quad \text{and} \quad R_0|_{\mathcal{L}} = 0). \tag{2.24}$$

Proof: Let s be of the form (2.23) for some $\sigma \in \mathcal{Z}(K_n)$. We introduce the Hankel block matrix

$$\widetilde{K}_n = \begin{pmatrix} s_0 & \dots & s_{n-1} & s_n \\ \vdots & & & \vdots \\ s_{n-1} & & & s_{2n-1} \\ s_n & \dots & s_{2n-1} & s \end{pmatrix} \tag{2.25}$$

which differs from K_n only by the block $\widetilde{s}_{2n} = s$. Thus, $\widetilde{K}_n \in \widehat{\mathcal{H}}_{m,n}$. Therefore, $\widetilde{K}_n \in \widetilde{\mathcal{H}}_{m,n}$, by Lemma 2.6. By Lemma 2.5, the block $\widetilde{s}_{2n} = s$ admits representation (2.24) for some $R_0 \geq 0$ vanishing on $\mathcal{L}$. The inequality $R_0 \leq R$ follows from (1.3) and (2.22)–(2.24). □

Lemma 2.8. *Let $K_n \in \mathcal{H}_{m,n}$ be of the form* (1.1)*, let $\mathcal{L}$ be the subspace given by* (2.11)*, let s_{2n}, s and $\widetilde{K}_n$ be matrices defined by* (2.22)*,* (2.24) *and* (2.25) *respectively, and let the positive semidefinite $R_0 : \mathbb{C}^m \to \mathbb{C}^m$ be defined by*

$$R_0 h = \begin{cases} 0 & for \quad h \in \mathcal{L}, \\ Rh & for \quad h \in \mathcal{L}^\perp. \end{cases} \tag{2.26}$$

Then the Hamburger moment problems associated with the sets of matrices $\{s_0, \dots, s_{2n-1}, s_{2n}\}$ and $\{s_0, \dots, s_{2n-1}, s\}$ have the same solutions: $\mathcal{Z}(K_n) = \mathcal{Z}(\widetilde{K}_n)$.

Proof: Let σ belong to $\mathcal{Z}(K_n)$. By Lemma 2.7, the matrix $\widehat{s} = \int_{-\infty}^{\infty} \lambda^{2n} d\sigma(\lambda)$ admits a representation (2.24) with a positive semidefinite matrix $\widehat{R}_0 \leq R$ vanishing on $\mathcal{L}$. In view of (2.26), $\widehat{R}_0 \leq R_0$. Therefore, $\widehat{s} \leq s$ and $\sigma \in \mathcal{Z}(\widetilde{K}_n)$. So, $\mathcal{Z}(K_n) \subseteq \mathcal{Z}(\widetilde{K}_n)$. The converse inclusion follows from the inequality $s \leq s_{2n}$. □

Remark 2.9. *By Lemmas 2.5 and 2.8, we can assume without loss of generality that the Pick matrix of the* **HMP** *belongs to $\widetilde{\mathcal{H}}_{m,n}$.*

Otherwise we replace the block s_{2n} (which is necessarily of the form (2.22)) by the block $\widetilde{s}_{2n} = s$ defined by (2.24), (2.26). By Lemma 2.5, $\widetilde{K}_n \in \widetilde{\mathcal{H}}_{m,n}$ and we describe the set $\mathcal{Z}(\widetilde{K}_n)$ of solutions of this new moment problem, which coincides, by Lemma 2.8, with $\mathcal{Z}(K_n)$.

3. The coefficient matrix of the problem

The coefficient matrix Θ of the nondegenerate **HMP** given by the formula (1.13) is the matrix polynomial of *deg* $\Theta = n+1$ and (1.13) is a realization of Θ with state space equal $\mathbb{C}^{m(n+1)}$. In this section we obtain some special decomposition (see formula (3.13) below) of the state space which will allow us to construct the analogue of (1.13) for K_n not strictly positive (formula (3.23)). The idea is simple: to replace in (1.13) the inverse of the matrix K_n (which does not exist for the degenerate case) by its pseudoinverse. However after this replacement the function Θ may lose its J–properties (1.10) which are essential for the description (1.14) to be in force. This suggests the following question: is there exist a pseudoinverse matrix $K_n^{[-1]}$ of the form (2.8) such that the function

$$\Theta(z) = I_{2m} + z \begin{pmatrix} M^* \\ -U^* \end{pmatrix} (I - zF_{m,n}^*)^{-1} K_n^{[-1]}(U,\ M)$$

still belongs to the class **W**? We show in Lemmas 3.2 and 3.3 below that such a pseudoinverse exists if (and in fact, only if) the Pick matrix K_n belongs to the class $\widetilde{\mathcal{H}}_{m,n}$. Recall that for the degenerate matricial Schur problem such a pseudoinverse always exists (see [4]).

Lemma 3.1. *Let $T_n = (t_{i+j})_{i,j=0}^n \in \widetilde{\mathcal{H}}_{l,n}$ ($t_i \in \mathbb{C}^{l\times l}$), let $t_0 > 0$ and let $\widehat{T}_{n-1}$ be the block matrix defined as*

$$\widehat{T}_{n-1} = D_n^{-1} \left\{ \mathbf{S} - \mathcal{T}_n t_0^{-1} \mathcal{T}_n^* \right\} D_n^{-*} \tag{3.1}$$

where

$$D_n=\begin{pmatrix} t_0 & 0 & \dots & 0 \\ t_1 & & & \\ \vdots & \ddots & \ddots & 0 \\ t_{n-1} & \dots & t_1 & t_0 \end{pmatrix}, \quad \mathbf{S}=(t_{i+j})_{i,j=1}^n, \quad \mathcal{T}_n=\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}. \tag{3.2}$$

Then $\widehat{T}_{n-1}$ *is a Hankel block matrix:*

$$\widehat{T}_{n-1}=(\widehat{t}_{i+j})_{i,j=0}^{n-1} \tag{3.3}$$

and moreover, $\widehat{T}_{n-1}\in\widetilde{\mathcal{H}}_{l,n-1}$.

