Integr. equ. oper. theory 99 (9999) 1–33
0378-620X/990001-33 $ 1.50+0.20/0

Integral Equations and Operator Theory

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On the Carathéodory–Fejér interpolation problem for generalized Schur functions

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Abstract. The solutions of the Carathéodory–Fejér interpolation problem for generalized Schur functions can be parametrized via a linear fractional transformation over the class of classical Schur functions. The linear fractional transformation of some of these functions may have a pole (simple or multiple) in one or more of the interpolation points or not satisfy one or more interpolation conditions, hence not all Schur functions can serve as a parameter. The set of excluded parameters is characterized in terms of the related Pick matrix.

Mathematics Subject Classification (2000). Primary 41A05; Secondary 32A35.

Keywords. Carathéodory–Fejér interpolation, generalized Schur functions.

1. Introduction

The objective of this paper is to study some aspects of the Carathéodory–Fejér interpolation problem for generalized Schur functions.

Definition 1.1. A function S belongs to the generalized Schur class $\mathcal{S}_\kappa$ if it is meromorphic on the unit disk $\mathbb{D}$ and the kernel

$$K_S(z,w) := \frac{1 - S(z)\overline{S(w)}}{1 - z\bar{w}} \tag{1.1}$$

has κ negative squares on $\mathbb{D} \cap \rho(S)$ ($\rho(S)$ stands for the domain of analyticity of S); in formulas: $\text{sq}_-(K_S) = \kappa$.

The last equality means that for every choice of an integer r and of r points $z_1, \ldots, z_r \in \mathbb{D} \cap \rho(S)$, the Hermitian matrix $[K_S(z_j, z_i)]_{i,j=1}^r$ has at most κ and for at least one such choice it has exactly κ negative eigenvalues counted with multiplicities.

The class $\mathcal{S}_0$ is the classical Schur class consisting of functions S such that the kernel in (1.1) is positive (that is, has no negative squares). This turns to be equivalent to the property of S to be analytic and less than one in modulus on $\mathbb{D}$.

The classes $\mathcal{S}_\kappa$ appeared implicitly in [23] in connection with interpolation problems (see discussion in [6, Chapter 19]), and were comprehensively studied by

[1]Research was supported by the Summer Research Grant from the College of William and Mary

Krein and Langer [20], [21]. It was shown in particular, that a function S belongs to the class $\mathcal{S}_\kappa$ if and only if it admits a representation of the form

$$S(z) = \frac{S_0(z)}{B(z)}, \tag{1.2}$$

for some Schur function $S_0 \in \mathcal{S}_0$ and a Blaschke product B of degree κ, having disjoint zeroes in $\mathbb{D}$ (quite different proofs of this factorization can be found in [18], [16], [10]). This representation in turn, leads to a characterization of $\mathcal{S}_\kappa$ as the class of all functions S such that

1. S is meromorphic in $\mathbb{D}$ and has κ poles inside $\mathbb{D}$ counted with multiplicities.
2. S is bounded on an annulus $\{z : \rho < |z| < 1\}$ for some $\rho \in (0, 1)$.
3. Boundary nontangential limits $S(t) := \lim_{z\to t} S(z)$ exist and satisfy $|S(t)| \leq 1$ for almost all $t \in \mathbb{T}$.

Various interpolation problems for generalized Schur functions (as well as for their matrix and operator valued analogues) were considered in [4], [5], [7], [22], [18], [6], [12], [13], [2], [3]. In the present paper we focus on some aspects of the Carathéodory–Fejér interpolation problem which will be denoted by $\mathbf{CF}_\kappa$ and which consists of the following:

$\mathbf{CF}_\kappa$: *Given k distinct points $z_1, \ldots, z_k \in \mathbb{D}$, equally many nonnegative integers $n_1, \ldots, n_k$ and $N := \sum_{i=1}^k n_i$ complex numbers $S_{i,j}$ $(0 \leq j \leq n_i - 1;\ 1 \leq i \leq k)$, find all functions $S \in \mathcal{S}_\kappa$ which are analytic at z_i and satisfy*

$$S^{(j)}(z_i) = j!\, S_{i,j} \qquad (i = 1, \ldots, k;\ j = 0, \ldots, n_i - 1). \tag{1.3}$$

In other words, it is required to find all functions $S \in \mathcal{S}_\kappa$ with prescribed Taylor expansions

$$S(z) = S_{i,0} + (z - z_i)S_{i,1} + \ldots + (z - z_i)^{n_i - 1} S_{i,n_i-1} + \ldots$$

at z_i for $i = 1, \ldots, k$. Throughout the paper $J_n(a)$ denotes the $n \times n$ Jordan block with the number a on the main diagonal and L_n stands for the row vector of the length n with the first coordinate equals one and other coordinates equal zero:

$$J_n(a) = \begin{bmatrix} a & 1 & 0 & \ldots & 0 \\ 0 & a & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & & & a & 1 \\ 0 & & \ldots & 0 & a \end{bmatrix}, \quad L_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Associated with interpolation data are matrices

$$T = \begin{bmatrix} J_{n_1}(z_1) & & \\ & \ddots & \\ & & J_{n_k}(z_k) \end{bmatrix}, \quad E = \begin{bmatrix} L_{n_1} & \cdots & L_{n_k} \end{bmatrix} \tag{1.4}$$

and

$$C = \begin{bmatrix} C_1 & \cdots & C_k \end{bmatrix}, \quad \text{where} \quad C_i = \begin{bmatrix} S_{i,0} & S_{i,1} & \cdots & S_{i,n_i-1} \end{bmatrix}, \tag{1.5}$$

which contain all the data and in turn, can be considered as interpolation data. Since $|z_i| < 1$, the series

$$P = \sum_{j=0}^{\infty} (T^*)^k (E^*E - C^*C) T^k \tag{1.6}$$

converges and defines a unique solution P of that the Stein equation

$$P - T^*PT = E^*E - C^*C, \tag{1.7}$$

which is referred to as the *Pick matrix* of the $\mathbf{CF}_\kappa$ problem.

The $\mathbf{CF}_\kappa$ was studied in [6, Chapter 19] in a more general bitangential matrix setting. We recall some needed results from [6] adopted to the present situation. It was shown that if a meromorphic function S meets interpolation conditions (1.3), then

$$\text{sq}_-(K_S) \geq \text{sq}_-(P) \tag{1.8}$$

which therefore, is a necessary condition for the $\mathbf{CF}_\kappa$ problem to have a solution. On the other hand, if

$$\text{sq}_-(P) = \kappa \quad \text{and} \quad \det P \neq 0, \tag{1.9}$$

then the $\mathbf{CF}_\kappa$ problem has infinitely many solutions, which are parametrized by a linear fractional transformation. Let $\widehat{\Theta}(z)$ be the $\mathbb{C}^{2\times 2}$-valued function defined by

$$\begin{aligned} \widehat{\Theta}(z) &= \begin{bmatrix} \widehat{\Theta}_{11}(z) & \widehat{\Theta}_{12}(z) \\ \widehat{\Theta}_{21}(z) & \widehat{\Theta}_{22}(z) \end{bmatrix} \\ &= I_2 + (1-z) \begin{bmatrix} C \\ E \end{bmatrix} (zI - T)^{-1} P^{-1} (I - T^*)^{-1} \begin{bmatrix} -C^* & E^* \end{bmatrix} \end{aligned} \tag{1.10}$$

and let

$$\Theta(z) := \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix} = \left(\prod_{i=1}^{k} (z - z_i)^{n_i} \right) \widehat{\Theta}(z). \tag{1.11}$$

The following theorem (for the proof see [6, Section 19.3]) is our starting point.

Theorem 1.2. *Let the Pick matrix P of the $\mathbf{CF}_\kappa$ problem meet conditions* (1.9) *and let Θ be the function given in* (1.11)*. Then all solutions S of the $\mathbf{CF}_\kappa$ problem are parametrized by the linear fractional transformation*

$$S(z) = \mathbf{T}_\Theta[\mathcal{E}] := \frac{\Theta_{11}(z)\mathcal{E}(z) + \Theta_{12}(z)}{\Theta_{21}(z)\mathcal{E}(z) + \Theta_{22}(z)}, \tag{1.12}$$

where the parameter $\mathcal{E}$ runs through the set of those functions in $\mathcal{S}_0$ which satisfy

$$\Theta_{21}(z_i)\mathcal{E}(z_i) + \Theta_{22}(z_i) \neq 0 \quad (i = 1, \ldots, k). \tag{1.13}$$

Theorem 1.2 gives a complete description of all solutions of the $\mathbf{CF}_\kappa$ problem. However, an important issue (below we point out some reasons for that) is to characterize the range of the linear fractional transformation (1.12) with parameter $\mathcal{E}$ varying on the whole Schur class $\mathcal{S}_0$.

This question was considered in [19] in connection with the problem of finding the Poincaré distance of a rational L^∞ function to S_0 (the latter problem amounts to finding of a smallest in L^∞ norm function in the range of a linear fractional transformation of the form (1.13)).

In [7] the range of (1.13) was characterized in the model-matching form as

$$\text{Range}\,(\mathbf{T}_\Theta) = \{T_1 + T_2 Q : \ Q \in H^\infty_\kappa\} \cap \mathcal{B}L^\infty, \tag{1.14}$$

where T_1 is any function analytic on the closed unit disk $\overline{\mathbb{D}}$ and satisfying interpolation conditions (1.3) and

$$T_2(z) = \prod_{i=1}^{k} \left(\frac{z - z_i}{1 - z\bar{z}_i}\right)^{n_i}. \tag{1.15}$$

Furthermore, $\mathcal{B}L^\infty$ stands for the closed unit ball of L^∞ and H^∞_κ is the set of all fractions $\frac{\widetilde{Q}}{B}$ with $\widetilde{Q} \in H^\infty$ and B a Blaschke product of degree κ. The latter characterization shows in fact that the transformation (1.13) with the Schur-class free parameters describes all solutions of the following truncated Takagi–Sarason problem.

$\mathbf{TS}_\kappa$: *Given a function T_1 analytic on $\overline{\mathbb{D}}$ and the Blaschke product T_2 of the form* (1.15), *find all functions $S \in \{T_1 + T_2 H^\infty_\kappa\} \cap \mathcal{B}L^\infty$.*

Interpolation data for S is now determined by values of T_1 (and its derivatives of appropriate orders) at the zeroes z_i's of T_2. If the Pick matrix P defined by the formula (1.6) from the interpolation data is invertible and has κ negative eigenvalues, then all solutions of the $\mathbf{TS}_\kappa$ problem are parametrized by (1.12) with the parameter $\mathcal{E}$ varying over $\mathcal{S}_0$ (see [6, Chapter 19] for more details).

Thus, every solution S of the $\mathbf{CF}_\kappa$ problem is a solution of the $\mathbf{TS}_\kappa$ problem, but not conversely. In the contrast to the $\mathbf{CF}_\kappa$ problem, solutions of the $\mathbf{TS}_\kappa$ problem are allowed to have poles at z_i's and thus, to miss some of interpolation conditions. All solutions S of the $\mathbf{TS}_\kappa$ problem that do not solve the $\mathbf{CF}_\kappa$ problem are obtained via (1.12) from parameters $\mathcal{E}$ that satisfy not all of the conditions (1.13). Such functions seem to be of certain interest. This interest becomes greater if one considers an interpolation problem for pseudomultipliers of the Hardy space H^2, that is, for functions S that are not necessarily meromorphic on $\mathbb{D}$, but with the associated kernel $K_S(z,w)$ still having κ negative squares (see e.g., [1], [8], [9]).

A Schur function $\mathcal{E}$ is called an *excluded parameter* of the transformation (1.12) if it does not satisfy at least one condition in (1.13). The notion of an excluded parameter was introduced in [18] in the context of the Nevanlinna–Pick problem (with no derivatives involved in the interpolation conditions). In [15, Chapter 5], excluded parameters were studied in the context of the Nevanlinna–Pick problem for generalized Nevanlinna functions. It was shown that excluded parameters can be classified in terms of diagonal entries of the inverse P^{-1} of the Pick matrix P of the problem.

The purpose of the present paper is to give a more detailed analysis of excluded parameters and of functions S obtained from such parameters via (1.12). Mostly we shall be interested in two questions: how many negative squares S may lose and how many and which interpolation conditions it may not satisfy. It turns out that the answers depend not only on at how many interpolating points the denominator in (1.12) takes zero values, but also on the multiplicity of this zero. We come up with the following

Definition 1.3. A function $\mathcal{E} \in \mathcal{S}_0$ is said to be an excluded parameter of multiorder $\mathbf{m} = (m_1, \dots, m_k)$ of the transformation (1.12) if the function $\Theta_{21}\mathcal{E} + \Theta_{22}$ has zeroes of multiplicities m_i at z_i for $i = 1, \dots, k$.

