PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 5, Pages 1373–1383
S 0002-9939(02)06899-5
Article electronically published on December 6, 2002

INTERPOLATION FOR MULTIPLIERS ON REPRODUCING KERNEL HILBERT SPACES

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(Communicated by Joseph A. Ball)

ABSTRACT. All solutions of a tangential interpolation problem for contractive multipliers between two reproducing kernel Hilbert spaces of analytic vector-valued functions are characterized in terms of certain positive kernels. In a special important case when the spaces consist of analytic functions on the unit ball of $\mathbb{C}^d$ and the reproducing kernels are of the form $(1-\langle z,w\rangle^{-1})I_p$ and $(1-\langle z,w\rangle)^{-1}I_q$, the characterization leads to a parametrization of the set of all solutions in terms of a linear fractional transformation.

1. INTRODUCTION

A Hilbert space $\mathcal{H}$ of $\mathbb{C}^{p\times 1}$-valued functions which are defined on a domain $\Omega\in\mathbb{C}^d$ is said to be a *reproducing kernel Hilbert space* if there exists a $\mathbb{C}^{p\times p}$-valued function $K(z,\omega)$ such that for every point $\omega\in\Omega$ and every vector $c\in\mathbb{C}^p$, the function $K_wc:=K(\cdot\,,w)c$ belongs to $\mathcal{H}$ and $\langle f,\,K_wc\rangle_{\mathcal{H}}=c^*f(w)$ for every function $f\in\mathcal{H}$. The function $K(z,w)$ turns to be positive on Ω in the sense that $\sum_{j,\ell=1}^n c_j^*K(z^{(j)},z^{(\ell)})c_\ell\geq 0$ for every choice of an integer n, of vectors $c_1,\dots,c_n\in\mathbb{C}^p$ and of points $z^{(1)},\dots,z^{(n)}\in\Omega$ or, equivalently, if the Hermitian block matrix with ℓj-th entry $K(z^{(j)},z^{(\ell)})$ is positive semidefinite. This property will be denoted by $K(z,\omega)\succeq 0$. The function $K(z,\omega)$ is, furthermore, uniquely defined (as is easily verified), and is called *the reproducing kernel* of $\mathcal{H}$. The fundamental result of Aronszajn [4] states that for every positive kernel K on Ω, there is a unique reproducing kernel Hilbert space $\mathcal{H}(K)$ with K as its reproducing kernel. Moreover, the set $\mathcal{H}_0$ consisting of functions of the form $\sum K(\cdot\,,w_j)c_j$, where $\{c_j\}$ and $\{w_j\}$ are finite sequences in $\mathbb{C}^p$ and Ω, respectively, is a dense linear manifold in $\mathcal{H}(K)$. In what follows we shall write $K_w(z)$ rather than $K(z,w)$ if the last function will be considered as a function of z with a fixed point $w\in\Omega$.

Let $K^{(1)}(z,w)$ and $K^{(2)}(z,w)$ be two positive kernels on Ω, which are respectively, $\mathbb{C}^{q\times q}$- and $\mathbb{C}^{p\times p}$-valued and let $\mathcal{H}(K^{(1)})$ and $\mathcal{H}(K^{(2)})$ be the corresponding reproducing kernel Hilbert spaces. A $\mathbb{C}^{p\times q}$-valued function S defined on Ω is called a *contractive multiplier* from $\mathcal{H}(K^{(1)})$ to $\mathcal{H}(K^{(2)})$ if the multiplication operator $\mathbf{M}_S:\ \mathcal{H}(K^{(1)})\to\mathcal{H}(K^{(2)})$, defined by

$$\mathbf{M}_S(f(z))=S(z)f(z), \tag{1.1}$$

Received by the editors February 24, 2001 and, in revised form, March 23, 2001.
2000 *Mathematics Subject Classification.* Primary 41A05, 46E22.

is a contraction (if $K^{(1)} = K^{(2)}$, then S is called a contractive multiplier on $\mathcal{H}(K^{(1)})$). The latter means that $I_{\mathcal{H}(K^{(2)})} - \mathbf{M}_S\mathbf{M}_S^* \geq 0$ and is equivalent to

$$K_S(z,w) := K^{(2)}(z,w) - S(z)K^{(1)}(z,w)S(w)^* \succeq 0 \qquad (z,\, w \in \Omega). \tag{1.2}$$

The set of all contractive multipliers S from $\mathcal{H}(K^{(1)})$ to $\mathcal{H}(K^{(2)})$ will be denoted by $\mathcal{S}(K^{(1)}, K^{(2)})$.

In this paper we shall focus on the following interpolation

Problem 1.1. Given functions $f_1, \ldots, f_n$ in $\mathcal{H}(K^{(1)})$ and $h_1, \ldots, h_n$ in $\mathcal{H}(K^{(2)})$ find necessary and sufficient conditions which insure the existence of a function $S \in \mathcal{S}(K^{(1)}, K^{(2)})$ such that

$$(\mathbf{M}_S^* h_j)(z) = f_j(z), \qquad j = 1, \ldots, n. \tag{1.3}$$

We shall make frequent use of notations

$$H(z) = [h_1(z) \quad \ldots \quad h_n(z)] \quad \text{and} \quad F(z) = [f_1(z) \quad \ldots \quad f_n(z)], \tag{1.4}$$

which allows us to rewrite interpolation conditions (1.3) in a more compact form as

$$(\mathbf{M}_S^* H)(z) = F(z). \tag{1.5}$$

Note that the tangential Nevanlinna–Pick problem in the class $\mathcal{S}(K^{(1)}, K^{(2)})$ is a particular case of Problem 1.1. Indeed, a simple computation shows that

$$(\mathbf{M}_S^* K_w^{(2)})(z) = K_w^{(1)}(z)S(w)^*,$$

and thus a special choice of $h_j = K_{w_j}^{(2)} c_j$ and $f_j = K_{w_j}^{(1)} d_j$ in (1.3) leads to the left-sided interpolation conditions

$$K_{w_j}^{(1)}(z)S(w_j)^* c_j \equiv K_{w_j}^{(1)}(z) d_j, \qquad j = 1, \ldots, n.$$