Proof: Let $F_{l,n-1}$ be the matrix defined via formula (1.7) and let

$$\widetilde{U}:=(I_l,0,\dots,0)^*\in\mathbb{C}^{ln\times l}. \tag{3.4}$$

We begin with the identities

$$D_nF_{l,n-1}=F_{l,n-1}D_n, \qquad \widetilde{U}^*F_{l,n-1}=0, \qquad D_n\widetilde{U}-F_{l,n-1}\mathcal{T}_n=\widetilde{U}t_0 \tag{3.5}$$

and

$$F_{l,n-1}(\mathbf{S}-\mathcal{T}_nt_0^{-1}\mathcal{T}_n^*)-(\mathbf{S}-\mathcal{T}_nt_0^{-1}\mathcal{T}_n^*)F_{l,n-1}^*=\mathcal{T}_nt_0^{-1}\widetilde{U}^*D_n^*-D_n\widetilde{U}t_0^{-1}\mathcal{T}_n^*$$

which follow immediately from (1.7), (3.2) and (3.4). Using these identities we get

$$\begin{aligned}
&F_{l,n-1}^*(F_{l,n-1}\widehat{T}_{n-1}-\widehat{T}_{n-1}F_{l,n-1}^*)F_{l,n-1}\\
&=F_{l,n-1}^*D_n^{-1}\left\{F_{l,n-1}\left(\mathbf{S}-\mathcal{T}_nt_0^{-1}\mathcal{T}_n^*\right)-\left(\mathbf{S}-\mathcal{T}_nt_0^{-1}\mathcal{T}_n^*\right)F_{l,n-1}^*\right\}D_n^{-*}F_{l,n-1}\\
&=F_{l,n-1}^*D_n^{-1}\left\{\mathcal{T}_nt_0^{-1}\widetilde{U}^*D_n^*-D_n\widetilde{U}t_0^{-1}\mathcal{T}_n^*\right\}D_n^{-*}F_{l,n-1}\\
&=F_{l,n-1}^*D_n^{-1}\mathcal{T}_nt_0^{-1}\widetilde{U}^*F_{l,n-1}-F_{l,n-1}^*\widetilde{U}t_0^{-1}\mathcal{T}_n^*F_{l,n-1}=0
\end{aligned}$$

and (3.3) follows by Lemma 2.1. Since D_n is invertible, the factorization formula

$$T_n=\begin{pmatrix} I_l & 0 \\ \mathcal{T}_nt_0^{-1} & D_n \end{pmatrix}\begin{pmatrix} t_0 & 0 \\ 0 & \widehat{T}_{n-1} \end{pmatrix}\begin{pmatrix} I_l & t_0^{-1}\mathcal{T}_n^* \\ 0 & D_n^* \end{pmatrix} \tag{3.6}$$

implies that $\widehat{T}_{n-1}\geq 0$ and thus, $\widehat{T}_{n-1}\in\mathcal{H}_{l,n-1}$. It remains to verify that

$$\mathbf{P}_{Ker\widehat{T}_{n-2}}\begin{pmatrix} \widehat{t}_n \\ \vdots \\ \widehat{t}_{2n-2} \end{pmatrix}=0. \tag{3.7}$$

To this end, we first observe that

$$\mathbf{P}_{KerT_{n-1}}\left(\mathcal{T}_n,\ \mathbf{S}\right)=0 \tag{3.8}$$

since $T_n\geq 0$. Using the factorization of T_{n-1} similar to (3.6) we obtain

$$\begin{pmatrix} s_0 & 0 \\ 0 & \widehat{T}_{n-2} \end{pmatrix}=\begin{pmatrix} I & 0 \\ -D_{n-1}^{-1}\mathcal{T}_{n-1}t_0^{-1} & D_{n-1}^{-1} \end{pmatrix}T_{n-1}\begin{pmatrix} I & -t_0^{-1}\mathcal{T}_{n-1}^*D_{n-1}^{-*} \\ 0 & D_{n-1}^{-*} \end{pmatrix} \tag{3.9}$$

where D_{n-1} and $\mathcal{T}_{n-1}$ are defined via (3.2). Upon applying Lemma 2.2 to the matrices

$$K=T_{n-1}, \quad V=\begin{pmatrix} I & 0 \\ -D_{n-1}^{-1}\mathcal{T}_{n-1}t_0^{-1} & D_{n-1}^{-1} \end{pmatrix} \quad \text{and} \quad A=\left(\mathcal{T}_n,\ \mathbf{S}\right),$$

and making use of (3.8), (3.9) we obtain

$$\mathbf{P}_{Ker\widehat{T}_{n-2}}D_{n-1}^{-1}\left(-\mathcal{T}_{n-1}t_0^{-1},\ I_{mn}\right)\left(\mathcal{T}_n,\ \mathbf{S}\right)=0. \tag{3.10}$$

From the block decomposition $D_n = \begin{pmatrix} t_0 & 0 \\ \mathcal{T}_{n-1} & D_{n-1} \end{pmatrix}$ we have

$$D_n^{-1} = \begin{pmatrix} t_0^{-1} & 0 \\ -D_{n-1}^{-1}\mathcal{T}_{n-1}t_0^{-1} & D_{n-1}^{-1} \end{pmatrix}. \tag{3.11}$$

Substituting (3.11) into (3.1) we obtain

$$\begin{pmatrix} \widehat{t}_1 & \dots & \widehat{t}_n \\ \vdots & & \vdots \\ \widehat{t}_{n-1} & \dots & \widehat{t}_{2n-2} \end{pmatrix} = (0,\ I_{m(n-1)})\widehat{T}_{n-1} = D_{n-1}^{-1}\left(-\mathcal{T}_{n-1}t_0^{-1}, I_{mn}\right)\left(\mathbf{S} - \mathcal{T}_n t_0^{-1}\mathcal{T}_n^*\right)D_n^{-*}.$$