According to this definition, a nonexcluded parameter can be considered as an excluded parameter of multiorder zero.

Thus, each excluded parameter $\mathcal{E}$ of multiorder $\mathbf{m} = (m_1, \dots, m_k)$ is characterized by equalities

$$(\Theta_{21}\mathcal{E} + \Theta_{22})^{(j)}(z_i) = 0 \quad (i = 1, \dots, k;\ j = 0, \dots, m_i - 1) \tag{1.16}$$

and inequalities

$$(\Theta_{21}\mathcal{E} + \Theta_{22})^{(m_i)}(z_i) \neq 0 \quad (i = 1, \dots, k). \tag{1.17}$$

We shall use the standard notation

$$|\mathbf{m}| = m_1 + \ldots + m_k$$

and the following partial order $\succ$ on the set of multiorders: we shall say that

$$\widetilde{\mathbf{m}} = (\widetilde{m}_1, \dots, \widetilde{m}_k) \succ \mathbf{m} = (m_1, \dots, m_k)$$

if $\widetilde{m}_i \geq m_i$ for all $i = 1, \dots, k$ and $|\widetilde{\mathbf{m}}| > |\mathbf{m}|$. We also shall write $\widetilde{\mathbf{m}} \succeq \mathbf{m}$ if $\widetilde{\mathbf{m}} \succ \mathbf{m}$ or $\widetilde{\mathbf{m}} = \mathbf{m}$. We also fix the multiinteger

$$\mathbf{n} = (n_1, \dots, n_k), \quad |\mathbf{n}| = N.$$

With every multiindex $\mathbf{m} = (m_1, \dots, m_k)$ we associate the sets

$$\mathcal{Z}_{\mathbf{m}}^- = \{i \in \{1, \dots, k\} : \ m_i \leq n_i\}, \qquad \mathcal{Z}_{\mathbf{m}}^+ = \{i \in \{1, \dots, k\} : \ m_i > n_i\}, \tag{1.18}$$
$$\mathcal{Z}_{\mathbf{m}}^0 = \{i \in \{1, \dots, k\} : \ m_i = 0\} \subseteq \mathcal{Z}_{\mathbf{m}}^-,$$

and the positive integer

$$\gamma_{\mathbf{m}} := \sum_{i \in \mathcal{Z}_{\mathbf{m}}^-} m_i + \sum_{i \in \mathcal{Z}_{\mathbf{m}}^+} n_i = \sum_{i=1}^k \min\{m_i,\ n_i\}. \tag{1.19}$$

Now we can state the main result of the paper.

Theorem 1.4. *If $\mathcal{E}$ is an excluded parameter of multiorder $\mathbf{m} = (m_1, \dots, m_k)$, then the function $S = \mathbf{T}_\Theta[\mathcal{E}]$ belongs to the class $\mathcal{S}_{\kappa - \gamma_{\mathbf{m}}}$, where $\gamma_{\mathbf{m}}$ is given in* (1.19). *Furthermore, S has poles of multiplicities $m_i - n_i$ at z_i ($i \in \mathcal{Z}^+_{\mathbf{m}}$) and satisfies interpolation conditions*

$$S^{(j)}(z_i) = j!\, S_{i,j} \quad (j = 0, \dots, n_i - m_i - 1) \quad (i \in \mathcal{Z}^-_{\mathbf{m}}) \tag{1.20}$$

and constraints

$$S^{(n_i - m_i)}(z_i) \neq (n_i - m_i)!\, S_{i, n_i - m_i} \quad (i \in \mathcal{Z}^-_{\mathbf{m}} \setminus \mathcal{Z}^0_{\mathbf{m}}). \tag{1.21}$$

Remark 1.5. Note that for $\mathbf{m} = (0, \dots, 0) = \mathbf{0}$, the last theorem reduces to a part of Theorem 1.2.

Indeed, it follows by definitions (1.18), (1.19) that $\mathcal{Z}^-_{\mathbf{0}} = \{1, \dots, k\}$, $\mathcal{Z}^+_{\mathbf{0}} = \emptyset$, $\gamma_{\mathbf{0}} = 0$ and conditions (1.20) reduce to conditions (1.3). Thus, for $\mathbf{m} = \mathbf{0}$, Theorem 1.4 reads: for every nonexcluded parameter (i.e., for every excluded parameter of zero multiorder) $\mathcal{E}$, the function $S = \mathbf{T}_\Theta[\mathcal{E}]$ belongs to the class $\mathcal{S}_{\kappa - 0} = \mathcal{S}_\kappa$ and satisfies conditions (1.3).

Note also that Theorem 1.4 doesn't say anything about interpolation conditions (1.3) for $j = n_i - m_i + 1, \dots, n_i - 1$ (in the relevant case, when $1 < m_i \leq n_i$); these conditions may or may not be satisfied.

Following [15], we shall also characterize excluded parameters in terms of certain principal submatrices of the matrix P^{-1}, the inverse of the Pick matrix of the $\mathbf{CF}_\kappa$ problem. To formulate the result we first introduce several more objects associated with a multiinteger $\mathbf{m} = (m_1, \dots, m_k)$. Let

$$\begin{aligned} N_i &= \begin{cases} \begin{bmatrix} 0_{m_i \times (n_i - m_i)} & I_{m_i} \end{bmatrix}, & i \in \mathcal{Z}^-_{\mathbf{m}}, \\ I_{n_i}, & i \in \mathcal{Z}^+_{\mathbf{m}}, \end{cases} \\ M_i &= \begin{cases} \begin{bmatrix} I_{n_i - m_i} & 0_{(n_i - m_i) \times m_i} \end{bmatrix}, & i \in \mathcal{Z}^-_{\mathbf{m}}, \\ \text{0-dimensional matrix}, & i \in \mathcal{Z}^+_{\mathbf{m}}, \end{cases} \end{aligned} \tag{1.22}$$

so that

$$\begin{bmatrix} M_i \\ N_i \end{bmatrix} = I_{n_i} \qquad (i = 1, \dots, k) \tag{1.23}$$

and let

$$\widetilde{P}_{\mathbf{m}} = N_{\mathbf{m}} P^{-1} N^*_{\mathbf{m}} \quad \text{and} \quad P_{\mathbf{m}} = M_{\mathbf{m}} P M^*_{\mathbf{m}}, \tag{1.24}$$

where

$$N_{\mathbf{m}} = \begin{bmatrix} N_1 & & \\ & \ddots & \\ & & N_k \end{bmatrix}, \quad \text{and} \quad M_{\mathbf{m}} = \begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_k \end{bmatrix}. \tag{1.25}$$

It is easily seen that $\widetilde{P}_{\mathbf{m}}$ and $P_{\mathbf{m}}$ are $\gamma_{\mathbf{m}} \times \gamma_{\mathbf{m}}$ and $(N - \gamma_{\mathbf{m}}) \times (N - \gamma_{\mathbf{m}})$, respectively, where $\gamma_{\mathbf{m}}$ is the integer given by (1.19).

Theorem 1.6. *There exists an excluded parameter of multiorder at least* $\mathbf{m} \preceq \mathbf{n}$ *if and only if the matrix* $\widetilde{P}_{\mathbf{m}}$ *defined in* (1.24) *is negative semidefinite. Moreover, if* $\widetilde{P}_{\mathbf{m}}$ *is negative definite, then there are infinitely many excluded parameters of multiorder* $\mathbf{m}$*. If* $\widetilde{P}_{\mathbf{m}}$ *is negative semidefinite (singular), then there is only one excluded parameter of multiorder at least* $\mathbf{m}$*, which is a Blaschke product of degree* $r = \operatorname{rank}(\widetilde{P}_{\mathbf{m}})$*.*

In Section 5 we compare this result with classification of excluded parameters obtained in [15].

Note also that the matrix $P_{\mathbf{m}}$ (which is, by definition (1.24), a principle submatrix of P) is the Pick matrix of the reduced **CF** problem with interpolation conditions (1.20), and Theorem 1.4 claims that the function $S = \mathbf{T}_{\Theta}[\mathcal{E}]$ is a solution of this interpolation problem. It will be shown (see Corollary 2.5) that if P is invertible and $\widetilde{P}_{\mathbf{m}}$ is negative semidefinite, then

$$\operatorname{sq}_-(P_{\mathbf{m}}) = \operatorname{sq}_-(P) - \operatorname{rank}\widetilde{P}_{\mathbf{m}} = \kappa - \gamma_{\mathbf{m}}$$

and thus, by (1.8), $S \in \mathcal{S}_{\widetilde{\kappa}}$ with $\widetilde{\kappa} \geq \kappa - \gamma_{\mathbf{m}}$. However, Theorem 1.4 guarantees that actually, $\widetilde{\kappa} = \kappa - \gamma_{\mathbf{m}}$.

The paper is organized as follows. Section 2 contains some needed auxiliary results which can be found (probably in a different form) in many sources and are included for the sake of completeness. It will be shown in Section 4 that excluded parameters can be characterized as solutions of certain Carathéodory–Fejér interpolation problem $\mathbf{CF_0}$ for Schur functions. All the needed facts on this problem are recalled in Section 3. Proofs of Theorems 1.5 and 1.6 are presented in Section 4. Two particular cases of the $\mathbf{CF}_\kappa$ problem (the Nevanlinna–Pick problem and one point interpolation problem) are considered in Sections 5 and 6. The obtained results are illustrated in Section 7 by two numerical examples.

2. Some auxiliary results

In this section we present some auxiliary results needed in the sequel.

Lemma 2.1. *Let* T*,* E *and* C *be given by* (1.4) *and* (1.5)*. Then the row vectors* $\widetilde{E}$ *and* $\widetilde{C}$ *defined by*

$$\begin{bmatrix} \widetilde{C} \\ \widetilde{E} \end{bmatrix} = \begin{bmatrix} C \\ E \end{bmatrix} (I - T)^{-1} P^{-1} (I - T^*) \tag{2.1}$$

satisfy the Stein identity

$$P^{-1} - TP^{-1}T^* = \widetilde{E}^*\widetilde{E} - \widetilde{C}^*\widetilde{C}. \tag{2.2}$$

Furthermore, the function $\widehat{\Theta}(z)$ *defined in* (1.10) *admits a representation*

$$\widehat{\Theta}(z) = \Upsilon + \begin{bmatrix} C \\ E \end{bmatrix} (zI - T)^{-1} \begin{bmatrix} -\widetilde{C}^* & \widetilde{E}^* \end{bmatrix} \tag{2.3}$$

where

$$\Upsilon = I_2 - \begin{bmatrix} C \\ E \end{bmatrix} (I-T)^{-1} \begin{bmatrix} -\widetilde{C}^* & \widetilde{E}^* \end{bmatrix}, \tag{2.4}$$

and the equality

$$\widehat{\Theta}(z) \begin{bmatrix} \widetilde{C} \\ \widetilde{E} \end{bmatrix} = \begin{bmatrix} C \\ E \end{bmatrix} (zI-T)^{-1} P^{-1} (I - zT^*) \tag{2.5}$$

holds at every point $z \notin \{z_1, \dots, z_k\}$.