Under the further assumption that $K^{(1)}$ is not degenerate (i.e., $K^{(1)}(z,z) > 0$ for all $z \in \Omega$), the latter conditions are equivalent to the classical Nevanlinna–Pick conditions

$$S(w_j)^* c_j = d_j, \qquad j = 1, \ldots, n.$$

In Section 2 all the solutions S of Problem 1.1 are characterized in terms of certain positive kernels constructed from the interpolation data. In Section 3 we consider a particular case of Problem 1.1 for multipliers on multivariable analogues of the Hardy space $\mathbf{H}_2$ of the unit disk, studied in [5]. For this case, the general result (Theorem 2.4) leads to a parametrization of the set of all solutions in terms of a linear fractional transformation. The Nevanlinna–Pick problem in this setting (see Remark 3.1) has been considered in [6].

2. The main result

As mentioned above, for a function $S \in \mathcal{S}(K^{(1)}, K^{(2)})$ the kernel $K_S(z,w)$ defined in (1.2) is positive on Ω. The corresponding reproducing kernel Hilbert space will be referred to as $\mathcal{H}(S)$.

The original characterization of $\mathcal{H}(S)$, as the space of all elements functions $f \in \mathcal{H}(K^{(2)})$ such that

$$\kappa(f) := \sup_{g \in \mathcal{H}(K^{(1)})} \left\{ \|f + Sg\|^2_{\mathcal{H}(K^{(2)})} - \|g\|^2_{\mathcal{H}(K^{(1)})} \right\} \tag{2.1}$$

is finite and $\|f\|^2_{\mathcal{H}(S)} = \kappa(f)$, is due to de Branges and Rovnyak [8].

On the other hand, the general complementation theory (see, e.g., [15, Ch.1]) applied to the contractive operator $\mathbf{M}_S$ provides the characterization of $\mathcal{H}(S)$ as the operator range

(2.2)
$$\mathcal{H}(S) = \mathrm{Range}(I - \mathbf{M}_S\mathbf{M}_S^*)^{\frac{1}{2}} \text{ with } \|(I - \mathbf{M}_S\mathbf{M}_S^*)^{\frac{1}{2}}h\|_{\mathcal{H}(S)} = \|(I-\pi)h\|_{\mathcal{H}_{\mathcal{H}(K^{(2)})}},$$

where π denotes the orthogonal projection onto $\mathrm{Ker}(I - \mathbf{M}_S\mathbf{M}_S^*)^{\frac{1}{2}}$.

To state the main theorem we need some preliminary results. The first one follows immediately from the characterization (2.2) upon setting $h = (I - \mathbf{M}_S\mathbf{M}_S^*)^{\frac{1}{2}}f$.

Lemma 2.1. *Let $S \in \mathcal{S}(K^{(1)}, K^{(2)})$ and $f \in \mathcal{H}(K^{(2)})$. Then $(I_{\mathcal{H}(K^{(2)})} - \mathbf{M}_S\mathbf{M}_S^*)f$ belongs to $\mathcal{H}(S)$ and*

$$\|(I - \mathbf{M}_S\mathbf{M}_S^*)f\|^2_{\mathcal{H}(S)} = \langle (I - \mathbf{M}_S\mathbf{M}_S^*)f,\ f\rangle_{\mathcal{H}(K^{(2)})}. \tag{2.3}$$

Let K be a positive $\mathbb{C}^{N\times N}$-valued kernel on Ω and let $\mathcal{H}(K)$ be the corresponding reproducing kernel Hilbert space consisting of $\mathbb{C}^N$-valued vector functions. The usage of matrix-valued functions with the columns in $\mathcal{H}(K)$ prompts us to introduce (besides the standard inner product) the following bilinear form:

$$[X,\ Y]_{\mathcal{H}(K)} = \left(\langle x_\ell,\ y_j\rangle_{\mathcal{H}(K)}\right)_{\ell=1,\dots,n}^{j=1,\dots,m} \tag{2.4}$$

which makes sense for every pair of functions

$$X(z) = [x_1(z)\dots x_n(z)] \in (\mathcal{H}(K))^{1\times n},\ Y(z) = [y_1(z)\dots y_m(z)] \in (\mathcal{H}(K))^{1\times m},$$

which are respectively, $\mathbb{C}^{N\times n}$- and $\mathbb{C}^{N\times m}$-valued.

Remark 2.2. The form (2.4) can be viewed as the matrix representation of the operator $\mathbf{M}_Y^*\mathbf{M}_X : \mathbb{C}^n \to \mathbb{C}^m$ with respect to the standard basis, where $\mathbf{M}_X : \mathbb{C}^n \to \mathcal{H}(K)$ and $\mathbf{M}_Y : \mathbb{C}^m \to \mathcal{H}(K)$ are the multiplication operators given by

$$\mathbf{M}_X c = X(z)c \quad \text{and} \quad \mathbf{M}_Y d = Y(z)d.$$

The next preliminary lemma characterizes $\mathcal{H}(K)$ in terms of positive kernels (see [9, Theorem 2.2] for scalar-valued kernels and [2, Lemma 2.2] for the matrix case):

Lemma 2.3. *A nonzero vector-valued function f defined on Ω belongs to $\mathcal{H}(K)$ and satisfies $\|f\|^2_{\mathcal{H}(K)} \le \gamma$ if and only if the kernel $K(z,w) - \gamma^{-1}f(z)f(w)^*$ is positive on Ω.*

The next theorem characterizes all the solutions S of Problem 1.1 in terms of positive kernels and in terms of the reproducing kernel Hilbert spaces $\mathcal{H}(S)$. The first develops Potapov's method (which characterizes the solutions of an interpolation problem in terms of a related fundamental matrix inequality [11]), and the second is related to reproducing kernel methods in interpolation theory [10].