The last equality both with (3.10) implies

$$\mathbf{P}_{Ker\widehat{T}_{n-2}} \begin{pmatrix} \widehat{t}_1 & \dots & \widehat{t}_n \\ \vdots & & \vdots \\ \widehat{t}_{n-1} & \dots & \widehat{t}_{2n-2} \end{pmatrix} = 0$$

and, in particular, (3.7), which completes the proof of lemma. □

Lemma 3.2. *Let* $K_n \in \widetilde{\mathcal{H}}_{m,n}$ *and let* $\operatorname{rank} K_n = r$. *Then there exists* $Q \in \mathbb{C}^{r\times(n+1)m}$ *such that*

$$QK_nQ^* > 0, \qquad \operatorname{rank} QK_nQ^* = \operatorname{rank} K_n, \qquad QF_{m,n} = NQ \tag{3.12}$$

for the shift $F_{m,n}$ *defined by* (1.7) *and some matrix* $N \in \mathbb{C}^{r\times r}$. *In other words, there exists a subspace* $\mathcal{Q} = \operatorname{Ran} Q \stackrel{\text{def}}{=} \{y \in \mathbb{C}^{m(n+1)} : y = fQ \text{ for some } f \in \mathbb{C}^r\}$ *coinvariant with respect to* $F_{m,n}$ *and such that*

$$\mathbb{C}^{m(n+1)} = Ker\ K \dotplus \mathcal{Q}. \tag{3.13}$$

Proof: We prove this lemma by induction. Let $n = 0$ and let $rank\ s_0 = l \le m$. Then there exists a unitary matrix $\mathbf{v} \in \mathbb{C}^{m\times m}$ such that

$$\mathbf{v}s_0\mathbf{v}^* = \begin{pmatrix} t_0 & 0 \\ 0 & 0_{m-l} \end{pmatrix} \qquad (t_0 > 0), \tag{3.14}$$

and the matrix

$$g = (I_l,\ \ 0)\mathbf{v} \in \mathbb{C}^{l\times m} \tag{3.15}$$

(considered as Q) clearly satisfies (3.12).

Let us suppose that the statement of the lemma holds for all integers up to $n-1$. Let as above, $rank\ s_0 = l$ and let $\mathbf{v}$ and g be matrices defined by (3.14), (3.15). Since $K_n \in \widetilde{\mathcal{H}}_{m,n}$, we have $Ker\ s_0 \subseteq Ker\ s_i$ for $i = 1, \dots, 2n$, and then we have from (3.14),

$$\mathbf{v}s_i\mathbf{v}^* = \begin{pmatrix} t_i & 0 \\ 0 & 0_{m-l} \end{pmatrix} \quad (t_i \in \mathbb{C}^{l\times l};\ \ i = 1, \dots, 2n). \tag{3.16}$$

In more detail, representations (3.16) for $i = 1, \dots, 2n-1$ follow from positivity of K_n along with its Hankel structure. Since K_n belongs to $\widetilde{\mathcal{H}}_{m,n}$, equality (2.12) holds. Upon substituting decompositions (3.14) and (3.16) (for $i = 1, \dots, 2n-1$) into (2.12), one can easily see that s_{2n} is necessarily of the form $\mathbf{v}s_{2n}\mathbf{v}^* = \begin{pmatrix} t_{2n} & \gamma \\ 0 & 0 \end{pmatrix}$ for some $\gamma \in \mathbb{C}^{l\times(m-l)}$. Since s_{2n} is Hermitian, $\gamma = 0$ and representation (3.16) for s_{2n} follows.

From (3.14)–(3.16) we obtain that $gs_ig^* = t_i$ $(i = 0, \dots, 2n)$ and

$$T_n \equiv (t_{i+j})_{i,j=0}^n = G_nK_nG_n^*, \qquad \operatorname{rank} T_n = \operatorname{rank} K_n, \tag{3.17}$$

where G_n is the $(n+1)l \times (n+1)m$ matrix defined by

$$G_n = \begin{pmatrix} g & & 0 \\ & \ddots & \\ 0 & & g \end{pmatrix}. \tag{3.18}$$

Since $K_n \in \widetilde{\mathcal{H}}_{m,n}$, then it is readily seen that $T_n \in \widetilde{\mathcal{H}}_{l,n}$. Let $\widehat{T}_{n-1}$, D_n and $\mathcal{T}_n$ be matrices defined by (3.1), (3.2). Multiplying K_n on the left by the matrix

$$\Phi = \begin{pmatrix} I_l & 0 \\ -D_n^{-1}\mathcal{T}_n t_0^{-1} & D_n^{-1} \end{pmatrix} G_n \tag{3.19}$$

and by Φ^* on the right we obtain, on account of of (3.18) and (3.6),

$$\Phi K_n \Phi^* = \begin{pmatrix} t_0 & 0 \\ 0 & \widehat{T}_{n-1} \end{pmatrix}. \tag{3.20}$$

By Lemma 3.1, $\widehat{T}_{n-1} \in \widetilde{\mathcal{H}}_{l,n-1}$, and it follows from (3.17), (3.19) and (3.20) that $rank\ \widehat{T}_{n-1} = rank\ K_n - rank\ t_0 = r - l$. Therefore, by the induction hypothesis, there exist matrices $\widetilde{Q} \in \mathbb{C}^{(r-l)\times ln}$ and $\widetilde{N} \in \mathbb{C}^{(r-l)\times(r-l)}$ such that

$$\widetilde{Q}\widehat{T}_{n-1}\widetilde{Q}^* > 0 \quad \text{and} \quad \widetilde{Q}F_{l,n} = \widehat{N}\widehat{Q}. \tag{3.21}$$

We show that the matrices

$$Q = \begin{pmatrix} I_l & 0 \\ 0 & \widetilde{Q} \end{pmatrix} \Phi \in \mathbb{C}^{r\times(n+1)}, \qquad N = \begin{pmatrix} 0_l & 0 \\ \widetilde{Q}\widetilde{U}t_0^{-1} & \widetilde{N} \end{pmatrix} \in \mathbb{C}^{r\times r} \tag{3.22}$$

(where $\widetilde{U}$ is the matrix given by (3.4)) satisfy (3.12). Indeed, by (3.20)–(3.22,)

$$QK_nQ^* = \begin{pmatrix} I_l & 0 \\ 0 & \widetilde{Q} \end{pmatrix} \begin{pmatrix} t_0 & 0 \\ 0 & \widehat{T}_{n-1} \end{pmatrix} \begin{pmatrix} I_l & 0 \\ 0 & \widetilde{Q}^* \end{pmatrix} > 0$$

and

$$\text{rank}\ QK_nQ^* = \text{rank}\ t_0 + \text{rank}\ \widetilde{Q}\widehat{T}_{n-1}\widetilde{Q}^* = l + (r-l) = \text{rank}\ K_n.$$