Proof: Under the assumption that P is invertible, identity (2.2) turns to be equivalent to (1.7). Indeed, by (2.1) and (1.7),

$$\begin{aligned}
\widetilde{E}^*\widetilde{E} - \widetilde{C}^*\widetilde{C} &= (I-T)P^{-1}(I-T^*)^{-1}\left[E^*E - C^*C\right](I-T)^{-1}P^{-1}(I-T^*) \\
&= (I-T)P^{-1}(I-T^*)^{-1}\left[P - T^*PT\right](I-T)^{-1}P^{-1}(I-T^*) \\
&= (I-T)P^{-1}\left[(I-T^*)^{-1}P + PT(I-T)^{-1}\right]P^{-1}(I-T^*) \\
&= (I-T)P^{-1} + TP^{-1}(I-T^*) \\
&= P^{-1} - TP^{-1}T^*.
\end{aligned}$$

By (1.10),

$$\begin{aligned}
\widehat{\Theta}(z) &= I_2 + (1-z)\begin{bmatrix} C \\ E \end{bmatrix}(zI-T)^{-1}(I-T)^{-1}\begin{bmatrix} -\widetilde{C}^* & \widetilde{E}^* \end{bmatrix} \\
&= I_2 + \begin{bmatrix} C \\ E \end{bmatrix}\left((zI-T)^{-1} - (I-T)^{-1}\right)\begin{bmatrix} -\widetilde{C}^* & \widetilde{E}^* \end{bmatrix},
\end{aligned} \tag{2.6}$$

which clearly is equivalent to (2.3). Finally, identity (2.5) follows immediately from (2.6), (2.2) and (2.1):

$$\begin{aligned}
\widehat{\Theta}(z)\begin{bmatrix} \widetilde{C} \\ \widetilde{E} \end{bmatrix} &= \begin{bmatrix} \widetilde{C} \\ \widetilde{E} \end{bmatrix} + (1-z)\begin{bmatrix} C \\ E \end{bmatrix}(zI-T)^{-1}(I-T)^{-1}\left(P^{-1} - TP^{-1}T^*\right) \\
&= \begin{bmatrix} C \\ E \end{bmatrix}(I-T)^{-1}P^{-1}(I-T^*) \\
&\quad + \begin{bmatrix} C \\ E \end{bmatrix}\left((zI-T)^{-1} - (I-T)^{-1}\right)\left(P^{-1} - TP^{-1}T^*\right) \\
&= -\begin{bmatrix} C \\ E \end{bmatrix}P^{-1}T^* + \begin{bmatrix} C \\ E \end{bmatrix}(zI-T)^{-1}\left(P^{-1} - TP^{-1}T^*\right) \\
&= \begin{bmatrix} C \\ E \end{bmatrix}(zI-T)^{-1}\left(-(zI-T)P^{-1}T^* + P^{-1} - TP^{-1}T^*\right) \\
&= \begin{bmatrix} C \\ E \end{bmatrix}(zI-T)^{-1}P^{-1}(I - zT^*).
\end{aligned}$$

In what follows, J denotes the signature matrix

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

A 2×2 matrix M is called *J–unitary* if $MJM^* = M$, or equivalently, $M^*JM = J$.

Lemma 2.2. *The function $\widehat{\Theta}(z)$ defined in (1.10) is J–unitary at every point on the unit circle $\mathbb{T}$. Moreover, for $z, w \notin \{z_1, \ldots, z_k\}$,*

$$J - \widehat{\Theta}(z)J\widehat{\Theta}(w)^* = (1 - z\bar{w}) \begin{bmatrix} C \\ E \end{bmatrix} (zI - T)^{-1} P^{-1} (\bar{w}I - T^*)^{-1} \begin{bmatrix} C^* & E^* \end{bmatrix}, \tag{2.7}$$

$$J - \widehat{\Theta}(w)^* J\widehat{\Theta}(z) = (1 - z\bar{w}) \begin{bmatrix} -\widetilde{C} \\ \widetilde{E} \end{bmatrix} (\bar{w}I - T^*)^{-1} P (zI - T)^{-1} \begin{bmatrix} -\widetilde{C}^* & \widetilde{E}^* \end{bmatrix} \tag{2.8}$$

and

$$\det \widehat{\Theta}(z) = \prod_{i=1}^{k} \left(\frac{(1 - z\bar{z}_i)(1 - z_i)}{(z - z_i)(1 - \bar{z}_i)} \right)^{n_i}. \tag{2.9}$$

Proof: Identities (2.7) and (2.8) are verified by straightforward computations using (1.7) and (2.3), respectively. Since $\widehat{\Theta}$ is rational, it follows from (2.7) that $\widehat{\Theta}(z)$ is J–unitary at every point $z \in \mathbb{T}$. Furthermore, it follows from the structure (1.4) of T that

$$\det(\alpha I_N - \beta T) = \prod_{i=1}^{k} \det(\alpha I_{n_i} - \beta J_{n_i}(z_i)) = \prod_{i=1}^{k} (\alpha - \beta z_i)^{n_i}.$$

Using (1.7) and the equality $\det(I + AB) = \det(I + BA)$, we get

$$\begin{aligned} \det \widehat{\Theta}(z) &= \det \left(I_2 + (1 - z)(zI - T)^{-1} P^{-1} (I - T^*)^{-1} \begin{bmatrix} -C^* & E^* \end{bmatrix} \begin{bmatrix} C \\ E \end{bmatrix} \right) \\ &= \det \left((zI - T)^{-1} P^{-1} (I - T^*)^{-1} \right) \\ &\quad \times \det \left((I - T^*)P(zI - T) + (1 - z)(P - T^*PT) \right) \\ &= \det \left((zI - T)^{-1} P^{-1} (I - T^*)^{-1} \right) \cdot \det \left((I - zT^*)P(I - T) \right) \\ &= \prod_{i=1}^{k} \left(\frac{(1 - z\bar{z}_i)(1 - z_i)}{(z - z_i)(1 - \bar{z}_i)} \right)^{n_i}. \quad \square \end{aligned}$$

Lemma 2.3. *Let $\Theta(z)$ be defined by (1.11). Then the function*

$$\Psi(z) = \begin{bmatrix} \Theta_{22}(z) & -\Theta_{12}(z) \\ -\Theta_{21}(z) & \Theta_{11}(z) \end{bmatrix} \begin{bmatrix} C \\ E \end{bmatrix} (zI - T)^{-1} \tag{2.10}$$

is analytic on $\mathbb{D}$.

Proof: By (2.5),

$$\widehat{\Theta}(z)^{-1} \begin{bmatrix} C \\ E \end{bmatrix} (zI - T)^{-1} = \begin{bmatrix} \widetilde{C} \\ \widetilde{E} \end{bmatrix} (I - zT^*)^{-1} P \tag{2.11}$$

Furthermore,

$$\widehat{\Theta}(z)^{-1} = \frac{1}{\det \widehat{\Theta}(z)} \begin{bmatrix} \widehat{\Theta}_{22}(z) & -\widehat{\Theta}_{12}(z) \\ -\widehat{\Theta}_{21}(z) & \widehat{\Theta}_{11}(z) \end{bmatrix}$$

and thus, by (1.11) and (2.9),

$$\widehat{\Theta}(z)^{-1} = \prod_{i=1}^{k} \left(\frac{1-\bar{z}_i}{(1-z\bar{z}_i)(1-z_i)} \right)^{n_i} \cdot \begin{bmatrix} \Theta_{22}(z) & -\Theta_{12}(z) \\ -\Theta_{21}(z) & \Theta_{11}(z) \end{bmatrix}. \tag{2.12}$$

Combining the last relation with (2.10) and (2.11) we conclude that

$$\Psi(z) = \prod_{i=1}^{k} \left(\frac{(1-z\bar{z}_i)(1-z_i)}{1-\bar{z}_i} \right)^{n_i} \begin{bmatrix} \widetilde{C} \\ \widetilde{E} \end{bmatrix} (I - zT^*)^{-1} P,$$

which completes the proof, since the function on the right hand side of the last equality is clearly analytic on $\mathbb{D}$. □

Lemma 2.4. *Let $P \in \mathbb{C}^{n\times n}$ be an invertible hermitian matrix and let*

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{bmatrix} \quad \textit{and} \quad P^{-1} = \begin{bmatrix} \alpha & \beta \\ \beta^* & \widetilde{P} \end{bmatrix}$$

be two conformal decompositions with P_3, $\widetilde{P} \in \mathbb{C}^{m\times m}$. Furthermore, let $\widetilde{P}$ be negative semidefinite. Then

$$\mathrm{sq}_-(P_1) = \mathrm{sq}_-(P) - m. \tag{2.13}$$

Proof: We start with the case when $\widetilde{P}$ is negative definite. It follows from the factorization

$$P^{-1} = \begin{bmatrix} \alpha & \beta \\ \beta^* & \widetilde{P} \end{bmatrix} = \begin{bmatrix} I & \beta\widetilde{P}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \alpha - \beta\widetilde{P}^{-1}\beta^* & 0 \\ 0 & \widetilde{P} \end{bmatrix} \begin{bmatrix} I & 0 \\ \beta^*\widetilde{P}^{-1} & I \end{bmatrix}, \tag{2.14}$$

that the matrix $\alpha - \beta\widetilde{P}^{-1}\beta^*$, the Schur complement of $\widetilde{P}$ in P^{-1}, is invertible. Taking inverses in (2.14), we get

$$P = \begin{bmatrix} I & 0 \\ -\beta^*\widetilde{P}^{-1} & I \end{bmatrix} \begin{bmatrix} (\alpha - \beta\widetilde{P}^{-1}\beta^*)^{-1} & 0 \\ 0 & \widetilde{P}^{-1} \end{bmatrix} \begin{bmatrix} I & -\beta\widetilde{P}^{-1} \\ 0 & I \end{bmatrix}$$

and conclude that the matrix

$$P_1 = (\alpha - \beta\widetilde{P}^{-1}\beta^*)^{-1} \tag{2.15}$$

is invertible. It follows from (2.14) that

$$\mathrm{sq}_-(P^{-1}) = \mathrm{sq}_-(\widetilde{P}) + \mathrm{sq}_-(\alpha - \beta\widetilde{P}^{-1}\beta^*),$$

which implies, on account of (2.15), that

$$\mathrm{sq}_-(P_1) = \mathrm{sq}_-(P_1^{-1}) = \mathrm{sq}_-(P^{-1}) - \mathrm{sq}_-(\widetilde{P}) = \mathrm{sq}_-(P) - m.$$

To prove (2.13) in the general case, we use the fact that a sufficiently small perturbation of an invertible matrix does not change the numbers of its positive and negative eigenvalues. Starting with the block decomposition (2.14), we introduce the matrix

$$R_\varepsilon := P^{-1} - \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta^* & \widetilde{P}_\varepsilon \end{bmatrix}, \quad \widetilde{P}_\varepsilon = \widetilde{P} - \varepsilon I_m.$$

Since

$$\|R_\varepsilon - P^{-1}\| = \varepsilon,$$

there exists $\varepsilon_0 > 0$ such that R_ε is invertible and has κ negative eigenvalues for every positive $\varepsilon < \varepsilon_0$. By continuity of singular values we can assume without loss of generality that

$$s_{\min}(R_\varepsilon) > \frac{s_{\min}(P^{-1})}{2} \qquad (0 < \varepsilon < \varepsilon_0),$$

where $s_{\min}(A)$ stands for the minimal singular value of a square matrix A. Then we have

$$\|R_\varepsilon^{-1} - P\| = \|R_\varepsilon^{-1}(P^{-1} - R_\varepsilon)P\| \leq \varepsilon \cdot \|R_\varepsilon^{-1}\| \cdot \|P\| = \frac{\varepsilon \cdot \|P\|}{s_{\min}(R_\varepsilon)} < \frac{2\varepsilon \cdot \|P\|}{s_{\min}(P^{-1})}. \tag{2.16}$$

Since $\widetilde{P}_\varepsilon$ is negative definite, we can apply the first part of the proof to conclude that the matrix

$$R_{\varepsilon,1} := \begin{bmatrix} I_{n-m} & 0 \end{bmatrix} R_\varepsilon^{-1} \begin{bmatrix} I_{n-m} \\ 0 \end{bmatrix} = (\alpha - \beta \widetilde{P}_\varepsilon^{-1} \beta^*)^{-1}$$

has $\mathrm{sq}_-(P) - m$ negative eigenvalues for every positive $\varepsilon < \varepsilon_0$. Since by (2.16)

$$\|R_{\varepsilon,1} - P_1\| \leq \|R_\varepsilon^{-1} - P\| < \varepsilon d,$$

(where d is a positive constant independent of ε) it follows that for ε small enough, $\mathrm{sq}_-(R_{\varepsilon,1}) = \mathrm{sq}_-(P_1)$, which completes the proof. □

Corollary 2.5. *Let P be the Pick matrix of the $\mathbf{CF}_\kappa$ problem (which is by assumption, invertible and with κ negative eigenvalues). Let $\mathbf{m} = (m_1, \ldots, m_k)$ be a multiinteger and let $P_{\mathbf{m}}$ and $\widetilde{P}_{\mathbf{m}}$ be the matrices given in* (1.24)*. If $\widetilde{P}_{\mathbf{m}}$ is negative semidefinite, then*

$$\mathrm{sq}_-(P_{\mathbf{m}}) = \mathrm{sq}_-(P) - \mathrm{rank}(\widetilde{P}_{\mathbf{m}}) = \kappa - \gamma_{\mathbf{m}}, \tag{2.17}$$

where $\gamma_{\mathbf{m}}$ is the integer defined in (1.19)*.*

Proof: Making use of matrices $N_{\mathbf{m}}$ and $M_{\mathbf{m}}$ given in (1.25) we introduce the permutation matrix

$$U = \begin{bmatrix} M_{\mathbf{m}} \\ N_{\mathbf{m}} \end{bmatrix}$$

(U is a permutation matrix due to (1.23)) and note that the following conformal block decompositions hold:

$$UPU^* = \begin{bmatrix} P_{\mathbf{m}} & * \\ * & * \end{bmatrix} \quad \text{and} \quad UP^{-1}U^* = (UPU^*)^{-1} = \begin{bmatrix} * & * \\ * & \widetilde{P}_{\mathbf{m}} \end{bmatrix}.$$

Upon applying Lemma 2.4 to the matrix UPU^* we conclude that if the matrix $\widetilde{P}_{\mathbf{m}} \in \mathbb{C}^{\gamma_{\mathbf{m}} \times \gamma_{\mathbf{m}}}$ is negative semidefinite, then

$$\text{sq}_-(P_{\mathbf{m}}) = \text{sq}_-(P) - \dim(\widetilde{P}_{\mathbf{m}}) = \kappa - \gamma_{\mathbf{m}}. \quad \square$$

3. Interpolation for Schur functions

In this section we recall some results on the Carathéodory–Fejér interpolation problem for Schur functions which will be the main tool in the investigation of the excluded parameters of the parametrization (1.12) in the next section. It would be natural to formulate this problem in the same terms as the $\mathbf{CF}_\kappa$ problem but with $\kappa = 0$. However, it is convenient to treat the case of $\mathbf{CF}_0$ in a slightly different form, which turns out to be more appropriate for our purposes.