Theorem 2.4. *Let $H \in (\mathcal{H}(K^{(2)}))^{1\times n}$ and $F \in (\mathcal{H}(K^{(1)}))^{1\times n}$ be as in (1.4), let S be a $p\times q$ matrix-valued function which is analytic in Ω, let K_S be defined by (1.2) and let*

$$P := [H,\ H]_{\mathcal{H}(K^{(2)})} - [F,\ F]_{\mathcal{H}(K^{(1)})} \tag{2.5}$$

and

$$B(z) = H(z) - S(z)F(z). \tag{2.6}$$

Then the following statements are equivalent:

(1) *S is a solution to Problem 1.1.*
(2) *For every choice of $x \in \mathbb{C}^n$, the function $B(z)x$ belongs to the space $\mathcal{H}(S)$ and*

$$\|Bx\|^2_{\mathcal{H}(S)} = x^*Px. \tag{2.7}$$

(3) *The following kernel is positive on Ω:*

$$\mathbf{K}(z,w) := \begin{bmatrix} P & B(w)^* \\ B(z) & K_S(z,w) \end{bmatrix} \succeq 0. \tag{2.8}$$

(4) *The following operator*

$$\mathbf{P} := \begin{bmatrix} P & \mathbf{M}_B^* \\ \mathbf{M}_B & I - \mathbf{M}_S\mathbf{M}_S^* \end{bmatrix} : \begin{bmatrix} \mathbb{C}^n \\ \mathcal{H}(K^{(2)}) \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^n \\ \mathcal{H}(K^{(1)}) \end{bmatrix} \tag{2.9}$$

is positive semidefinite.

Proof. (**1**) $\Rightarrow$ (**2**). Let S be a solution to Problem 1.1. Then $\mathbf{M}_S\mathbf{M}_S^* \le I_{\mathcal{H}(K^{(2)})}$ and (1.5) is in force. Substituting (1.5) into the right-hand side of (2.5) and (2.6) we get

$$P = [H,\ H]_{\mathcal{H}(K^{(2)})} - [\mathbf{M}_S^*H,\ \mathbf{M}_S^*H]_{\mathcal{H}(K^{(1)})} = [(I-\mathbf{M}_S\mathbf{M}_S^*)H,\ H]_{\mathcal{H}(K^{(2)})} \tag{2.10}$$

and

$$B(z) = H(z) - S(z)(\mathbf{M}_S^*H)(z) = (\{I-\mathbf{M}_S\mathbf{M}_S^*\}\,H)\,(z). \tag{2.11}$$

Since $Hx \in \mathcal{H}(K^{(2)})$ for every $x \in \mathbb{C}^n$, the last formula implies, by Lemma 2.1, that $Bx \in \mathcal{H}(S)$. Finally, by (2.3) and (2.10),

$$\|Bx\|^2_{\mathcal{H}(S)} = \|(I-\mathbf{M}_S\mathbf{M}_S^*)Hx\|^2_{\mathcal{H}(S)} = \langle (I-\mathbf{M}_S\mathbf{M}_S^*)Hx,\ Hx\rangle_{\mathcal{H}(K^{(2)})} = x^*Px.$$

(**2**) $\Rightarrow$ (**3**). By Lemma 2.3, equality (2.7) implies

$$K_S(z,w) - (x^*Px)^{-1}B(z)xx^*B(w)^* \succeq 0 \qquad (z,\, w \in \Omega)$$

for every vector $x \in \mathbb{C}^n$ such that $Px \ne 0$. The last inequality is obviously equivalent to

$$\begin{bmatrix} x^*Px & x^*B(w)^* \\ B(z)x & K_S(z,w) \end{bmatrix} \succeq 0 \qquad (z,\, w \in \Omega). \tag{2.12}$$

If $Px = 0$, then (2.7) implies $B(z)x \equiv 0$, and thus (2.12) is in force as well. Thus, (2.12) holds for every $x \in \mathbb{C}^n$, which is equivalent to (2.8).

(**3**) $\Rightarrow$ (**4**). By the reproducing kernel property,

$$(\mathbf{M}_S^*K_w^{(2)})(z) = \left[\mathbf{M}_S^*K_w^{(2)},\ K_z^{(1)}\right]_{\mathcal{H}(K^{(1)})} = \left[K_w^{(2)},\ SK_z^{(1)}\right]_{\mathcal{H}(K^{(2)})} = K_w^{(1)}(z)S(w)^*$$

and therefore,

$$\left[(I-\mathbf{M}_S\mathbf{M}_S^*)K_w^{(2)},\ K_z^{(2)}\right]_{\mathcal{H}(K^{(2)})} = K_w^{(2)}(z) - S(z)K_w^{(1)}(z)S(w)^* = K_S(z,w), \tag{2.13}$$

which shows, in particular, that the kernel K_S is positive. Fix a vector $f \in \mathbb{C}^n \oplus \mathcal{H}(K^{(2)})$ of the form

$$f = \sum_{j=1}^{r} \begin{bmatrix} c_j \\ K^{(2)}_{w^{(j)}} d_j \end{bmatrix} \qquad (c_j \in \mathbb{C}^n,\ d_j \in \mathbb{C}^p,\ w^{(j)} \in \Omega). \tag{2.14}$$