We next make use of (3.19)–(3.21) and of the block decompositions

$$G_n = \begin{pmatrix} g & 0 \\ 0 & G_{n-1} \end{pmatrix} \quad \text{and} \quad F_{m,n} = \begin{pmatrix} 0 & 0 \\ \widetilde{U} & F_{m,n-1} \end{pmatrix}$$

to compute

$$QF_{m,n} = \begin{pmatrix} 0 & 0 \\ \widetilde{Q}D_n^{-1}G_{n-1}\widetilde{U} & \widetilde{Q}D_n^{-1}G_{n-1}F_{m,n-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \widetilde{Q}D_n^{-1}\widetilde{U}g & \widetilde{Q}D_n^{-1}F_{l,n-1}G_{n-1} \end{pmatrix}$$

and

$$\begin{aligned} NQ &= \begin{pmatrix} 0 & 0 \\ (\widetilde{Q}\widetilde{U} - \widetilde{N}\widetilde{Q}D_n^{-1}\mathcal{T}_n)t_0^{-1}g & \widetilde{N}\widetilde{Q}D_n^{-1}G_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \widetilde{Q}D_n^{-1}(D_n\widetilde{U} - F_{l,n-1}\mathcal{T}_n)t_0^{-1}g & \widetilde{Q}F_{l,n-1}D_n^{-1}G_{n-1} \end{pmatrix}. \end{aligned}$$

We now invoke equalities (3.5) to verify that the right hand side matrices in the two last formulas coincide. Thus, $QF_{m,n} = NQ = 0$, and the matrices Q and N defined by (3.22) satisfy (3.12). This completes the proof. □

In what follows, the indeces will be omitted and by K and F we mean matrices K_n and $F_{m,n}$ given by (1.1) and (1.7) respectively.

Lemma 3.3. *Let $K \in \widetilde{\mathcal{H}}_{m,n}$, let Q be any matrix satisfying* (3.12) *and let F, U, M, J and $K^{[-1]}$ be matrices given by* (1.7), (1.8), (1.11) *and* (2.8). *Then the $\mathbb{C}^{2m\times 2m}$–valued function*

$$\Theta(z) = I_{2m} + z \begin{pmatrix} M^* \\ -U^* \end{pmatrix} (I - zF^*)^{-1} K^{[-1]}(U,\ M) \tag{3.23}$$

is of the class $\mathbf{W}$ *and moreover,*

$$\Theta(z)^* J\Theta(z) - J = i(\bar{z} - z) \begin{pmatrix} U^* \\ M^* \end{pmatrix} K^{[-1]}(I - \bar{z}F)^{-1} K (I - zF^*)^{-1} K^{[-1]}(U, M), \tag{3.24}$$

$$J - \Theta(z)^{-*} J\Theta^{-1}(z) = i(\bar{z} - z) \begin{pmatrix} U^* \\ M^* \end{pmatrix} (I - \bar{z}F^*)^{-1} K^{[-1]} (I - zF)^{-1}(U, M). \tag{3.25}$$

Observe that the two first relations in (3.12) enable us to construct the pseudoinverse matrix $K^{[-1]}$ according to (2.8) and the third equality guarantees (3.24) and (3.25) to be in force.

Proof: Using (3.23), (1.11) and (1.9) we have

$$\Theta(z)^* J\Theta(z) - J = i \begin{pmatrix} U^* \\ M^* \end{pmatrix} L(z)(U,\ M) \tag{3.26}$$

where

$$\begin{aligned} L(z) &= |z|^2 K^{[-1]}(I - \bar{z}F)^{-1} \{MU^* - UM^*\} (I - zF^*)^{-1} K^{[-1]} \\ &\quad + \bar{z}K^{[-1]}(I - \bar{z}F)^{-1} - z(I - zF^*)^{-1} K^{[-1]} \\ &= (\bar{z} - z) K^{[-1]}(I - \bar{z}F)^{-1} K (I - zF^*)^{-1} K^{[-1]} \\ &\quad + \bar{z}K^{[-1]}(I - \bar{z}F)^{-1}(I - KK^{[-1]}) - z(I - K^{[-1]}K)(I - zF^*)^{-1} K^{[-1]}. \end{aligned} \tag{3.27}$$

It follows from (3.12) that $QF^j = N^j Q$ $(j = 0, 1, \ldots)$ which both with (2.8) implies

$$K^{[-1]} F^j \left(I - KK^{[-1]}\right) = Q^*(QKQ^*)^{-1} N^j Q \left(I - KQ(QKQ^*)^{-1} Q\right) = 0 \tag{3.28}$$

for $j = 0, 1, \ldots$ Since $(I - zF^*)^{-1} = \sum_{j=0}^{n} z^j F^{*j}$, then also

$$K^{[-1]}(I - zF)^{-1} \left(I - KK^{[-1]}\right) \qquad (z \in \mathbb{C}). \tag{3.29}$$

Substituting (3.29) into (3.27) and (3.27) into (3.26), we obtain (3.24). Similarly,

$$\Theta(z) J\Theta(z)^* - J = i \begin{pmatrix} M^* \\ -U^* \end{pmatrix} (I - zF^*)^{-1} \widetilde{L}(z) (I - \bar{z}F)^{-1}(M,\ -U) \tag{3.30}$$

where

$$\begin{aligned} \widetilde{L}(z) &= \bar{z}(I - zF^*) K^{[-1]} - zK^{[-1]}(I - \bar{z}F) - |z|^2 K^{[-1]} \{MU^* - UM^*\} K^{[-1]} \\ &= (\bar{z} - z) K^{[-1]} + |z|^2 K^{[-1]} F \left(I - KK^{[-1]}\right) - |z|^2 \left(I - K^{[-1]}K\right) F^* K^{[-1]}. \end{aligned} \tag{3.31}$$

Using (3.28) for $j = 1$ we obtain from (3.31) that $\widetilde{L}(z) = (\bar{z} - z) K^{[-1]}$ and by (3.30),

$$\Theta(z) J\Theta(z)^* - J = i(\bar{z} - z) \begin{pmatrix} M^* \\ -U^* \end{pmatrix} (I - zF^*)^{-1} K^{[-1]} (I - \bar{z}F)^{-1}(M,\ -U). \tag{3.32}$$

Relations (1.10) follow from (3.32) and thus, $\Theta \in \mathbf{W}$. Since it Θ is J–unitary on $\mathbb{R}$, then by the symmetry principle,

$\Theta^{-1}(z) = J\Theta(\bar{z})^*J$ which both with (3.32) leads to

$$\begin{aligned} J - \Theta(z)^{-*}J\Theta^{-1}(z) &= J(J - \Theta(\bar{z})J\Theta(\bar{z})^*)J \\ &= i(z-\bar{z})J\begin{pmatrix} M^* \\ -U^* \end{pmatrix}(I - \bar{z}F^*)^{-1}K^{[-1]}(I - zF)^{-1}(M,\ -U)J \end{aligned} \tag{3.33}$$

and implies (3.25). □

4. Parametrization of all solutions

In this section we parametrize the set $\mathcal{R}(K_n)$ of all solutions of the degenerate **HMP** in terms of a linear fractional transformation. The following theorem can be found in [7, 9].