Problem 3.1. Given k distinct points $z_1, \ldots, z_k \in \mathbb{D}$, equally many nonnegative integers $m_1, \ldots, m_k$ and $2k$ functions $\mathbf{a}_i(z)$ and $\mathbf{b}_i(z)$ analytic at z_i (i=1,...,k), find all Schur functions $\mathcal{E} \in \mathcal{S}_0$ such that

$$(\mathbf{a}_i\mathcal{E} + \mathbf{b}_i)^{(j)}(z_i) = 0 \quad (i = 1, \ldots, k;\ j = 0, \ldots, m_i - 1). \tag{3.1}$$

Interpolation conditions (3.1) are equivalent to

$$\sum_{\ell=0}^{j} \binom{j}{\ell} \mathbf{a}_i^{(j-\ell)}(z_i)\mathcal{E}^{(\ell)}(z_i) = -\mathbf{b}_i^{(j)}(z_i) \quad (i = 1, \ldots, k;\ j = 0, \ldots, m_i - 1). \tag{3.2}$$

Associated with this interpolation problem are the matrices

$$F = \begin{bmatrix} J_{m_1}(z_1) & & \\ & \ddots & \\ & & J_{m_k}(z_k) \end{bmatrix}, \tag{3.3}$$

$$A = \begin{bmatrix} A_1 & \cdots & A_k \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & \cdots & B_k \end{bmatrix}, \tag{3.4}$$

where

$$A_i = \begin{bmatrix} \overline{\dfrac{\mathbf{a}_i^{(m_i-1)}(z_i)}{(m_i-1)!}} & \ldots & \overline{\mathbf{a}_i'(z_i)} & \overline{\mathbf{a}_i(z_i)} \end{bmatrix}, \tag{3.5}$$

and

$$B_i = \begin{bmatrix} \overline{\dfrac{\mathbf{b}_i^{(m_i-1)}(z_i)}{(m_i-1)!}} & \ldots & \overline{\mathbf{b}_i'(z_i)} & \overline{\mathbf{b}_i(z_i)} \end{bmatrix}. \tag{3.6}$$

Since $|z_i| < 1$, the series

$$K = \sum_{j=0}^{\infty} F^j (A^*A - B^*B)(F^*)^j \tag{3.7}$$

converges and defines a unique solution K of that the Stein equation

$$K - FKF^* = A^*A - B^*B, \tag{3.8}$$

which is the *Pick matrix* of Problem 3.1. Furthermore, Problem 3.1 has a solution if and only if K is positive semidefinite. The Carathéodory–Fejér problem for Schur functions is well known and well studied. The following theorem giving a description of all solutions can be found in many sources (see e.g., [6, Chapter 18], [17, Chapter 5], [11]).

Theorem 3.2. *If the matrix K defined in* (3.7) *is positive definite, the set of all solutions of Problem 3.1 is parametrized by the formula*

$$\mathcal{E}(z) = \mathbf{T}_\Phi[\widetilde{\mathcal{E}}] := \frac{\Phi_{11}(z)\widetilde{\mathcal{E}}(z) + \Phi_{12}(z)}{\Phi_{21}(z)\widetilde{\mathcal{E}}(z) + \Phi_{22}(z)}, \tag{3.9}$$

where $\Phi = (\Phi_{ij})$ is the 2×2 matrix valued function given by

$$\Phi(z) = I_2 - (1-z)\begin{bmatrix} -A \\ B \end{bmatrix}(I - zF^*)^{-1}K^{-1}(I-F)^{-1}\begin{bmatrix} A^* & B^* \end{bmatrix} \tag{3.10}$$

and $\widetilde{\mathcal{E}}(z)$ is the parameter varying over the Schur class $\mathcal{S}_0$. If K is positive semidefinite, then Problem 3.1 has a unique solution which is a Blaschke product of degree $r = \operatorname{rank} K$.

In the contrast to the indefinite case, the denominator in (3.10) does not vanish inside $\mathbb{D}$.

A calculation similar to that in the proof of Lemma 2.2 shows that

$$\det \Phi(z) = \prod_{i=1}^{k} \left(\frac{(z - z_i)(1 - \bar{z}_i)}{(1 - z\bar{z}_i)(1 - z_i)} \right)^{m_i}, \tag{3.11}$$

while the arguments used for the proof of Lemma 2.3 lead us to the conclusion that the function

$$(zI - F)^{-1}\begin{bmatrix} A^* & B^* \end{bmatrix}\Phi(z)$$

is analytic in $\mathbb{D}$. Due to the structure of matrices F, A and B, the latter is equivalent to

$$\left(\begin{bmatrix} \mathbf{a}_i & \mathbf{b}_i \end{bmatrix}\Phi\right)^{(j)} = 0 \quad (i = 1, \ldots, k;\ j = 0, \ldots, m_i - 1). \tag{3.12}$$

The following simple remark will be however, useful.

Remark 3.3. If the Pick matrix of Problem 3.1 is positive definite, then there are infinitely many solutions $\mathcal{E}$ which do not satisfy additional conditions

$$(\mathbf{a}_i\mathcal{E} + \mathbf{b}_i)^{(m_i)}(z_i) = 0 \quad (i = 1, \ldots, k). \tag{3.13}$$

Indeed, substituting (3.9) into (3.13) we rewrite these last conditions in terms of the parameter $\widetilde{\mathcal{E}}$ as

$$\left(\mathbf{a}_i\frac{\Phi_{11}\widetilde{\mathcal{E}}+\Phi_{12}}{\Phi_{21}\widetilde{\mathcal{E}}+\Phi_{22}}+\mathbf{b}_i\right)^{(m_i)}(z_i)=0 \quad (i=1,\ldots,k),$$

which is equivalent, since $\Phi_{21}\widetilde{\mathcal{E}}+\Phi_{22}$ does no vanish at z_i and in view of (3.12), to

$$\left(\mathbf{a}_i[\Phi_{11}\widetilde{\mathcal{E}}+\Phi_{12}]+\mathbf{b}_i[\Phi_{21}\widetilde{\mathcal{E}}+\Phi_{22}]\right)^{(m_i)}(z_i)=0 \quad (i=1,\ldots,k).$$

Making use of Leibniz's rule and again taking into account (3.12), we conclude that $\mathcal{E}$ of the form (3.9) satisfies conditions (3.13) if and only if the corresponding parameter $\widetilde{\mathcal{E}}$ satisfies

$$c_i\widetilde{\mathcal{E}}(z_i)=d_i \quad (i=1,\ldots,k) \tag{3.14}$$

where

$$c_i=(\mathbf{a}_i\Phi_{11}+\mathbf{b}_i\Phi_{21})^{m_i}(z_i) \quad \text{and} \quad d_i=(\mathbf{a}_i\Phi_{12}+\mathbf{b}_i\Phi_{22})^{m_i}(z_i) \quad (i=1,\ldots,k). \tag{3.15}$$

Assuming that $c_i=d_i=0$ (for some i) we get from (3.12) and (3.15)

$$\left(\begin{bmatrix} \mathbf{a}_i & \mathbf{b}_i \end{bmatrix}\Phi\right)^{(j)}=0 \quad (j=0,\ldots,m_i),$$

which in turn, implies that the function $\begin{bmatrix} \mathbf{a}_i & \mathbf{b}_i \end{bmatrix}\Phi$ has at z_i the zero of multiplicity at least m_i+1. But then the determinant of Φ has at z_i zero of multiplicity at least m_i+1, which is impossible, in view of (3.11). Thus,

$$|c_i|+|d_i|>0 \quad (i=1,\ldots,k).$$

But then it is obvious that there are infinitely many Schur functions $\widetilde{\mathcal{E}}$ which do not satisfy (3.14) for $i=1,\ldots,k$. Each such function leads via (3.9) to a Schur function $\mathcal{E}$ that is a solution of Problem 3.1 and does not satisfy (3.13).

4. Excluded parameters and interpolation conditions

In what follows,

$$U_{\mathcal{E}}(z)=\Theta_{11}(z)\mathcal{E}(z)+\Theta_{12}(z), \qquad V_{\mathcal{E}}(z)=\Theta_{21}(z)\mathcal{E}(z)+\Theta_{22}(z), \tag{4.1}$$

for a fixed Schur function $\mathcal{E}$, so that (1.12) takes the form

$$S(z)=\frac{U_{\mathcal{E}}(z)}{V_{\mathcal{E}}(z)}. \tag{4.2}$$

By (4.1),

$$\begin{bmatrix} V_{\mathcal{E}}(z) & -U_{\mathcal{E}}(z) \end{bmatrix}=\begin{bmatrix} 1 & -\mathcal{E}(z) \end{bmatrix}\begin{bmatrix} \Theta_{22}(z) & -\Theta_{12}(z) \\ -\Theta_{21}(z) & \Theta_{11}(z) \end{bmatrix}, \tag{4.3}$$

and

$$\begin{bmatrix} U_{\mathcal{E}}(z) \\ V_{\mathcal{E}}(z) \end{bmatrix}=\Theta(z)\begin{bmatrix} \mathcal{E}(z) \\ 1 \end{bmatrix}. \tag{4.4}$$

Note also that excluded parameters $\mathcal{E}$ of multiorder $\mathbf{m} = (m_1, \dots, m_k)$ are characterized by conditions

$$V_{\mathcal{E}}(z_i) = V'_{\mathcal{E}}(z_i) = \dots = V_{\mathcal{E}}^{(m_i-1)}(z_i) = 0 \quad (i = 1, \dots, k) \tag{4.5}$$

and

$$V_{\mathcal{E}}^{(m_i)}(z_i) \neq 0 \quad (i = 1, \dots, k). \tag{4.6}$$

Theorem 4.1. *Let P be invertible with* $\text{sq\_}P = \kappa$, *let $\mathcal{E}$ be a Schur function, let Θ, $U_{\mathcal{E}}$ and $V_{\mathcal{E}}$ be given as in* (1.11) *and* (4.1). *Then*

1. *The nontangential boundary limits $U_{\mathcal{E}}(t)$ and $V_{\mathcal{E}}(t)$ exist at almost every point $t \in \mathbb{T}$ and satisfy*

$$|U_{\mathcal{E}}(t)| \le |V_{\mathcal{E}}(t)| \quad a.e. \text{ on } \quad \mathbb{T}.$$

2. *It holds that*

$$N\{V_{\mathcal{E}}\} = N\{\Theta_{21}\mathcal{E} + \Theta_{22}\} = N\{\Theta_{22}\} = \kappa, \tag{4.7}$$

where $N(f)$ stands for the total number of zeroes of a function f that fall inside $\mathbb{D}$.