By (2.13),

$$\left\langle (I - \mathbf{M}_S \mathbf{M}_S^*) K^{(2)}_{w^{(\ell)}} d_\ell,\ K^{(2)}_{w^{(j)}} d_j \right\rangle_{\mathcal{H}(K^{(2)})} = d_j^* K_S(w^{(j)}, w^{(\ell)}) d_\ell$$

and by the reproducing kernel property,

$$\left\langle \mathbf{M}_B c_\ell,\ K^{(2)}_{w^{(j)}} d_j \right\rangle_{\mathcal{H}(K^{(2)})} = d_j^* B(w^{(j)}) c_\ell.$$

Using the two last equalities and taking into account partitionings (2.8) and (2.9) of $\mathbf{K}$ and $\mathbf{P}$ we get

$$\left\langle \mathbf{P} \begin{bmatrix} c_\ell \\ K^{(2)}_{w^{(\ell)}} d_j \end{bmatrix}, \begin{bmatrix} c_j \\ K^{(2)}_{w^{(j)}} d_j \end{bmatrix} \right\rangle_{\mathbb{C}^n \oplus \mathcal{H}(K^{(2)})} = \begin{bmatrix} c_j^* & d_j^* \end{bmatrix} \mathbf{K}(w^{(j)}, w^{(\ell)}) \begin{bmatrix} c_\ell \\ d_\ell \end{bmatrix}.$$

By linearity and in view of (2.14),

$$\langle \mathbf{P} f,\ f \rangle_{\mathbb{C}^n \oplus \mathcal{H}(K^{(2)})} = \sum_{j,\ell=1}^{r} \begin{bmatrix} c_j^* & d_j^* \end{bmatrix} \mathbf{K}(w^{(j)}, w^{(\ell)}) \begin{bmatrix} c_\ell \\ d_\ell \end{bmatrix}.$$

Since the kernel $\mathbf{K}(z, w)$ is positive on Ω, the expression on the right-hand side of the last equality is nonnegative. Thus, $\langle \mathbf{P} f,\ f \rangle_{\mathbb{C}^n \oplus \mathcal{H}(K^{(2)})} \geq 0$ for every vector f of the form (2.14). Since the set of all such vectors is dense in $\mathbb{C}^n \oplus \mathcal{H}(K^{(2)})$, $\mathbf{P}$ is positive semidefinite.

(**4**) $\Rightarrow$ (**1**). If $\mathbf{P}$ is positive semidefinite, then in particular, $\mathbf{M}_S \mathbf{M}_S^* \leq I$ and therefore, $S \in \mathcal{S}(K^{(1)}, K^{(2)})$. It remains to show that the interpolation condition (1.5) is valid. To this end let us consider the block operator

$$\widehat{\mathbf{P}} = \begin{bmatrix} I_{\mathcal{H}(K^{(1)})} & \mathbf{M}_F & \mathbf{M}_S^* \\ \mathbf{M}_F^* & \mathbf{M}_H^* \mathbf{M}_H & \mathbf{M}_H^* \\ \mathbf{M}_S & \mathbf{M}_H & I_{\mathcal{H}(K^{(2)})} \end{bmatrix} : \begin{bmatrix} \mathcal{H}(K^{(1)}) \\ \mathbb{C}^n \\ \mathcal{H}(K^{(2)}) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(K^{(1)}) \\ \mathbb{C}^n \\ \mathcal{H}(K^{(2)}) \end{bmatrix}.$$

Here we use a somewhat sloppy notation: the domain and range of a multiplication operator $\mathbf{M}_X$ depends on the X. Specifically, we have

$$\mathbf{M}_F \colon \mathbb{C}^n \to \mathcal{H}(K^{(1)}), \quad \mathbf{M}_H \colon \mathbb{C}^n \to \mathcal{H}(K^{(2)})$$

with therefore

$$\mathbf{M}_F^* \colon \mathcal{H}(K^{(1)}) \to \mathbb{C}^n, \quad \mathbf{M}_H^* \colon \mathcal{H}(K^{(2)}) \to \mathbb{C}^n,$$

but

$$\mathbf{M}_S \colon \mathcal{H}(K^{(1)}) \to \mathcal{H}(K^{(2)})$$

with therefore

$$\mathbf{M}_S^* \colon \mathcal{H}(K^{(2)}) \to \mathcal{H}(K^{(1)}).$$

Taking advantage of Remark 2.2, we obtain from (2.5) and (2.6) the representations

$$P = \mathbf{M}_H^* \mathbf{M}_H - \mathbf{M}_F^* \mathbf{M}_F, \qquad \mathbf{M}_B = \mathbf{M}_H - \mathbf{M}_S \mathbf{M}_F,$$

which allow us to conclude that the operator $\mathbf{P}$ given in (2.9) is the Schur complement of the $(1,1)$ block entry of $\widehat{\mathbf{P}}$:

$$\mathbf{P} = \begin{bmatrix} \mathbf{M}_H^*\mathbf{M}_H & \mathbf{M}_H^* \\ \mathbf{M}_H & I_{\mathcal{H}(K^{(2)})} \end{bmatrix} - \begin{bmatrix} \mathbf{M}_F^* \\ \mathbf{M}_S \end{bmatrix} \begin{bmatrix} \mathbf{M}_F & \mathbf{M}_S^* \end{bmatrix}.$$