Theorem 4.1. *Let* $\Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$ *be the block decomposition of a* $\mathbb{C}^{2m\times 2m}$*–valued function* $\Theta \in \mathbf{W}$ *into four* $\mathbb{C}^{m\times m}$*–valued blocks. Then all* $\mathbb{C}^{m\times m}$*–valued analytic in* $\mathbb{C}\backslash\mathbb{R}$ *solutions* w *to the inequality*

$$(w(z)^*,\ I_m)\frac{\Theta(z)^{-*}J\Theta^{-1}(z)}{i(\bar{z}-z)}\begin{pmatrix} w(z) \\ I_m \end{pmatrix} \geq 0 \tag{4.1}$$

are parametrized by the formula

$$w(z) = (\theta_{11}(z)p(z) + \theta_{12}(z)q(z))(\theta_{21}(z)p(z) + \theta_{22}(z)q(z))^{-1} \tag{4.2}$$

when the parameter $\{p,\ q\}$ *varies in the set* $\mathbf{N}_m$ *of all Nevanlinna pairs and satisfies*

$$\det\,(\theta_{21}(z)p(z) + \theta_{22}(z)q(z)) \not\equiv 0; \tag{4.3}$$

Moreover, two Nevanlinna pairs lead via (4.2) *to the same function* w *if and only if these pairs are equivalent.*

Lemma 4.2. *Let* $\{p,\ q\} \in \mathbf{N}_m$ *be a Nevanlimma pair. Then*

$$\det\,(p(z) + iq(z)) \not\equiv 0, \tag{4.4}$$

the function

$$S(z) = (p(z) - iq(z))(p(z) + iq(z))^{-1} \tag{4.5}$$

is a $\mathbb{C}^{m\times m}$*–valued contraction in* $\mathbb{C}_+$ *and moreover, two different pairs lead by* (4.5) *to the same* s *if and only if they are equivalent.*

The proof is given in [7]. Observe that by (4.4), every Nevanlinna pair $\{p,\ q\}$ satisfies the dual nondegeneracy property (compare with Definition 1.2)

$$\det\,(p(z)p(z)^* + q(z)q(z)^*) \not\equiv 0. \tag{4.6}$$

Lemma 4.3. *Let* $\{p,\ q\} \in \mathbf{N}_m$ *be a Nevanlimma pair such that* $(I_\nu,\ 0)\ p(z) \equiv 0$ $(\nu \leq m)$*. Then* $\{p,\ q\}$ *is equivalent to a pair*

$$\left\{\begin{pmatrix} 0_\nu & 0 \\ 0 & \widetilde{p}(z) \end{pmatrix},\ \begin{pmatrix} I_\nu & 0 \\ 0 & \widetilde{q}(z) \end{pmatrix}\right\} \quad \text{for some} \quad \{\widetilde{p},\ \widetilde{q}\} \in \mathbf{N}_{m-\nu}. \tag{4.7}$$

Proof: By the assumption assumption, p and q are of the form

$$p(z) = \begin{pmatrix} 0_\nu & 0 \\ p_{21}(z) & p_{22}(z) \end{pmatrix}, \qquad q(z) = \begin{pmatrix} q_{11}(z) & q_{12}(z) \\ q_{21}(z) & q_{22}(z) \end{pmatrix} \tag{4.8}$$

and in view of (4.6), *rank* $(q_{11}(z),\ q_{12}(z)) = m$ at almost all $z \in \mathbb{C}_+$. Multiplying $(q_{11}(z),\ q_{12}(z))$ by an appropriate unitary matrix U on the right we obtain

$$(q_{11}(z),\ q_{12}(z))\ U = (\widetilde{q}_{11}(z),\ \widetilde{q}_{12}(z)), \qquad \det\ \widetilde{q}_{11}(z) \not\equiv 0.$$

The pair $\{p,\ q\}$ is equivalent to the pair $\{p_1,\ q_1\}$ defined as

$$\begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix} = \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} U\Phi(z) \quad \text{where} \quad \Phi(z) = \begin{pmatrix} \widetilde{q}_{11}^{-1}(z) & -\widetilde{q}_{11}^{-1}(z)\widetilde{q}_{12}(z) \\ 0 & I_{m-\nu} \end{pmatrix}.$$

It follows from (4.8) that the functions p_1 and q_1 are of the form

$$p_1(z) = \begin{pmatrix} 0_\nu & 0 \\ \widetilde{p}_1(z) & \widetilde{p}(z) \end{pmatrix} \qquad q_1(z) = \begin{pmatrix} I_\nu & 0 \\ \widetilde{q}_1(z) & \widetilde{q}(z) \end{pmatrix} \tag{4.9}$$

and it remains to show that $\{p_1,\ q_1\}$ is equivalent to the pair defined in (4.7). Indeed, (4.9) implies that $\{\widetilde{p},\ \widetilde{q}\} \in \mathbf{N}_{m-\nu}$ and therefore, det $(\widetilde{p}(z) + i\widetilde{q}(z)) \not\equiv 0$. Substituting the pair (4.9) into (4.5) gives

$$\begin{aligned} S(z) &= (p_1(z) - iq_1(z))(p_1(z) + iq_1(z))^{-1} \\ &= \begin{pmatrix} -iI & 0 \\ \widetilde{p}_1 - i\widetilde{q}_1 & \widetilde{p} - i\widetilde{q} \end{pmatrix} \begin{pmatrix} iI & 0 \\ \widetilde{p}_1 + i\widetilde{q}_1 & \widetilde{p} + i\widetilde{q} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -I & 0 \\ i(\widetilde{p} - i\widetilde{q})(\widetilde{p} + i\widetilde{q})^{-1}(\widetilde{p}_1 + i\widetilde{q}_1) - i(\widetilde{p}_1 - i\widetilde{q}_1) & (\widetilde{p} - i\widetilde{q})(\widetilde{p} + i\widetilde{q})^{-1} \end{pmatrix} \\ &= \begin{pmatrix} -I & 0 \\ 0 & (\widetilde{p} - i\widetilde{q})(\widetilde{p} + i\widetilde{q})^{-1} \end{pmatrix} \end{aligned}$$