3. *$U_{\mathcal{E}}$ and $V_{\mathcal{E}}$ can have a common zero at no point inside $\mathbb{D}$, but $z_1, \dots, z_d$.*
4. *$U_{\mathcal{E}}$ and $V_{\mathcal{E}}$ cannot have a common zero at z_j of multiplicity greater than n_j.*
5. *If $V_{\mathcal{E}}$ has the zero of multiplicity $m_j > n_j$ at z_j, then $U_{\mathcal{E}}$ has the zero of multiplicity n_j at z_j.*

Proof: The existence of boundary limits for $U_{\mathcal{E}}$ and $V_{\mathcal{E}}$ follows from definitions (4.1), since $\mathcal{E}$ is a Schur function and Θ_{ij}'s are polynomials. By (4.4),

$$\begin{aligned} |V_{\mathcal{E}}(z)|^2 - |U_{\mathcal{E}}(z)|^2 &= \begin{bmatrix} \overline{U_{\mathcal{E}}(z)} & \overline{V_{\mathcal{E}}(z)} \end{bmatrix} J \begin{bmatrix} U_{\mathcal{E}}(z) \\ V_{\mathcal{E}}(z) \end{bmatrix} \\ &= \begin{bmatrix} \overline{\mathcal{E}(z)} & 1 \end{bmatrix} \Theta(z)^* J \Theta(z) \begin{bmatrix} \mathcal{E}(z) \\ 1 \end{bmatrix}. \end{aligned} \tag{4.8}$$

Upon multiplying (2.8) (evaluated at $z = w$) by $\prod_{i=1}^{k} |z - z_i|^{2n_i}$ and letting $z \to t \in \mathbb{T}$, we get

$$\Theta(t)^* J \Theta(t) = \prod_{i=1}^{k} |t - z_i|^{2n_i} J.$$

Now we let $z \to t$ in (4.8) to get

$$\begin{aligned} |V_{\mathcal{E}}(t)|^2 - |U_{\mathcal{E}}(t)|^2 &= \prod_{i=1}^{k} |t - z_i|^{2n_i} \begin{bmatrix} \overline{\mathcal{E}(t)} & 1 \end{bmatrix} J \begin{bmatrix} \mathcal{E}(t) \\ 1 \end{bmatrix} \\ &= \prod_{i=1}^{k} |t - z_i|^{2n_i} (1 - |\mathcal{E}(t)|^2) \ge 0, \end{aligned}$$

since $\mathcal{E}$ is a Schur function and therefore $|\mathcal{E}(t)| \le 1$ a.e. on $\mathbb{T}$. This completes the proof of the first statement.

Upon multiplying (2.7) (evaluated at $z=w$) by $\prod_{i=1}^k |z-z_i|^{2n_i}$ and letting $z\to t\in\mathbb{T}$, we get

$$\Theta(t)J\Theta(t)^* = \prod_{i=1}^k |t-z_i|^{2n_i} J.$$

Taking into account block decompositions of Θ and J, we compare the right bottom elements in the latter matrix equality to get

$$|\Theta_{21}(t)|^2 - |\Theta_{22}(t)|^2 = -\prod_{i=1}^k |t-z_i|^{2n_i}.$$

Thus, $|\Theta_{22}(t)| > |\Theta_{21}(t)|$ on $\mathbb{T}$ and since $\mathcal{E}\in\mathcal{S}_0$, it follows that

$$|\Theta_{22}(t)| > |\Theta_{21}(t)\mathcal{E}(t)|$$

at almost every point $t\in\mathbb{T}$. Then, by Rouchè's theorem, the functions $\Theta_{21}\mathcal{E}+\Theta_{22}$ and Θ_{22} have the same number of zeroes in the disk $\{z:\ |z|<r\}$ for every r close enough to 1. Since the polynomial Θ_{22} has finitely many zeroes in $\mathbb{D}$, we let $r\to 1$ to conclude that

$$N\{\Theta_{21}\mathcal{E}+\Theta_{22}\} = N\{\Theta_{22}\}.$$

The last equality shows that $V_{\mathcal{E}}$ has the same number of zeroes inside $\mathbb{D}$ for every $\mathcal{E}\in\mathcal{S}_0$. But it follows by Theorem 1.2 that

$$N\{V_{\mathcal{E}}\} = N\{\Theta_{21}\mathcal{E}+\Theta_{22}\} = \kappa$$

for every nonexcluded parameter $\mathcal{E}$ (in this case, the function $S=\mathbf{T}_\Theta[\mathcal{E}]$ belongs to $\mathcal{S}_\kappa$ and therefore, it has κ poles all of which are zeroes of $V_{\mathcal{E}}$). This completes the proof of the second statement.

To prove the third statement, note that by (1.11) and (2.9),

$$\det\,\Theta(z) = \left(\prod_{i=1}^k (z-z_i)^{2n_i}\right)\det\,\widehat{\Theta}(z) = \prod_{i=1}^k \left(\frac{(z-z_i)(1-z\bar{z}_i)(1-z_i)}{1-\bar{z}_i}\right)^{n_i}. \tag{4.9}$$

Assuming that $U_{\mathcal{E}}(w)=V_{\mathcal{E}}(w)=0$ at some point $w\in\mathbb{D}$, we get from (4.4) that

$$\Theta(w)\begin{bmatrix}\mathcal{E}(w)\\ 1\end{bmatrix} = 0$$

and therefore, that $\det\,\Theta(w)=0$. But by (4.9), $z_1,\ldots,z_k$ are the only zeroes of $\det\,\Theta$, which completes the proof of the third statement.

Assuming that $U_{\mathcal{E}}$ and $V_{\mathcal{E}}$ have the common zero of order $m_j>n_j$ at z_j, we conclude by (4.4) that the vector valued function

$$\Theta(z)\begin{bmatrix}\mathcal{E}(z)\\ 1\end{bmatrix}$$

has the zero of multiplicity $m_j>n_j$ at z_j. But then, $\det\,\Theta(z)$ has the zero of multiplicity $m_j>n_j$ at z_j, which contradicts to (4.9) and completes the proof of the theorem. $\square$

The first statement in Theoren 4.1 can be completed by the following

Remark 4.2. There exists $\rho \in (0,\, 1)$ and $K_\rho < \infty$ such that

$$|U_{\mathcal{E}}(z)| \leq K_\rho \cdot |V_{\mathcal{E}}(z)| \qquad \forall z \in A_\rho = \{z:\ \rho < |z| < 1\}. \tag{4.10}$$

Proof: It follows from (2.8) (evaluated at $z = w$ and multiplied by $\prod_{i=1}^k |t - z_i|^{2n_i}$), that

$$\Theta(z)^* J \Theta(z) = \prod_{i=1}^k |t - z_i|^{2n_i} J - (1 - |z|^2)\Theta(z)^* F(z)^* P F(z)\Theta(z), \tag{4.11}$$

where

$$F(z) = (zI - T)^{-1} \begin{bmatrix} -\widetilde{C}^* & \widetilde{E}^* \end{bmatrix} \widehat{\Theta}(z)^{-1}.$$

Since $\widehat{\Theta}$ is J-unitary on $\mathbb{T}$, it follows by the symmetry principle that $\widehat{\Theta}(z)^{-1} = J\widehat{\Theta}(1/\bar{z})^* J$, which together with (1.10) implies

$$\widehat{\Theta}(z)^{-1} = I_2 + (z-1)\begin{bmatrix} C \\ E \end{bmatrix}(I - T)^{-1}P^{-1}(I - zT^*)^{-1}\begin{bmatrix} -C^* & E^* \end{bmatrix}. \tag{4.12}$$

Thus, $\widehat{\Theta}^{-1}$ is analytic on $\mathbb{D}$ and therefore, the function F may have poles inside $\mathbb{D}$ only at $z_1, \ldots, z_d$. Fix $r < 1$ such that $r > \max\{|z_1|, \ldots, |z_d|\}$ and let

$$\beta := \sup_{r<|z|<1} \|F(z)\| = \max_{r<|z|<1} \|F(z)\|.$$

Then

$$-\beta^2 \|P\|\, I_2 \leq F(z)^* P F(z) \leq \beta^2 \|P\|\, I_2 \qquad r < |z| < 1). \tag{4.13}$$

Now we choose $\rho \in (r,\, 1)$ such that

$$\delta := (1 - \rho^2)\beta^2 \|P\| < 1. \tag{4.14}$$

Now we conclude from (4.11) and (4.13) that

$$\Theta(z)^* J \Theta(z) \geq \prod_{i=1}^k |t - z_i|^{2n_i} J - \delta\Theta(z)^*\Theta(z)$$

for every z such that $\rho < |z| < 1$. Substituting the last inequality into (4.8), we get

$$\begin{aligned} |V_{\mathcal{E}}(z)|^2 - |U_{\mathcal{E}}(z)|^2 &\geq \prod_{i=1}^k |x - z_i|^{2n_i} \begin{bmatrix} \overline{\mathcal{E}(z)} & 1 \end{bmatrix} J \begin{bmatrix} \mathcal{E}(z) \\ 1 \end{bmatrix} \\ & \quad -\delta \begin{bmatrix} \overline{\mathcal{E}(z)} & 1 \end{bmatrix} \Theta(z)^*\Theta(z) \begin{bmatrix} \mathcal{E}(z) \\ 1 \end{bmatrix} \\ &= \prod_{i=1}^k |z - z_i|^{2n_i}(1 - |\mathcal{E}(z)|^2) - \delta \begin{bmatrix} \overline{U_{\mathcal{E}}(z)} & \overline{V_{\mathcal{E}}(z)} \end{bmatrix} \begin{bmatrix} U_{\mathcal{E}}(z) \\ V_{\mathcal{E}}(z) \end{bmatrix} \\ &= \prod_{i=1}^k |z - z_i|^{2n_i}(1 - |\mathcal{E}(z)|^2) - \delta|U_{\mathcal{E}}(z)|^2 - \delta|V_{\mathcal{E}}(z)|^2. \end{aligned}$$

Therefore,

$$(1+\delta)|V_{\mathcal{E}}(z)|^2 - (1-\delta)|U_{\mathcal{E}}(z)|^2 \geq \prod_{i=1}^k |z - z_i|^{2n_i}(1 - |\mathcal{E}(z)|^2) \geq 0 \quad (\rho < |z| < 1)$$

which implies (4.10) with $K_\rho := \frac{1+\delta}{1-\delta}$, which depends on ρ and is positive and finite, due to (4.14). □

Corollary 4.3. *There are no excluded parameters of multiorder* $\mathbf{m}$ *with* $|\mathbf{m}| > N = |\mathbf{n}|$.

Indeed, if $V_{\mathcal{E}}$ has zeroes of multiplicities m_j at z_j, then by (4.7),

$$|\mathbf{m}| = m_1 + \ldots + m_k \leq N(V_{\mathcal{E}}) = \kappa \leq N$$

(the last inequality is clear, since the number κ of negative squares of P cannot exceed the size N of P).

However, one cannot guarantee that $m_j \leq n_j$ for all $j = 1, \ldots, k$ or, in other words, that $\mathbf{m} \preceq \mathbf{n}$. In the case when $m_j > n_j$, the last statement in Theorem 4.1 implies that the function $S = \frac{U_{\mathcal{E}}}{V_{\mathcal{E}}}$ will have the pole of multiplicity $m_j - n_j$ at z_j.

Equalities (1.16) mean that $\mathcal{E}$ is a solution of Problem 3.1 with

$$\mathbf{a}_i(z) = \frac{\Theta_{21}(z)}{\prod_{\ell \neq i}(z - z_\ell)^{n_\ell}} \quad \text{and} \quad \mathbf{b}_i(z) = \frac{\Theta_{22}(z)}{\prod_{\ell \neq i}(z - z_\ell)^{n_\ell}} \quad (i = 1, \ldots, k). \tag{4.15}$$

Of course, we could choose $\mathbf{a}_i = \Theta_{21}(z)$ and $\mathbf{b}_i = \Theta_{22}(z)$ for all $i = 1, \ldots, k$, but the choice (4.15) makes some subsequent calculations easier. By (1.11) and (2.3),

$$\frac{\Theta(z)}{\prod_{\ell \neq i}^k (z - z_\ell)^{n_\ell}} = (z - z_i)^{n_i}\Upsilon + \begin{bmatrix} C \\ E \end{bmatrix} (z - z_i)^{n_i}(zI - T)^{-1} \begin{bmatrix} -\widetilde{C}^* & \widetilde{E}^* \end{bmatrix}.$$

Let

$$\widetilde{C} = \begin{bmatrix} \widetilde{C}_1 & \cdots & \widetilde{C}_k \end{bmatrix} \text{and} \quad \widetilde{E} = \begin{bmatrix} \widetilde{E}_1 & \cdots & \widetilde{E}_k \end{bmatrix}$$

be decompositions of $\widetilde{C}$ and $\widetilde{E}$ conformal with decompositions (1.4) and (1.5). It follows by the structure (1.4) of T that

$$\frac{\Theta(z)}{\prod_{\ell\neq i}^{k}(z-z_\ell)^{n_\ell}} = \begin{bmatrix} C_i \\ L_{n_i} \end{bmatrix} (z-z_i)^{n_i}(zI-J_{n_i}(z_i))^{-1} \begin{bmatrix} -\widetilde{C}_i^* & \widetilde{E}_i^* \end{bmatrix} + O((z-z_i)^{n_i}).$$