Since $\mathbf{P}$ is positive semidefinite, it follows that $\widehat{\mathbf{P}} \geq 0$. Therefore the Schur complement of the $(3,3)$ block entry of $\widehat{\mathbf{P}}$ is positive semidefinite:

$$\begin{aligned} &\begin{bmatrix} I_{\mathcal{H}(K^{(1)})} & \mathbf{M}_F \\ \mathbf{M}_F^* & \mathbf{M}_H^*\mathbf{M}_H \end{bmatrix} - \begin{bmatrix} \mathbf{M}_S^* \\ \mathbf{M}_H^* \end{bmatrix} \begin{bmatrix} \mathbf{M}_S & \mathbf{M}_H \end{bmatrix} \\ &\quad = \begin{bmatrix} I - \mathbf{M}_S^*\mathbf{M}_S & \mathbf{M}_F - \mathbf{M}_S^*\mathbf{M}_H \\ \mathbf{M}_F^* - \mathbf{M}_H^*\mathbf{M}_S & 0 \end{bmatrix} \geq 0. \end{aligned}$$

The last relation implies that $\mathbf{M}_F - \mathbf{M}_S^*\mathbf{M}_H = 0$, which is equivalent to (1.5). □

3. Example

In this section we apply the preceding analysis to a class $\mathcal{S}^{p\times q}$ of $\mathbb{C}^{p\times q}$-valued functions S analytic in the unit ball $\mathbb{B}^d = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d : \sum_1^d |z_j|^2 < 1 \right\}$ of $\mathbb{C}^d$ and such that

$$K_S(z,w) = \frac{I_p - S(z)S(w)^*}{1 - \langle z, w\rangle} \succeq 0 \qquad (z,\, w \in \mathbb{B}^d). \tag{3.1}$$

The Nevanlinna–Pick problem for these functions (in the operator-valued version) has been recently considered in [6]. It was shown that every solution of the problem corresponds to a unitary extension of a partially defined isometric operator, which led to a parametrization of all solutions given in terms of a Redheffer linear fractional transformation. We shall pose a more general interpolation problem and, upon including it in the general scheme of Problem 1.1, shall get a different parametrization of all its solutions.

We shall use standard notations: points in $\mathbb{C}^d$ will be denoted by $z = (z_1, \ldots, z_d)$, where $z_j \in \mathbb{C}$ and $\langle z, w\rangle = \sum_{j=1}^d z_j\bar{w}_j$ will stand for the standard inner product in $\mathbb{C}^d$. For multiindices $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ we set

$$n_1 + n_2 + \ldots + n_d = |\mathbf{n}|, \qquad n_1!n_2!\ldots n_d! = \mathbf{n}!, \qquad z_1^{n_1}z_2^{n_2}\ldots z_d^{n_d} = z^{\mathbf{n}}.$$

The kernel

$$\mathbf{k}(z,w) = \frac{1}{1 - \langle z, w\rangle} \tag{3.2}$$

is positive on $\mathbb{B}^d$. It can be shown (see, e.g., [5, Lemma 3.8]) that in the metric of $\mathcal{H}(\mathbf{k})$,

$$\langle z^{\mathbf{n}}, z^{\mathbf{m}}\rangle_{\mathcal{H}(\mathbf{k})} = \begin{cases} \dfrac{\mathbf{n}!}{|\mathbf{n}|!} & \text{if } \mathbf{n} = \mathbf{m}, \\ 0 & \text{otherwise,} \end{cases}$$

which leads to the following characterization of $\mathcal{H}(\mathbf{k})$:

$$\mathcal{H}(\mathbf{k}) = \left\{ f(z) = \sum_{\mathbf{n}\in\mathbb{N}^d} f_{\mathbf{n}} z^{\mathbf{n}}, \ \text{with} \ \|f\|^2_{\mathcal{H}(\mathbf{k})} = \sum_{\mathbf{n}\in\mathbb{N}^d} \frac{\mathbf{n}!}{|\mathbf{n}|!}|f_{\mathbf{n}}|^2 < \infty \right\}. \tag{3.3}$$

The space $\mathcal{H}(\mathbf{k}I_p)$ can be viewed as the tensor product Hilbert space $\mathcal{H}(\mathbf{k})\otimes\mathbb{C}^{p\times 1}$ and we denote it $\mathcal{H}^p(\mathbf{k})$ for short. Similarly, we use the notation $\mathcal{H}^{p\times q}(\mathbf{k})$ for the space of $\mathbb{C}^{p\times q}$-valued functions with entries in $\mathcal{H}(\mathbf{k})$. Note that the bilinear form defined in (2.4) takes in this context the form

$$[H,\, F]_{\mathcal{H}(\mathbf{k})} = \sum_{\mathbf{n}\in\mathbb{N}^d} \frac{\mathbf{n}!}{|\mathbf{n}|!}\, F_{\mathbf{n}}^* H_{\mathbf{n}}, \tag{3.4}$$

and makes sense for every choice of $H\in\mathcal{H}^{p\times m}(\mathbf{k})$ and $F\in\mathcal{H}^{p\times \ell}(\mathbf{k})$.

The kernel K_S defined in (3.1) is a particular case of (1.5) corresponding to the particular choice of $K^{(1)}=\mathbf{k}I_q$ and $K^{(2)}=\mathbf{k}I_p$, and condition (3.1) means that S is a contractive multiplier from $\mathcal{H}^q(\mathbf{k})$ to $\mathcal{H}^p(\mathbf{k})$.