(to obtain the last equality we used the following: if the function $S = \begin{pmatrix} s_1 & 0 \\ s_2 & -I \end{pmatrix}$ is contractive valued, then $s_2 \equiv 0$). It is easily seen that the pair (4.7) being substituted into (4.5), leads to the same function S. By Lemma 4.2, the pairs (4.5) and (4.9) are equivalent. ☐

Lemma 4.4. *Let $R \in \mathbb{C}^{l\times 2m}$ be a J–neutral matrix (i.e. $RJR = 0$) and let* $\operatorname{rank} R = \nu \leq \min(m,\ l)$. *Then there exist a J–unitary matrix Ψ and an invertible T such that*

$$TR\Psi = \begin{pmatrix} I_\nu & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.10}$$

Proof: Since $\operatorname{rank} R = \nu$, there exists an invertible matrix T such that

$$TR = \begin{pmatrix} \widehat{R} \\ 0_{(m-\nu)\times 2m} \end{pmatrix} \tag{4.11}$$

where $\widehat{R}$ is a full rank J–neutral matrix. Let us endow the space $\mathbb{C}^{1\times 2m}$ with the indefinite inner product $[x,\ y] = yJx^*$. By (1.11), the subspace

$$\mathcal{G} = \left\{g \in \mathbb{C}^{1\times 2m} :\ \ g = (\widehat{g},\ 0)\ \text{ for some }\ \widehat{g} \in \mathbb{C}^{1\times\nu}\right\}$$

is J–neutral. The subspace $\mathcal{F} = \left\{f \in \mathbb{C}^{1\times 2m} :\ f = \widehat{g}\widehat{R},\ \widehat{g} \in \mathbb{C}^{1\times\nu}\right\}$ J–neutral as well. Let us introduce the operator $\widehat{\mathbf{\Psi}}\ :\ \mathcal{F} \to \mathcal{G}$ by $\widehat{g}\widehat{R}\ \widehat{\mathbf{\Psi}} = (\widehat{g},\ 0)$. Since $\mathcal{F}$ and $\mathcal{G}$ are J–neutral and dim $\mathcal{F}$ = dim $\mathcal{G}$, the operator $\widehat{\mathbf{\Psi}}$ is J–isometric and has equal defect numbers. Furthermore, $\widehat{\mathbf{\Psi}}$ is invertible and therefore, it admits a J–unitary extension $\mathbf{\Psi}$ to all of $\mathbb{C}^{1\times 2m}$ ([6]). The matrix Ψ of this extended operator in the standard basis is J–unitary and satisfies $\widehat{R}\Psi = (I_\nu,\ 0)$ which both with (4.11) implies (4.10). ☐

Remark 4.5. Let $R = (R_1,\ R_2) \in \mathbb{C}^{l\times 2m}$ be a J–neutral matrix: $R_1R_2^* - R_2R_1^* = 0$. Then rank R = rank $(R_1 + iR_2)$. Indeed,

$$\operatorname{rank}(R_1 + iR_2) = \operatorname{rank}(R_1 + iR_2)(R_1 + iR_2)^* = \operatorname{rank}(R_1R_1^* + R_2R_2^*) = \operatorname{rank} RR^* = \operatorname{rank} R.$$

The following theorem is the degenerate analogue of Theorem 1.3.

Theorem 4.6. *Let the Pick matrix K_n of the* **HMP** *be in the class $\widetilde{\mathcal{H}}_{m,n}$ and let Θ be the $\mathbb{C}^{2m\times 2m}$–valued function defined by (3.23). Then, there exists a J-unitary matrix $\Psi \in \mathbb{C}^{2m\times 2m}$ such that*

(1) *All the functions $w \in \mathcal{R}(K_n)$ are obtained by the formula*

$$w(z) = (a_{11}(z)p(z) + a_{12}(z)q(z))(a_{21}(z)p(z) + a_{22}(z)q(z))^{-1} \tag{4.12}$$

with the coefficient matrix $A(z) = (a_{ij}(z)) = \Theta(z)\Psi \in \mathbf{W}$ when the parameter $\{p, q\}$ varies in the set of all Nevanlinna pairs of the form

$$\{p(z),\ q(z)\} = \left\{ \begin{pmatrix} 0_\nu & 0 \\ 0 & \widetilde{p}(z) \end{pmatrix},\ \begin{pmatrix} I_\nu & 0 \\ 0 & \widetilde{q}(z) \end{pmatrix} \right\} \tag{4.13}$$

where $\{\widetilde{p},\ \widetilde{q}\} \in \mathbf{N}_{m-\mu}$ and ν is the integer given by

$$\nu = rank\ \{(I_m, is_0, \ldots, is_{n-1})\mathbf{P}_{KerK_n}\}.$$

(2) *Two pairs lead to the same function w if and only if they are equivalent.*

Proof: According to Theorem 1.1 the set $\mathcal{R}(K_n)$ coincides with the set of all solutions to the inequality (1.6) which is equivalent, by Lemma 2.4, to the following system

$$\frac{w(z) - w(z)^*}{z - \bar{z}} - (Uw(z) + M)^*(I - zF)^{-*}K^{[-1]}(I - zF)^{-1}(Uw(z) + M) \geq 0, \tag{4.14}$$

$$\mathbf{P}_{KerK}(I - zF)^{-1}\{Uw(z) + M\} \equiv 0. \tag{4.15}$$

It is easily seen that (4.14) can be written as

$$(w(z)^*,\ I)\left\{\frac{J}{i(\bar{z} - z)} - \begin{pmatrix} U^* \\ M^* \end{pmatrix}(I - zF^*)^{-1}K^{[-1]}(I - zF)^{-1}(U,\ M)\right\}\begin{pmatrix} w(z) \\ I \end{pmatrix} \geq 0$$

and is equivalent, in view of (3.25), to the inequality (4.1) with the function Θ defined by (3.23) which is of the class **W** by Lemma 3.3. According to Theorem 4.1, all solutions w to the inequality (4.14) are parametrized by the linear fractional transformation (4.2) when the parameter $\{p,\ q\}$ varies in the set $\mathbf{N}_m$ of all Nevanlinna pairs and satisfies (4.3). It remains to choose among these solutions all functions w which satisfy also identity (4.15). The rest of the proof is broken into four steps which we now specify.