Note that

$$\begin{aligned} T_i(z) &:= (z-z_i)^{n_i}(zI-J_{n_i}(z_i))^{-1} \\ &= \begin{bmatrix} (z-z_i)^{n_i-1} & \dots & z-z_i & 1 \\ 0 & (z-z_i)^{n_i-1} & & z-z_i \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & (z-z_i)^{n_i-1} \end{bmatrix} \end{aligned}$$

and thus,

$$\begin{aligned} \begin{bmatrix} \mathbf{a}_i(z) & \mathbf{b}_i(z) \end{bmatrix} &= \frac{1}{\prod_{\ell\neq i}^{k}(z-z_\ell)^{n_\ell}} \begin{bmatrix} \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix} \\ &= L_{n_i}T_i(z) \begin{bmatrix} -\widetilde{C}_i^* & \widetilde{E}_i^* \end{bmatrix} + O((z-z_i)^{n_i}) \quad (i=1,\dots,k). \end{aligned}$$

Now it is clear that for $j < n_i$, $\frac{1}{j!}\mathbf{a}_i^{(j)}(z_i)$ and $\frac{1}{j!}\mathbf{b}_i^{(j)}(z_i)$ are equal to (n_i-j)-th entries of the columns $-\widetilde{C}_i^*$ and $\widetilde{E}_i^*$, respectively. Therefore, in the setting of (4.15), the matrices A_i, B_i defined in (3.5), (3.6), take the form

$$A_i = -\widetilde{C}_i \begin{bmatrix} 0 \\ I_{m_i} \end{bmatrix} \quad \text{and} \quad B_i = \widetilde{E}_i \begin{bmatrix} 0 \\ I_{m_i} \end{bmatrix}.$$

Assuming furthermore, that $\mathbf{m} \preceq \mathbf{n}$ (i.e., $m_i \le n_i$ for $i=1,\dots,k$) and making use of matrices $N_{\mathbf{m}}$ defined in (1.25), we conclude that in the setting of (4.15), the matrices A and B given in (3.4), take the form

$$A = -\widetilde{C}N_{\mathbf{m}}^* \quad \text{and} \quad B = \widetilde{E}N_{\mathbf{m}}^*. \tag{4.16}$$

The next step is to compute the Pick matrix K (given by (3.7)) in the setting of (4.15). Note that matrices T and F given in (1.4) and (3.3) respectively, are intertwined by $N_{\mathbf{m}}$:

$$N_{\mathbf{m}}T = FN_{\mathbf{m}}.$$

Making use of (4.16), (2.4) and of the last commutation relation, we get

$$\begin{aligned} A^*A - B^*B &= N_{\mathbf{m}}\left(\widetilde{C}^*\widetilde{C} - \widetilde{E}^*\widetilde{E}\right)N_{\mathbf{m}}^* \\ &= N_{\mathbf{m}}\left(TP^{-1}T^* - P^{-1}\right)N_{\mathbf{m}}^* \\ &= FN_{\mathbf{m}}P^{-1}N_{\mathbf{m}}^* - N_{\mathbf{m}}P^{-1}N_{\mathbf{m}}^*. \end{aligned}$$

Thus, the matrix

$$N_{\mathbf{m}}P^{-1}N_{\mathbf{m}}^* = -\widetilde{P}_{\mathbf{m}}$$

satisfies the Stein equation (3.8) and since this equation has a unique solution, it follows that

$$K = -\widetilde{P}_{\mathbf{m}}. \tag{4.17}$$

Proof of Theorem 1.6: First we note that excluded parameters $\mathcal{E} \in \mathcal{S}_0$ of multi-orders $\widetilde{\mathbf{m}} \succeq \mathbf{m}$ are characterized by equalities (4.5) regardless restrictions (4.6), which means that each such $\mathcal{E}$ is a solution of Problem 3.1 in the setting of (4.15). But this latter problem has a solution if and only if its Pick matrix K is positive semidefinite. The first statement in Theorem 1.6 now follows from (4.17). Furthermore, it follows by Theorem 3.2, that if $\widetilde{P}_{\mathbf{m}}$ is negative semidefinite (singular), then Problem 3.1 has only one solution $\mathcal{E}$ which is a Blaschke product of degree $r = \text{rank}(-\widetilde{P}_{\mathbf{m}}) = \text{rank}(\widetilde{P}_{\mathbf{m}})$. The existence of infinitely many excluded parameters of multiorder $\mathbf{m}$, that is, solutions of Problem 3.1 which in addition satisfy constraints (1.17), follows by Remark 3.3. □

Proof of Theorem 1.4: It follows by Lemma 2.3, that the function

$$Q(z) := \begin{bmatrix} 1 & -\mathcal{E}(z) \end{bmatrix} \begin{bmatrix} \Theta_{22}(z) & -\Theta_{12}(z) \\ -\Theta_{21}(z) & \Theta_{11}(z) \end{bmatrix} \begin{bmatrix} C \\ E \end{bmatrix} (zI - T)^{-1}$$

is analytic at $z_1, \ldots, z_k$. Making use of (4.3), we represent Q as

$$Q(z) = [V_{\mathcal{E}}(z)C - U_{\mathcal{E}}(z)E]\,(zI - T)^{-1}.$$

The block structure of matrices T, C and E leads to the conformal block structure of Q:

$$Q(z) = \begin{bmatrix} Q_1(z) & \ldots & Q_k(z) \end{bmatrix},$$

where

$$Q_i(z) = [V_{\mathcal{E}}(z)C_i - U_{\mathcal{E}}(z)L_{n_i}]\,(zI - J_{n_i}(z_i))^{-1} \tag{4.18}$$

and to the conclusion that $Q_i(z)$ is analytic at z_i for $z = 1, \ldots, k$. It is readily seen from the definition of Q_i that the residue of Q_i at z_i equals

$$\begin{aligned} \text{Res}_{z=z_i} Q_i(z) \;\; &= \;\; \begin{bmatrix} V_{\mathcal{E}}(z_i) & \ldots & \dfrac{V_{\mathcal{E}}^{(n_i-1)}(z_i)}{(n_i-1)!} \end{bmatrix} \begin{bmatrix} S_{i,0} & S_{i,1} & \ldots & S_{i,n_i-1} \\ 0 & S_{i,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & S_{i,1} \\ 0 & \ldots & 0 & S_{i,0} \end{bmatrix} \\ &\quad - \begin{bmatrix} U_{\mathcal{E}}(z_i) & U_{\mathcal{E}}'(z_i) & \ldots & \dfrac{U^{(n_i-1)}(z_i)}{(n_i-)!} \end{bmatrix}. \end{aligned}$$

Since Q_i is analytic at z_i and therefore, $\text{Res}_{z=z_i}Q_i(z) = 0$, the last displayed equality implies

$$\begin{bmatrix} U_{\mathcal{E}}(z_i) & \dots & \frac{U^{(n_i-1)}(z_i)}{(n_i-1)!} \end{bmatrix}$$

$$= \begin{bmatrix} V_{\mathcal{E}}(z_i) & \dots & \frac{V_{\mathcal{E}}^{(n_i-1)}(z_i)}{n!} \end{bmatrix} \begin{bmatrix} S_{i,0} & S_{i,1} & \dots & S_{i,n_i-1} \\ 0 & S_{i,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & S_{i,1} \\ 0 & \dots & 0 & S_{i,0} \end{bmatrix}.$$

Thus, if $m_i \leq n_i$, then conditions (4.5) force

$$U_{\mathcal{E}}^{(j)}(z_i) = 0 \quad \text{for; } j = 0, \dots m_i - 1,$$

which means that $U_{\mathcal{E}}$ has the zero at z_i of at least the same multiplicity as $V_{\mathcal{E}}$ does and therefore, the function $S = \mathbf{T}_{\Theta}[\mathcal{E}]$ admits an analytic continuation to z_i. If $m_i > n_i$, then the same arguments show that $U_{\mathcal{E}}$ has zero of multiplicity $\widetilde{m}_i \geq n_i$ at z_i. If $\widetilde{m}_j > n_i$, then z_i is a common zero of $U_{\mathcal{E}}$ and $V_{\mathcal{E}}$ of multiplicity greater than n_j, which is impossible, by Statement 4 of Theorem 4.1. Thus, $\widetilde{m}_i = n_i$, which means that $U_{\mathcal{E}}$ has zeroes of multiplicities n_i at z_i for $i \in \mathcal{Z}_{\mathbf{m}}^+$.

By (4.7), $V_{\mathcal{E}}$ has κ zeroes inside $\mathbb{D}$. Since $\mathcal{E}$ is an excluded parameter of multiorder $\mathbf{m}$, $V_{\mathcal{E}}$ has m_i zeroes at z_i (for $i = 1, \dots, k$) and the remaining $\kappa - |\mathbf{m}|$ zeroes in $\mathbb{D} \setminus \{z_1, \dots, z_k\}$. We have already shown that for every $i \in \mathcal{Z}_{\mathbf{m}}^-$, all the m_i zeroes of $V_{\mathcal{E}}$ at z_i are canceled by zeroes of $U_{\mathcal{E}}$ and n_i zeroes of $V_{\mathcal{E}}$ at z_i are canceled if $i \in \mathcal{Z}_{\mathbf{m}}^+$. After all cancellations, the function $V_{\mathcal{E}}$ will have $m_i - n_i$ zeroes at z_i (for all $i \in \mathcal{Z}_{\mathbf{m}}^+$) and still $\kappa - |\mathbf{m}|$ zeroes in $\mathbb{D} \setminus \{z_1, \dots, z_k\}$ which has not been canceled, by Statement 3 of Theorem 4.1. Thus, the function $S = \frac{U_{\mathcal{E}}}{V_{\mathcal{E}}}$ will have

$$\kappa - |\mathbf{m}| + \sum_{i \in \mathcal{Z}_+} (m_i - n_i - 1) = \kappa - \sum_{i \in \mathcal{Z}_-} m_i - \sum_{i \in \mathcal{Z}_+} (n_i + 1) = \kappa - \gamma_{\mathbf{m}}$$

poles inside $\mathbb{D}$. Moreover, by the first statement in Theorem 4.1, $|S(t)| \leq 1$ a.e. on $\mathbb{T}$ and by Remark 4.2, $S(z) \leq K_\rho < \infty$ for every $z \in A_\rho = \{z : \ \rho < |z| < 1$. Thus, S belongs to $\mathcal{S}_{\kappa - \gamma_{\mathbf{m}}}$.

The next step is to show that S satisfies interpolation conditions (1.20). To this end, we make use of (4.2) to represent the function Q_i from (4.18) as

$$Q_i(z) = V_{\mathcal{E}}(z) \left[C_i - S(z) L_{n_i} \right] (zI - J_{n_i}(z_i))^{-1},$$

and to conclude that

$$\begin{aligned} \text{Res}_{z=z_i} Q_i(z) &= \begin{bmatrix} V_{\mathcal{E}}(z_i) & \dots & \frac{V_{\mathcal{E}}^{(n_i-1)}(z_i)}{(n_i-1)!} \end{bmatrix} \begin{bmatrix} R_{i,0} & R_{i,1} & \dots & R_{i,n_1-1} \\ 0 & R_{i,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & R_{i,1} \\ 0 & \dots & 0 & R_{i,0} \end{bmatrix} \\ &= 0, \end{aligned} \tag{4.19}$$

where

$$R_{i,j} = S_{i,j} - \frac{S^{(j)}(z_i)}{j!} \quad (j = 0, \dots, n_i - 1). \tag{4.20}$$

Since $V_{\mathcal{E}}^{(m_i)}(z_0) \neq 0$, it follows from (4.19) that

$$R_{i,j} = 0 \quad (j = 0, \dots, n_i - m_i;\ i = 1, \dots, k),$$

which is equivalent to (1.20), by (4.20).

It remains to show that if $m_i \le n_i$, then S is subject to the corresponding inequality in (1.21).