Let matrices $C_1\in\mathbb{C}^{p\times m}$, $C_2\in\mathbb{C}^{q\times m}$ and $A_1,\dots,A_d\in\mathbb{C}^{m\times m}$ be such that

1. The joint spectrum of $A_1,\dots,A_d$ sits inside $\mathbb{B}^d$:

$$\sigma_{\text{joint}}(A_1,\dots,A_d)\subset\mathbb{B}^d. \tag{3.5}$$

2. For every two products ("words") $W_{\mathbf{n}}(A_1,\dots,A_d)$ and $W'_{\mathbf{n}}(A_1,\dots,A_d)$ containing the same number n_j of a letter A_j (for all $j=1,\dots,d$),

$$\begin{bmatrix} C_1\\ C_2\end{bmatrix} W_{\mathbf{n}}(A_1,\dots,A_d) = \begin{bmatrix} C_1\\ C_2\end{bmatrix} W'_{\mathbf{n}}(A_1,\dots,A_d) \tag{3.6}$$

(for instance, the last condition is clearly satisfied if $A_\ell A_j = A_j A_\ell$ ($\ell, j=1,\dots,d$)). We set

$$C=\begin{bmatrix} C_1\\ C_2\end{bmatrix},\qquad \mathbf{A}=\begin{bmatrix} A_1\\ \vdots\\ A_d\end{bmatrix}\quad\text{and}\quad Z(z)=\begin{bmatrix} z_1I_m & z_2I_m & \dots & z_dI_m\end{bmatrix}, \tag{3.7}$$

and we shall use the shorthand notation

$$C\mathbf{A}^{\mathbf{n}} = CA_1^{n_1}A_2^{n_2}\cdots A_d^{n_d} \tag{3.8}$$

for a multiindex $\mathbf{n}=(n_1,n_2,\dots,n_d)$, when the order of multipliers is not essential. We assume furthermore that

3. The series $\sum_{\mathbf{n}\in\mathbb{N}^d}\frac{|\mathbf{n}|!}{\mathbf{n}!}(\mathbf{A}^{\mathbf{n}})^*C^*C\mathbf{A}^{\mathbf{n}}$ converges.

Note that the assumptions 1 and 3 provide that the function

$$G(z)=(I_m - Z(z)\mathbf{A})^{-1} \tag{3.9}$$

is analytic in $\overline{\mathbb{B}}_d$ and belongs to $\mathcal{H}^{m\times m}(\mathbf{k})$. Making use of notation (3.8) one can write

$$CG(z)=C\left(I_m-\sum_{j=1}^d z_jA_j\right)^{-1} = C\sum_{\mathbf{n}\in\mathbb{N}^d}\frac{|\mathbf{n}|!}{\mathbf{n}!}\mathbf{A}^{\mathbf{n}}z^{\mathbf{n}}. \tag{3.10}$$

The symbol $\mathbf{IP}(\mathbf{A},\, C)$ will be used to denote the following interpolation problem:

Given $\mathbf{A}$, C *and* G *as above, find necessary and sufficient conditions which insure the existence of a function* $S\in\mathcal{S}^{p\times q}$ *such that*

$$\mathbf{M}_S^*(C_1G(z)) = C_2G(z) \tag{3.11}$$

and describe the set of all such functions.

Remark 3.1. It can be easily seen that the particular choice of

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} b_1 & \dots & b_m \\ c_1 & \dots & c_m \end{bmatrix}, \qquad A_j = \begin{bmatrix} \bar{w}_1^{(j)} & & \\ & \ddots & \\ & & \bar{w}_m^{(j)} \end{bmatrix} \quad (j = 1, \dots, d)$$

in (3.11) for prescribed m points $w^{(\ell)} = (w_1^{(\ell)}, \dots, w_d^{(\ell)}) \in \mathbb{B}^d$ and vectors $b_\ell \in \mathbb{C}^p$, $c_\ell \in \mathbb{C}^q$, leads to left–sided Nevanlinna–Pick conditions $b_\ell^* S(w^{(\ell)}) = c_\ell^*$ $(\ell = 1, \dots, m)$.

The $\mathbf{IP}(\mathbf{A}, C)$ can be included in the general scheme of Problem 1.1 upon setting

$$H(z) = C_1 G(z) \in \mathcal{H}^{p \times m}(\mathbf{k}) \quad \text{and} \quad F(z) = C_2 G(z) \in \mathcal{H}^{q \times m}(\mathbf{k}). \tag{3.12}$$

Then the function B defined in (2.6) takes the form $B(z) = (C_1 - S(z)C_2)G(z)$. By Theorem 2.4 we get that a $\mathbb{C}^{p \times q}$-valued function S analytic in $\mathbb{B}^d$ is a soluttion to the $\mathbf{IP}(\mathbf{A}, C)$ if and only if

$$\mathbf{K}(z, w) := \begin{bmatrix} P & G(\omega)^*(C_1^* - C_2^* S(w)^*) \\ (C_1 - S(z)C_2)G(z) & \dfrac{I_p - S(z)S(w)^*}{1 - \langle z, w \rangle} \end{bmatrix} \succeq 0 \quad (z,\, w \in \mathbb{B}^d), \tag{3.13}$$

where

$$P := [C_2 G,\ C_2 G]_{\mathcal{H}(\mathbf{k})} - [C_1 G,\ C_1 G]_{\mathcal{H}(\mathbf{k})}. \tag{3.14}$$

Making use of the signature matrix

$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \tag{3.15}$$

we can represent P as

$$P = [JCG,\ CG]_{\mathcal{H}(\mathbf{k})}. \tag{3.16}$$

In contrast to the general case, inequality (3.13) admits a nice description of all its solutions. First we note that the Pick matrix P defined in (3.14) satisfies the generalized Stein equation

$$P - \sum_{j=1}^d A_j^* P A_j = C^* J C. \tag{3.17}$$

Indeed, due to conditions (3.6) one can solve (3.17) iteratively to get a (unique) solution P of (3.17) in the form of a uniformly converging series

$$P = \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} (\mathbf{A}^{\mathbf{n}})^* C^* J C \mathbf{A}^{\mathbf{n}}.$$