Step 1: The function $w(z)$ of the form (4.2) stisfies the identity (4.15) if and only if the corresponding parameter $\{p,\ q\}$ satisfies

$$\mathbf{P}_{KerK}\{Up(z) + Mq(z)\} \equiv 0. \tag{4.16}$$

Step 2: If a pair $\{p,\ q\} \in \mathbf{N}_m$ satisfies (4.16) then it also satisfies (4.3).

Step 3: If a pair $\{p,\ q\} \in \mathbf{N}_m$ satisfies (4.16) then it is equivalent to some pair $\{p_1,\ q_1\}$ of the form

$$\begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix} = \Psi\begin{pmatrix} 0_\nu & 0 \\ 0 & \widetilde{p}(z) \\ I_\nu & 0 \\ 0 & \widetilde{q}(z) \end{pmatrix} \sim \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \tag{4.17}$$

for some J–unitary matrix $\Psi \in \mathbb{C}^{2m\times 2m}$ which depends only on K_n and a pair $\{\widetilde{p},\ \widetilde{q}\} \in \mathbf{N}_{m-\nu}$, where $\nu = rank\ \mathbf{P}_{KerK}(U, M) = rank\ \mathbf{P}_{KerK}(U + iM)$.

Proof of Step 1: Let $\Theta = (\theta_{ij})$ be the function defined by (3.23) and let w be a function of the form (4.2) for some pair $\{p,\ q\} \in \mathbf{N}_m$ which satisfies (4.3). Then

$$\begin{pmatrix} w(z) \\ I \end{pmatrix} = \Theta(z)\begin{pmatrix} p(z) \\ q(z) \end{pmatrix}(\theta_{21}(z)p(z) + \theta_{22}(z)q(z))^{-1}$$

and therefore, identity (4.15) is equivalent to

$$\mathbf{P}_{KerK}(I-zF)^{-1}(U,\ M)\Theta(z)\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \equiv 0. \tag{4.18}$$

Using (1.9), (3.23) and the identity

$$K(I-zF^*)^{-1}-(I-zF)^{-1}K = z(I-zF)^{-1}(KF^*-FK)(I-zF^*)^{-1}$$

we get

$$(I-zF)^{-1}(U,\ M)\Theta(z) = K(I-zF^*)^{-1}K^{[-1]}(U,\ M)+(I-zF)^{-1}\left\{I-KK^{[-1]}\right\}(U,\ M).$$

Substituting the latter equality into (4.18) gives

$$\mathbf{P}_{KerK}(I-zF)^{-1}\{I-KK^{[-1]}\}\left(Up(z)+Mq(z)\right)\equiv 0$$

which on account of (2.9), can be written as

$$\{I+z\mathbf{P}_{KerK}F(I-zF)^{-1}(I-KK^{[-1]})\}\mathbf{P}_{KerK}\left(Up(z)+Mq(z)\right)\equiv 0. \tag{4.19}$$

Since the matrix $\{I+z\mathbf{P}_{kerK}F(I-zF)^{-1}(I-KK^{[-1]})\}$ is nondegenerate, (4.19) implies (4.16).

Proof of Step 2: Let a pair $\{p,\ q\}\in \mathbf{N}_m$ satisfy the condition (4.16). We introduce the pair

$$\begin{pmatrix} p_0(z) \\ q_0(z) \end{pmatrix} = \Theta(z)\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \tag{4.20}$$

and show that $\ det\ q_0(z)\not\equiv 0$. Indeed, suppose that the point $\lambda\in\mathbb{C}_+$ and the nonzero vector $\mathbf{h}\in\mathbb{C}^m$ are such that $\det\ \Theta(\lambda)\neq 0$ and

$$q_0(\lambda)\mathbf{h}=0. \tag{4.21}$$

Since $\mathbf{h}^*\left(p(\lambda)^*,\ q(\lambda)^*\right)\Theta(\lambda)^*J\Theta(\lambda)\begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix}\mathbf{h} = \mathbf{h}^*\left(p_0(\lambda)^*,\ 0\right)J\begin{pmatrix} p_0(\lambda) \\ 0 \end{pmatrix}\mathbf{h}=0$, then

$$0\leq \mathbf{h}^*\left(p(\lambda)^*,\ q(\lambda)^*\right)J\begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix}\mathbf{h} = \mathbf{h}^*\left(p(\lambda)^*,\ q(\lambda)^*\right)\{J-\Theta(\lambda)^*J\Theta(\lambda)\}\begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix}\mathbf{h},$$

due to (1.12). Substituting (3.24) into this last inequality leads us to

$$K(I-\lambda F^*)^{-1}K^{[-1]}\{Up(\lambda)+Mq(\lambda)\}\mathbf{h}=0. \tag{4.22}$$

It follows from (3.23) and (4.20) that

$$p_0(\lambda)=p(\lambda)+\lambda M^*(I-\lambda F^*)^{-1}K^{[-1]}\{Up(\lambda)+Mq(\lambda)\}. \tag{4.23}$$

Since $M=FKU$ (see (1.8)), then $\lambda M^*(I-\lambda F^*)^{-1}=U^*K(I-\lambda F^*)^{-1}-U^*K$. Substituting this last equality into (4.23) and taking into account (2.9), (4.16), (4.22) and the evident equalities $U^*U=I_m$ and $U^*M=0$ we receive

$$\begin{aligned} p_0(\lambda)\mathbf{h} &= p(\lambda)\mathbf{h}-U^*KK^{[-1]}\{Up(\lambda)+Mq(\lambda)\}\mathbf{h}+U^*K(I-zF^*)^{-1}K^{[-1]}\{Up(\lambda)+Mq(\lambda)\}\mathbf{h} \\ &= U^*(I-KK^{[-1]})\{Up(\lambda)+Mq(\lambda)\}\mathbf{h}+(I-UU^*)p(\lambda)-U^*Mq(\lambda) \\ &= U^*(I-KK^{[-1]})\mathbf{P}_{KerK}\{Up(\lambda)+Mq(\lambda)\}\mathbf{h}=0. \end{aligned}$$

Since $\ det\,\Theta(\lambda)\neq 0$, the equality $p_0(\lambda)h=0$ both with (4.20) and (4.21) implies

$$\begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix}\mathbf{h}=\Theta(\lambda)^{-1}\begin{pmatrix} p_0(\lambda) \\ q_0(\lambda) \end{pmatrix}\mathbf{h}=0$$

and since λ is a arbitrary point, the latter equality contradicts to the nondegeneracy of the pair $\{p,\ q\}$.