Assuming that one of the inequalities (say, the first, for definiteness) in (1.21) fails, i.e., that

$$S^{(n_1-m_1)}(z_1) = (n_1 - m_1)!\, S_{1,n_1-m_1}, \tag{4.21}$$

we come to contradiction as follows. Let

$$\mathbf{m}' = (m_1 - 1, m_2, \dots, m_k) \tag{4.22}$$

and let $P_{\mathbf{m}'}$ and $\widetilde{P}_{\mathbf{m}'}$ be defined via (1.24). By Theorem 1.6, $\widetilde{P}_{\mathbf{m}} \le 0$. Since $m_1 \le n_1$, it follows by construction, that $\widetilde{P}_{\mathbf{m}'}$ is a principal submatrix of $\widetilde{P}_{\mathbf{m}}$ (of the size $(\gamma_{\mathbf{m}} - 1) \times (\gamma_{\mathbf{m}} - 1)$) and therefore, it is also negative semidefinite. Then by (2.17) and (4.22),

$$\mathrm{sq}_-(P_{\mathbf{m}'}) = \mathrm{sq}_-(P) - \dim(\widetilde{P}_{\mathbf{m}'}) = \kappa - \gamma_{\mathbf{m}} + 1. \tag{4.23}$$

On the other hand, $P_{\mathbf{m}'}$ is the Pick matrix of a reduced Carathéodory–Fejér problem with interpolation conditions (1.20) and (4.21). By (1.20) and by assumption (4.21), the function $S = \mathbf{T}_\Theta[\mathcal{E}]$ is a solution of this problem and thus, by (1.8) and (4.23),

$$\mathrm{sq}_-(K_S) \ge \mathrm{sq}_-(P_{\mathbf{m}'}) = \kappa - \gamma_{\mathbf{m}} + 1.$$

But it has been already shown that S belongs to $\mathcal{S}_{\kappa-\gamma_{\mathbf{m}}}$, which is a contradiction. □

The following three statements are simple corollaries of Theorem 1.6.

Corollary 4.4. *Let*

$$P^{-1} = ((P^{-1})_{ij})_{i,j=1}^{k}, \quad (P^{-1})_{ij} \in \mathbb{C}^{n_i \times n_j} \tag{4.24}$$

be the decomposition of P^{-1} conformal to (1.4) and let p_{jj} denote the bottom diagonal entry of the diagonal block $(P^{-1})_{jj}$ in this decomposition.

1. *If $p_{jj} > 0$, then for every excluded parameter of multiorder $\mathbf{m} = (m_1, \dots, m_k)$ it holds that $m_j = 0$. If $p_{jj} > 0$ for $j = 1, \dots, k$, then there are no excluded parameters in the transformation (1.12).*
2. *If $p_{jj} = 0$, then there is only one excluded parameter $\mathcal{E}_0$ of multiorder $\mathbf{m}$ with $m_j \ge 1$, which is a unimodular constant.*
3. *If $p_{jj} < 0$, then there are infinitely many excluded parameters of multiorder $\mathbf{m}$ with $m_j \ge 1$.*

Proof: If $m_j > 0$, then $\widetilde{P}_{\mathbf{m}}$ contains p_{jj} as a diagonal entry and therefore, it is not negative semidefinite. Thus, by Theorem 1.6, there are no excluded parameters with $m_j > 0$. Furthermore, if $m_j = 0$ for all $j = 1, \ldots, k$, then every parameter $\mathcal{E}$ is of multiorder $\mathbf{m} = \mathbf{0}$, that is, it is not excluded.

By Theorem 1.6, the only excluded parameter $\mathcal{E}_0$ with partial order $m_j > 0$ is a Blaschke product of degree zero, i.e., a unimodular constant. This proves the second statement. The last statement follows by Theorem 1.6. □

Note that Theorem 1.6 does not characterize excluded parameters of multiorders $\mathbf{m} \not\preceq \mathbf{n}$. In fact, these parameters still can be characterized in terms of the Pick matrix K of Problem 3.1 in the setting of (4.15). But in the case when $\mathbf{m} \not\preceq \mathbf{n}$, this matrix is not anymore a principal submatrix of $(-P^{-1})$ and the characterization loses its beauty.

Note also that Theorem 1.6 does not establish necessary and sufficient conditions for existence of a unique excluded parameter of multiorder $\mathbf{m}$ (it gives such conditions for existence of a unique excluded parameter of multiorder at least $\mathbf{m}$). The following theorem partially completes this gap.

Theorem 4.5. *Let $m_i < n_i$ for $i = 1, \ldots, k$ and let the matrix $\widetilde{P}_{\mathbf{m}}$ given by (1.24) is negative semidefinite (singular). Then there is an excluded parameter of multiorder $\mathbf{m} = (m_1, \ldots, m_k)$ if and only if $\widetilde{P}_{\mathbf{m}}$ is "maximal" in the sense that the matrix $\widetilde{P}_{\widetilde{\mathbf{m}}}$ is not negative semidefinite, whenever $\widetilde{\mathbf{m}} \succ \mathbf{m}$.*

Proof: By Theorem 1.6, there is only one excluded parameter $\mathcal{E}_0$ of multiorder not less than $\mathbf{m}$. Furthermore, $\widetilde{P}_{\mathbf{m}}$ is not "maximal" if and only if $\widetilde{P}_{\widetilde{\mathbf{m}}} \leq 0$ for some $\widetilde{\mathbf{m}}$ such that

$$\widetilde{\mathbf{m}} \succ \mathbf{m} \quad \text{and} \quad |\widetilde{\mathbf{m}}| = |\mathbf{m}| + 1.$$

This means, again by the solvability criteria for the $\mathbf{CF}_0$ problem, that at least one inequality in (1.17) fails, which means in its turn, that multiorder of $\mathcal{E}_0$ is not less that $\widetilde{\mathbf{m}} \succ \mathbf{m}$. Thus, if $\widetilde{P}_{\mathbf{m}} \leq 0$, the only excluded parameter $\mathcal{E}_0$ is of multiorder $\mathbf{m}$ if and only if $\widetilde{P}_{\mathbf{m}}$ is "maximal". □

A similar characterization can be obtained for any multiorder, but again, not in terms of the Pick matrix P of the problem.

We conclude the section with a brief discussion of some connections between Theorem 1.4 and characterization (1.14).

First we note that a part of statements in Theorem 1.4 are intuitively clear by (1.14): if S belongs to the range of the transformation (1.12), it admits a representation

$$S = T_1 + T_2 \frac{\widetilde{Q}}{B}, \tag{4.25}$$

where T_1 is a fixed H^∞ interpolant, T_2 is the Blaschke product defined in (1.15), B is a Blaschke product of degree κ and $\widetilde{Q}$ is a function from H^∞. The norm constraint

$$\|S\|_{L^\infty} \leq 1 \tag{4.26}$$

imposes certain restrictions on the choice of $\widetilde{Q}$ and B in (4.25), which we ignore for the moment.

If B has the zero of multiplicity $m_j > n_j$ at z_j, then S has in fact (after cancellation of the factor $(z - z_j)^{n_j}$ in T_2 and B) the pole of multiplicity $m_j - n_j$ at z_j. In the case when $m_j \leq n_j$, a similar cancellation leads us to the conclusion that S admits the analytic continuation at z_j and since the second term in the representation (4.25) has the zero of multiplicity at least $n_j - m_j$ at z_j, it follows that the first $n_j - m_j$ Taylor coefficients of S and T_1 at z_j coincide and we come in fact, to conditions (1.20) in Theorem (1.5).

On the other hand, conditions (1.21) mean (in the context of the $\mathbf{TS}_\kappa$ problem) that the $n_j - m_j + 1$-th Taylor coefficients of S and T_1 at z_j are different, i.e., that the second term in (4.25) has the zero of multiplicity exactly $n_j - m_j$ at z_j, or equivalently, that

$$\widetilde{Q}(z_j) \neq 0 \quad \text{if} \;\; n_j > m_j. \tag{4.27}$$

It follows also by Theorem 1.4 that the functions $\widetilde{Q}$ and B cannot have common zeroes, which means by the Krein-Langer representation (1.2) that

$$\frac{1}{\|\widetilde{Q}\|_{H^\infty}} \cdot \frac{\widetilde{Q}}{B} \in \mathcal{S}_\kappa. \tag{4.28}$$

Otherwise S of the form (4.25) would have less than κ poles satisfying at the same time all the interpolation conditions (1.3), which is impossible, by Theorem 1.4. Restrictions (4.27) and (4.28) are not imposed by (4.25); thus, they shoud be implied by the norm constraint (4.26). However, direct proofs of these implications do not seem to be trivial.

Note also that in the coset characterization (1.14) one can replace T_1 by any solution T_1' of the corresponding $\mathbf{CF}_\kappa$ problem (so that the choice of $Q = 0$ in (4.25) would certainly lead to a function $S = T_1' \in \mathcal{B}L^\infty$. In this case the norm constraint (4.26) imposes quite different restrictions on the choice of $\widetilde{Q}$ and B some of which also can be clarified with help of Theorem 1.4.

5. Nevanlinna–Pick interpolation problem

In this section we discuss a particular case of the $\mathbf{CF}_\kappa$ problem, when $n_i = 1$ $(i = 1, \ldots, k)$ in (1.3). The corresponding problem is called the *Nevanlinna–Pick* interpolation problem $\mathbf{NP}_\kappa$:

Given points $z_1, \dots, z_k \in \mathbb{D}$ and complex numbers $S_1, \dots, S_k$, find all functions $S \in \mathcal{S}_\kappa$ which are analytic at $z_1, \dots, z_k$ and satisfy

$$S(z_i) = S_i, \qquad (i = 1, \dots, k). \tag{5.1}$$

In this case (1.4) and (1.5) take the form

$$T = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_k \end{bmatrix}, \quad E = \begin{bmatrix} 1 & \dots 1 \end{bmatrix}, \quad C = \begin{bmatrix} S_1 & \dots S_k \end{bmatrix}. \tag{5.2}$$

The Pick matrix $P \in \mathbb{C}^{k\times k}$ of $\mathbf{NP}_\kappa$ problem can be derived from the Stein equation (1.7):

$$P = \left[\frac{1 - \bar{S}_i S_j}{1 - \bar{z}_i z_j} \right]_{i,j=1}^k .$$

If P is invertible and has κ negative eigenvalues, then (again by the general result [6, Chapter 19]), the $\mathbf{NP}_\kappa$ problem has infinitely many solutions which are parametrized by the linear fractional transformation (1.12) with the matrix of coefficients

$$\Theta(z) := \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix} = \left(\prod_{i=1}^k (z - z_i) \right) \widehat{\Theta}(z),$$

where $\widehat{\Theta}$ is given via (1.10) in the setting of (5.2). Let p_{ij} denote the ij-th entry in the matrix P^{-1}:

$$P^{-1} = [p_{ij}]_{i,j=1}^k , \tag{5.3}$$

and let $\mathbf{m} = (m_1, \dots, m_k)$ be a fixed multiinteger. Then (1.24) takes the form

$$\widetilde{P}_{\mathbf{m}} = [p_{ij}]_{i,j \notin \mathcal{Z}_{\mathbf{m}}^0} \tag{5.4}$$

(where $\mathcal{Z}_{\mathbf{m}}^0$ is the set defined in (1.18)) and excluded parameters are characterized in terms of the latter matrix by Theorem 1.6, which in the present context reads as follows.

Theorem 5.1. *If $\mathcal{E}$ is an excluded parameter of multiorder $\mathbf{m}$, then the function $S = \mathbf{T}_\Theta[\mathcal{E}]$ belongs to the class $\mathcal{S}_{\kappa-\gamma_{\mathbf{m}}}$. It has poles of multiplicities $m_i - 1$ at z_i (if $m_i > 1$) and can is extended analytically to all other interpolating points. Moreover,*

$$S(z_i) = S_i \quad (\text{if } \ m_i = 0) \quad \text{and} \quad S(z_i) \neq S_i \quad (\text{if } \ m_i = 1).$$

Note that the entries p_{jj} appearing in Corollary 4.4 are precisely the diagonal entries of the matrix P^{-1} in (5.3). Thus, in the context of the $\mathbf{NP}_\kappa$ problem, Corollary 4.4 leads to certain "local" classification of excluded parameters:

1. If $p_{jj} > 0$ then for every excluded parameter of multiorder $\mathbf{m} = (m_1, \dots, m_k)$, it holds that $m_j = 0$.

2. If $p_{jj} = 0$, there is only one excluded parameter $\mathcal{E}_0$, which is a unimodular constant.
3. If $p_{jj} < 0$, then there are infinitely many excluded parameters with $m_j = 1$.

This classification was first obtained in [15] in the context of generalized Nevanlinna functions.

6. One point interpolation problem

In this section we consider another particular case of the $\mathbf{CF}_\kappa$ problem, when $k = 1$:

Given a point $z_0 \in \mathbb{D}$ and complex numbers $S_0, \ldots, S_{n-1}$, find all functions $S \in \mathcal{S}_\kappa$ which are analytic at z_0 and satisfy

$$S^{(j)}(z_0) = j!\, S_j, \qquad (j = 0, \ldots, n-1). \tag{6.1}$$

In this case (1.4) and (1.5) take the form

$$T = J_{n+1}(z_0), \quad E = L_{n+1} \quad \text{and} \quad C = \begin{bmatrix} S_0 & \cdots & S_{n-1} \end{bmatrix}, \tag{6.2}$$

In this case, we say that $\mathcal{E}$ is an excluded parameter of order m (multiorder is not needed anymore) if the function $V_{\mathcal{E}}$ has the zero of multiplicity m at z_0.