On the other hand, substituting the Taylor expansion (3.10) for G into the left-hand side in (3.16) and making use of (3.4) we come to the same expression for P:

$$\begin{aligned} [JCG,\ CG]_{\mathcal{H}(\mathbf{k})} &= \left[JC \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} \mathbf{A}^{\mathbf{n}} z^{\mathbf{n}},\ C \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} \mathbf{A}^{\mathbf{n}} z^{\mathbf{n}} \right]_{\mathcal{H}(\mathbf{k})} \\ &= \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} (\mathbf{A}^{\mathbf{n}})^* C^* J C \mathbf{A}^{\mathbf{n}}. \end{aligned}$$

Let us assume that P is positive definite. Then the $\mathbb{C}^{(p+q)\times(md+p+q)}$–valued function

$$\begin{aligned}\Theta(z) &= \begin{bmatrix}0 & I_{p+q}\end{bmatrix} \\ &\quad + CG(z)P^{-1}\begin{bmatrix}(z_1I_m - A_1^*)P^{\frac{1}{2}} & \dots & (z_dI_m - A_d^*)P^{\frac{1}{2}} & -C^*J\end{bmatrix}\end{aligned} \tag{3.18}$$

is analytic in $\mathbb{B}^d$ and satisfies

$$\frac{J - \Theta(z)\mathbf{J}\Theta(w)^*}{1 - \langle z,\, w\rangle} = CG(z)P^{-1}G(w)^*C^*, \quad \text{where} \quad \mathbf{J} = \begin{bmatrix} I_{md} & 0 \\ 0 & J\end{bmatrix}, \tag{3.19}$$

for every choice of $z = (z_1, \dots, z_d)$ and $w = (w_1, \dots, w_d)$ in $\mathbb{B}_d$. Indeed, it follows readily from (3.19) that

$$J - \Theta(z)\mathbf{J}\Theta(w)^* = CG(z)P^{-1}T(z,w)P^{-1}G(w)^*C, \tag{3.20}$$

where

$$\begin{aligned}T(z,w) = &\left(I - \sum_{j=1}^{d}\bar{w}_jA_j^*\right)P + P\left(I - \sum_{j=1}^{d}z_jA_j\right) \\ &- \sum_{j=1}^{d}(z_jI - A_j^*)P(\bar{w}_jI - A_j) - C^*JC.\end{aligned}$$

Making use of (3.17), we get

$$T(z,w) = 2P - \sum_{j=1}^{d}\left(z_j\bar{w}_jP + A_j^*PA_j\right) - C^*JC = (1 - \langle z,\, w\rangle)P,$$

which together with (3.20) imply (3.19).

Still assuming that P is positive definite, we conclude that (3.13) is equivalent to

$$\frac{I_p - S(z)S(w)^*}{1 - \langle z,\, w\rangle} - (C_1 - S(z)C_2)G(z)P^{-1}G(\omega)^*(C_1^* - C_2^*S(w)^*) \succeq 0 \quad (z,\, w \in \mathbb{B}^d),$$

which in its turn, can be written as

$$\begin{bmatrix}I_p & -S(z)\end{bmatrix}\left\{\frac{J}{1 - \langle z,\, w\rangle} - CG(z)P^{-1}G(w)^*C\right\}\begin{bmatrix} I_p \\ -S(w)^*\end{bmatrix} \succeq 0 \qquad (z,\, w \in \mathbb{B}^d).$$

Taking advantage of (3.19), we rewrite the last inequality as

$$\begin{bmatrix}I_p & -S(z)\end{bmatrix}\frac{\Theta(z)\mathbf{J}\Theta(w)^*}{1 - \langle z,\, w\rangle}\begin{bmatrix} I_p \\ -S(w)^*\end{bmatrix} \succeq 0 \qquad (z,\, w \in \mathbb{B}^d). \tag{3.21}$$

Theorem 3.2. *Let P be positive definite and let*

$$\Theta = \begin{bmatrix}\Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22}\end{bmatrix} : \begin{bmatrix}\mathbb{C}^{md+p} \\ \mathbb{C}^q\end{bmatrix} \to \begin{bmatrix}\mathbb{C}^p \\ \mathbb{C}^q\end{bmatrix} \tag{3.22}$$

be the partition of the function Θ given by (3.18) into four blocks of the indicated sizes. Then the set of all solutions S of the $\mathbf{IP}(\mathbf{A},\, C)$ *are parametrized by the linear fractional transformation*

$$S(z) = \left(\Theta_{11}(z)\mathcal{E}(z) + \Theta_{12}(z)\right)\left(\Theta_{21}(z)\mathcal{E}(z) + \Theta_{22}(z)\right)^{-1}, \tag{3.23}$$

when the parameter $\mathcal{E}$ varies on the set $\mathcal{S}^{(md+p)\times q}$.

Proof. It follows from (3.19) and (3.22) that

$$-I_q - \Theta_{21}(z)\Theta_{21}(z)^* + \Theta_{22}(z)\Theta_{22}(z)^* \geq 0 \qquad (z \in \mathbb{B}^d).$$

Therefore, $\Theta_{22}(z)$ is invertible at every point $z \in \mathbb{B}^d$ and $\|\Theta_{22}(z)^{-1}\Theta_{21}(z)\| < 1$. Then the function

$$\Theta_{21}(z)\mathcal{E}(z) + \Theta_{22}(z) = \Theta_{22}(z)(\Theta_{22}(z)^{-1}\Theta_{21}(z)\mathcal{E}(z) + I_q)$$

is invertible in $\mathbb{B}^d$ for every $\mathcal{E} \in \mathcal{S}^{(md+p)\times q}$, which means that the transformation (3.2) is well defined on the set $\mathcal{S}^{(md+p)\times q}$.