Proof of Step 3: Using (1.9) we obtain that the matrix $\mathbf{P}_{KerK}(U,\ M)$ is J–neutral:

$$\mathbf{P}_{KerK}(U,\ M)J\begin{pmatrix} U^* \\ M^* \end{pmatrix}\mathbf{P}_{KerK}=i\mathbf{P}_{KerK}(KF^*-FK)\mathbf{P}_{KerK}=0.$$

Thus, by Remark 4.5,

$$\mu = \text{rank}\,(\mathbf{P}_{KerK}(U,\ M)) = \text{rank}\,(\mathbf{P}_{KerK}(U+iM)) = \text{rank}\ \{(I_m, is_0, \ldots, is_{n-1})\mathbf{P}_{KerK_n}\}\,.$$

According to Lemma 4.4, there exist a J–unitary matrix Ψ and an invertible T such that

$$T\mathbf{P}_{KerK}(U,\ M)\Psi = \begin{pmatrix} I_\nu & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.24}$$

Let $\{p_2,\ q_2\}$ be the pair defined by

$$\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} = \Psi \begin{pmatrix} p_2(z) \\ q_2(z) \end{pmatrix}. \tag{4.25}$$

On account of (4.24) and (4.25), condition (4.3) can be rewritten as $(I_\nu,\ 0)\ p_2(z) \equiv 0$ and by Lemma 4.3, the pair $\{p_2,\ q_2\}$ is equivalent to some pair of the form (4.7), i.e.,

$$\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} = \Psi \begin{pmatrix} p_2(z) \\ q_2(z) \end{pmatrix} \sim \Psi \begin{pmatrix} 0_\nu & 0 \\ 0 & \widehat{p}(z) \\ I_\nu & 0 \\ 0 & \widehat{q}(z) \end{pmatrix} = \begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix}$$

which completes the proof of Step 3.

Substituting (4.17) into (4.2) and taking into account that the equivalent pairs lead under the linear fractional transformation to the same function $w(z)$, we finish the proof of theorem. □

By Remark 2.9, the condition $K_n \in \widetilde{\mathcal{H}}_{m,n}$ is not restrictive and hence, the received in Theorem 3.3 description is applicable to the general situation $K_n \in \mathcal{H}_{m,n}$.

5. Correction of erratae in [2]

The following result was formulated in [2] (see Lemmas 2.5, 2.10 and 2.11 there).

Lemma 5.1. *Let* $K_n = (s_{i+j})_{i,j=0}^n \in \mathcal{H}_{m,n}$ *and let* $\mathcal{L}$ *be the subspace of* $\mathbb{C}^{1\times m}$ *given in* (2.11). *The following are equivalent:*

1. K_n *admits a positive semidefinite Hankel extension.*
2. $\mathbf{P}_{KerK_{n-1}} \begin{pmatrix} s_{n+1} \\ \vdots \\ s_{2n} \end{pmatrix} = 0.$
3. *The block* s_{2n} *is of the form*
$$s_{2n} = (s_n, \ldots, s_{2n-1})K_{n-1}^{[-1]}(s_n, \ldots, s_{2n-1})^* + R \tag{5.1}$$
for some positive semidefinite matrix $R \in \mathbb{C}^{m\times m}$ *which vanishes on the subspace* $\mathcal{L}$ *and does not depend on the choice of* $K_{n-1}^{[-1]}$.
4. *The associated truncated Hamburger moment problem admits an "exact" solution* σ *such that*
$$\int_{-\infty}^{\infty} \lambda^k d\sigma(\lambda) = s_k \qquad (k = 0, \ldots, 2n).$$

The proofs of implications (1) ⇒ (4) ⇒ (2) ⇔ (3) presented in [2] are correct; they are reproduced in Lemmas 2.10 and 2.11 above. To complete the proof, it suffices to justify (2) ⇒ (1), that is, in our current terminology, to show that

$$\widetilde{\mathcal{H}}_{m,n} \subseteq \mathcal{H}^+_{m,n}. \tag{5.2}$$

This inclusion together with (2.17) implies that all three classes introduced in Section 2 coincide.

Proof of (5.2): Let $K_n \in \widetilde{\mathcal{H}}_{m,n}$. Plug in the Nevanlinna pair $\{p, q\} = \{0_m, I_m\}$ (which is certainly of the form (4.13)) into formula (4.12) to get a solution $w(z) = a_{12}(z)a_{22}(z)^{-1}$ from $\mathcal{R}(K_n)$. This Pick function w is rational (since A is) and takes Hermitian values at every real point at which it is analytic (since A is J-unitary on $\mathbb{R}$). Then the measure σ from the Herglotz representation (1.4) of w is finitely atomic and therefore, the integrals $\int_{-\infty}^{\infty} \lambda^N d\sigma(\lambda)$ exists for every $N \geq 0$. Since this measure solves the associated **HMP**, it satisfies (1.2) and 1.3. By virtue of (2.18), the Hankel block matrix

$$\widetilde{K}_n = \begin{pmatrix} & & & s_n & s_{n+1} \\ & K_{n-1} & & \vdots & \vdots \\ & & & s_{2n-1} & s_{2n} \\ s_n & \cdots & s_{2n-1} & s & s_{2n+1} \\ s_{n+1} & \cdots & s_{2n} & s_{2n+1} & s_{2n+2} \end{pmatrix}$$

is positive semidefinite, where we have set

$$s = \int_{-\infty}^{\infty} \lambda^{2n} d\sigma(\lambda), \quad s_{2n+1} = \int_{-\infty}^{\infty} \lambda^{2n+1} d\sigma(\lambda), \quad s_{2n+2} = \int_{-\infty}^{\infty} \lambda^{2n+2} d\sigma(\lambda).$$

The Hankel block matrix $K_{n+1} := (s_{i+j})_{i,j=0}^{n+1}$ extends K_n and is positive semidefinie. Indeed, by (1.3), we have $K_{n+1} \geq \widetilde{K}_n \geq 0$. Thus $K_n \in \mathcal{H}_{m,n}^{+}$ which completes the proof. □

Remark 5.2. The proof of implication (2) ⇒ (1) presented in [2] does not rely on interpolation Theorem 4.6. The extending matrices s_{2n+1} and s_{2n+2} were constructed directly in terms of the given $s_0, \ldots, s_{2n}$. Unfortunately, the construction turned out to be wrong. The author was very glad to learn that correct explicit proofs of the above implication have been recently obtained [5, 10].

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