According to the preceding analysis S is a solution of the $\mathbf{IP}(\mathbf{A},\, C)$ if and only if it satisfies the inequality (3.21). Setting

$$\begin{bmatrix} u(z) & -v(z) \end{bmatrix} = \begin{bmatrix} I_p & -S(z) \end{bmatrix} \Theta(z) \tag{3.24}$$

where u and v are respectively, $\mathbb{C}^{p\times(md+p)}$- and $\mathbb{C}^{p\times q}$-valued, one can rewrite (3.21) as

$$\frac{u(z)u(w)^* - v(z)v(w)^*}{1 - \langle z,\, w\rangle} \succeq 0 \qquad (z,\, w \in \mathbb{B}^d),$$

which is equivalent (see, e.g., [3, Theorem 3.1]) to the existence of a function $\mathcal{E} \in \mathcal{S}^{(md+p)\times q}$ such that

$$u(z)\mathcal{E}(z) = v(z) \qquad (z \in \mathbb{B}^d).$$

By (3.24), we conclude that S is a solution of the $\mathbf{IP}(\mathbf{A},\, C)$ if and only if

$$\begin{bmatrix} I_p & -S(z) \end{bmatrix} \Theta(z) = u(z) \begin{bmatrix} I_{md+p} & -\mathcal{E}(z) \end{bmatrix}$$

for some bounded analytic function u and a function $\mathcal{E} \in \mathcal{S}^{(md+p)\times q}$. The latter is equivalent to

$$\begin{bmatrix} I_p & -S(z) \end{bmatrix} \Theta(z) \begin{bmatrix} \mathcal{E}(z) \\ I_q \end{bmatrix} = 0,$$

which, being rewritten as

$$\Theta_{11}(z)\mathcal{E}(z) + \Theta_{12}(z) - S(z)\left(\Theta_{21}(z)\mathcal{E}(z) + \Theta_{22}(z)\right) = 0,$$

is evidently equivalent to (3.23). □

As a consequence of the last theorem we get that under the assumption $P > 0$, the $\mathbf{IP}(\mathbf{A},\, C)$ has infinitely many solutions. Using the standard approximation argument it can be easily shown that if P is positive semidefinite, then there exists a solution to the $\mathbf{IP}(\mathbf{A},\, C)$ (the questions about paremetrization of all solutions and uniqueness criteria are more delicate and will be considered elsewhere). Therefore, the condition $P \geq 0$ is necessary and sufficient for the $\mathbf{IP}(\mathbf{A},\, C)$ to be solvable (the necessity of this condition follows readily from (3.13)). This conclusion is not surprising in light of recent papers [1], [14], [12], [13], [6] where this result has been established (for a wide class of reproducing kernel Hilbert spaces) for the Nevanlinna–Pick problem.

I am indebted to the referee whose suggestions led to a substantial simplification of the proof of Theorem 2.4.

References

1. J. Agler and J. E. McCarthy, *Complete Nevanlinna-Pick kernels*, J. Funct. Anal., **175** (2000), 111–124. MR **2001h:**47019
2. D. Alpay and V. Bolotnikov, *On tangential interpolation in reproducing kernel Hilbert modules and applications*, in: *Topics in Interpolation Theory* (H. Dym, B. Fritzsche, V. Katsnelson and B. Kirstein, eds.), Oper. Theory Adv. Appl., **OT95**, Birkhäuser Verlag, Basel, 1997, pp. 37–68. MR **99b:**30055
3. D. Alpay, V. Bolotnikov and H. T. Kaptanoğlu, *The Schur algorithm and reproducing kernel Hilbert spaces in the ball*, Linear Algebra Appl., **342** (2002), 163–186. MR **2002m:**47019
4. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc., **68** (1950), 337–404. MR **14:**479c
5. W. Arveson, *Subalgebras of C^*-algebras. III. Multivariable operator theory*, Acta Math. **181** (1998), no. 2, 159–228. MR **2000e:**47013
6. J. A. Ball, T. T. Trent and V. Vinnikov, *Interpolation and commutant lifting for multipliers on reproducing kernels Hilbert spaces*, Oper. Theory Adv. Appl. **122** (2001), 89–138. MR **2002f:**47028
7. V. Bolotnikov and H. Dym, *On boundary interpolation for matrix Schur functions*, Preprint, 1999.
8. L. de Branges and J. Rovnyak, *Square summable power series*, Holt, Rinehart and Winston, New York, 1966. MR **35:**5909
9. F. Beatrous and J. Burbea, *Positive-definiteness and its applications to interpolation problems for holomorphic functions*, Trans. Amer. Math. Soc., **284** (1984), no.1, 247–270. MR **85e:**32020
10. H. Dym, *J contractive matrix functions, reproducing kernel spaces and interpolation*, CBMS Lecture Notes, vol. 71, Amer. Math. Soc., Rhode Island, 1989. MR **90g:**47003
11. I. V. Kovalishina and V. P. Potapov, *Seven Papers Translated from the Russian*, Amer. Math. Soc. Transl. (2), **138**, Providence, R.I., 1988. MR **89f:**00030
12. S. McCullough, *The local de Branges-Rovnyak construction and complete Nevanlinna-Pick kernels*, in *Algebraic methods in operator theory* (Ed. R. Curto and P. E. T. Jorgensen), Birkhäuser–Verlag, Boston, 1994, pp. 15–24. MR **95j:**47016
13. G. Popescu, *Interpolation problems in several variables*, J. Math. Anal. Appl., **227** (1998), 227–250. MR **99i:**47028
14. P. Quiggin, *For which reproducing kernel Hilbert spaces is Pick's theorem true?*, Integral Equations Operator Theory **16** (1993), no. 2, 244–266. MR **94a:**47026
15. D. Sarason, *Sub-Hardy Hilbert spaces in the unit disk*, John Wiley and Sons Inc., New York, 1994. MR **96k:**46039

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