By Corollary 4.3, there are no excluded parameters of order greater than n. According to (1.18), $\gamma_m = m$ and (1.24) takes the form

$$\widetilde{P}_m = \begin{bmatrix} 0 & I_m \end{bmatrix} P^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix},$$

i.e., $\widetilde{P}_m$ is just the $m \times m$ bottom principal submatrix $\widetilde{P}$ of P^{-1}. Theorems 1.4 and 1.6 in the present context lead to

Theorem 6.1. *Let the Pick matrix P of the one point $\mathbf{CF}_\kappa$ problem be invertible and have κ negative eigenvalues. Then*

1. *There exists an excluded parameter of order m of the transformation* (1.12) *if and only if either the matrix $\widetilde{P}_m$ is negative definite or it is the maximal negative semidefinite (singular) bottom principal submatrix of P^{-1} (i.e., the matrix $\widetilde{P}_{m+1}$ has one positive eigenvalue).*
2. *If $\widetilde{P}_m < 0$, then there are infinitely many excluded parameters $\mathcal{E}$ of order m.*
3. *If $\widetilde{P}_m$ is the maximal negative semidefinite (singular) bottom principal submatrix of P^{-1}, then there is a unique excluded parameter $\mathcal{E}$ of order m, which is a Blaschke product of degree $r = \operatorname{rank}\widetilde{P}$.*
4. *If $\mathcal{E}$ is an excluded parameter of order m of the transformation* (1.12), *then the function $S = \mathbf{T}_\Theta[\mathcal{E}]$ belongs to the class $\mathcal{S}_{\kappa-m}$, admits an analytic continuation at z_0, satisfies interpolation conditions*

$$S^{(j)}(z_0) = j!\, S_j \quad (j = 0, \ldots, n-m-1). \tag{6.3}$$

and is subject to

$$S^{(n-m)}(z_0) \neq (n-m)!\, S_{n-m}. \tag{6.4}$$

7. Examples

In this section we illustrate the preceding results by two simple numerical examples. In the first example we consider a one point interpolation problem with three interpolation conditions.

Example 7.1. Let $z_0 = 0$, $S_0 = 1$, $S_1 = 1$ and $S_2 = 0$. Then Then the matrices in (1.4) and (1.5) take the form

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

and the unique solution P of the Stein equation (1.7) equals

$$P = -\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Furthermore,

$$P^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

and the formula (1.11) takes the form (we omit simple straightforward calculations)

$$\Theta(z) = \begin{bmatrix} 1+z-z^2 & z^3 - z \\ 1 - z^2 & z^3 + z^2 - z \end{bmatrix}.$$

Since $\mathrm{sq}_-(P) = 2$, it follows by Theorem 1.2, that all functions $S \in \mathcal{S}_2$ such that

$$S(0) = 1, \quad S'(0) = 1 \quad \text{and} \quad S''(0) = 0, \tag{7.1}$$

are parametrized by the formula

$$S(z) = \frac{(1+z-z^2)\mathcal{E}(z) + z^3 - z}{(1-z^2)\mathcal{E}(z) + z^3 + z^2 - z}, \tag{7.2}$$

where the parameter $\mathcal{E}$ is an arbitrary Schur function, such that $\mathcal{E}(0) \neq 0$. Since the right bottom entry in P^{-1} is negative and since the 2×2 bottom principal submatrix of P^{-1} is the maximal negative semidefinite submatrix of P^{-1}, it follows by Theorem 1.6 that there are infinitely many excluded parameters of order one and only one excluded parameter of order two of the transformation (7.2).

Excluded parameters $\mathcal{E}$ in the transformation (7.2) are characterized by equality $\mathcal{E}(0) = 0$ and therefore, are of the form

$$\mathcal{E}(z) = z\widetilde{\mathcal{E}}(z), \quad \widetilde{\mathcal{E}} \in \mathcal{S}_0. \tag{7.3}$$

A function $\mathcal{E}$ of the form (7.3) is an excluded parameter of the first order if and only if

$$\widetilde{\mathcal{E}}(0) \neq 1. \tag{7.4}$$

Substituting (7.3) into (7.2) we get

$$S(z) = \frac{(1+z-z^2)\widetilde{\mathcal{E}}(z) + z^2 - 1}{(1-z^2)\widetilde{\mathcal{E}}(z) + z^2 + z - 1}.$$

It is easily verified, that under assumption (7.4),

$$S(0) = S'(0) = \frac{\widetilde{\mathcal{E}}(0) - 1}{\widetilde{\mathcal{E}}(0) - 1} = 1, \quad S''(0) = -\frac{1}{\widetilde{\mathcal{E}}(0) - 1} \neq 0,$$

and thus, S satisfies only the two first interpolation conditions in (7.1) and does not satisfy the third. All these conclusions follow immediately from the last statement in Theorem 1.3.

The only excluded parameter $\mathcal{E}$ of the second order is subject to

$$\mathcal{E}(0) = 0 \quad \text{and} \quad \mathcal{E}'(0) = 1$$

and since $\mathcal{E} \in \mathcal{S}_0$, it follows that $\mathcal{E}(z) = z$. The corresponding function S equals

$$S(z) = \frac{1 + z - z^2 + z^2 - 1}{1 - z^2 + z^2 + z - 1} \equiv 1$$

and has zero negative squares. Furthermore, it satisfies the first condition in (7.1) and does not satisfy the second. Note that the third condition is satisfied, which is not guaranteed by Theorem 1.3, but may happen.

The next example treats a two-point Nevanlinna–Pick problem.

Example 7.2. Let $z_1 = 0$, $z_2 = -\frac{1}{2}$, $S_{1,0} = 2$ and $S_{2,0} = \frac{3}{2}$. Then the matrices in (1.4) and (1.5) take the form

$$T = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & \frac{3}{2} \end{bmatrix}$$

and the unique solution P of the Stein equation (1.7) equals

$$P = \begin{bmatrix} -3 & -2 \\ -2 & -\frac{5}{3} \end{bmatrix}.$$

It is easily seen that $\mathrm{sq}_-(P) = 2$ and thus, the corresponding multipoint interpolation problem consists of finding all functions $S \in \mathcal{S}_2$ such that

$$S(0) = 2 \quad \text{and} \quad S(-1/2) = 3/2. \tag{7.5}$$

Furthermore,

$$P^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -\frac{5}{3} \end{bmatrix}$$

and according to (1.11),

$$\Theta(z) = \begin{bmatrix} -(z-4)(z+2) & 2(z-1)(2z+1) \\ -2(z-1)(z+2) & (4z-1)(2z+1) \end{bmatrix}.$$

All functions $S \in \mathcal{S}_2$ satisfying interpolation conditions (7.5) are parametrized by the formula

$$S(z) = \mathbf{T}_\Theta[\mathcal{E}] = \frac{-(z-4)(z+2)\mathcal{E}(z) + 2(z-1)(2z+1)}{-2(z-1)(z+2)\mathcal{E}(z) + (4z-1)(2z+1)}, \tag{7.6}$$

where the parameter $\mathcal{E}$ is an arbitrary Schur function, such that

$$\mathcal{E}(0) \neq 1/4 \quad \text{and} \quad \mathcal{E}(-1/2) \neq 0.$$

By Corollary 4.3 there are no excluded parameters of multiorder $\mathbf{m}$ with $|\mathbf{m}| > 2$. Since the diagonal entries in P^{-1} are negative, there are infinitely many excluded parameters of multiorders $(1,0)$ and $(0,1)$. Furthermore, since $\widetilde{P}_{(1,1)} = P^{-1} < 0$, there are infinitely many excluded parameters $\mathcal{E} \in \mathcal{S}_0$ of multiorder $(1,1)$. Each such parameter is subject to interpolation conditions

$$\mathcal{E}(0) = 1/4 \quad \text{and} \quad \mathcal{E}(-1/2) = 0$$

and thus, all excluded parameters of multiorder $(1,1)$ are parametrized by the formula

$$\mathcal{E}(z) = \mathbf{T}_\Phi[\widetilde{\mathcal{E}}] = \frac{2z+1}{z+2} \cdot \frac{(4z-1)\widetilde{\mathcal{E}}(z) + 2(1-z)}{2(z-1)\widetilde{\mathcal{E}}(z) - (z-4)},$$

when $\widetilde{\mathcal{E}}$ varies over $\mathcal{S}_0$. It is easily seen that the last transformation is inverse to the transformation (7.6) and therefore,

$$S = \mathbf{T}_\Theta[\mathcal{E}] = \mathbf{T}_\Theta[\mathbf{T}_\Phi[\widetilde{\mathcal{E}}]] = \widetilde{\mathcal{E}}.$$

Thus, every excluded parameter of multiorder $(1,1)$ leads to a Schur function $S \in \mathcal{S}_0$ which furthermore, satisfies no condition in (7.5).

Next, we note that there is no excluded parameter $\mathcal{E}$ of multiorder $(0,2)$; if it existed, the denominator in (7.6) would have the zero of multiplicity two at $z = -\frac{1}{2}$, or equivalently, $\mathcal{E}$ would meet conditions

$$\mathcal{E}(-1/2) = 0 \quad \text{and} \quad \mathcal{E}'(-1/2) = 4/3,$$

which is impossible, since $\mathcal{E} \in \mathcal{S}_0$.

Finally, it turns out that the denominator in (7.6) may have zero of order two at $z = 0$, or equivalently, there are Schur functions $\mathcal{E}$ such that

$$\mathcal{E}(0) = 1/4 \quad \text{and} \quad \mathcal{E}'(0) = -3/8.$$

They are parametrized by the formula

$$\mathcal{E}(z) = \frac{(20z^2 - 2z)\widetilde{\mathcal{E}}(z) + 5 - 8z}{(5z^2 - 8z)\widetilde{\mathcal{E}}(z) + 20 - 2z}. \tag{7.7}$$

Substituting (7.7) into (7.6) we get

$$S(z) = \frac{1}{z} \cdot \frac{18z\widetilde{\mathcal{E}}(z) + 7z - 10}{(7-10z)\widetilde{\mathcal{E}}(z) + 18}. \tag{7.8}$$

Evaluating the last equality at $z = -1/2$, we get $S(-1/2) = 3/2$. Thus, every excluded parameter of multiorder $(2,0)$ leads to a function S which has a simple pole at the origin and satisfies the interpolation condition at the point $z = -1/2$.

8. Conclusive remarks

In conclusion we give some justification of the choice (1.11) of the matrix Θ of coefficients of the linear fractional transformation (1.12). The matrix of coefficients of a linear fractional transformation is not defined uniquely; any multiple of Θ by a scalar valued function (e.g., the function $\widehat{\Theta}$ given by (1.10)) obviously gives rise to the same linear fractional transformation. Looking back, one can see, however, that the following three properties of Θ were helpful for obtaining characterization of excluded parameters $\mathcal{E}$ and investigation of corresponding functions $\mathbf{T}_\Theta[\mathcal{E}]$:

1. Θ is a scalar multiple of a J-unitary (on $\mathbb{T}$) matrix function.
2. Θ is analytic on the closed unit disk $\overline{\mathbb{D}}$, which guarantees that the numerator $U_\mathcal{E}$ and the denominator $V_\mathcal{E}$ in (1.12) are analytic on $\overline{\mathbb{D}}$.
3. $\det\Theta$ has zeroes of multiplicities n_i at z_i for $i = 1,\dots,k$ which implies $|\Theta_{21}(z_i)|^2 + |\Theta_{22}(z_i)|^2 > 0$ and thus, the denominator in (1.12) vanishes at z_i just for some (not all) parameters $\mathcal{E}$.

To get a function Θ with all these properties we have multiplied the J-unitary (on $\mathbb{T}$) function $\widehat{\Theta}$ by an appropriate polynomial in (1.11). Furthermore, any function $\phi\Theta$ where $\phi(z)$ is a scalar valued function analytic on $\overline{\mathbb{D}}$ which does not vanish at z_i $(i = 1,\dots,k)$ has properties $1-3$ and therefore, can be used as the matrix of coefficients in the parametrization formula (1.12) and all the above constructions. We would like to emphasize one special choice of ϕ. In what follows, $A^t = (a_{ji})$ and $\overline{A} = (\bar{a}_{ij})$ stand for the transpose and for the complex conjugate of a matrix $A = (a_{ij})$.

Lemma 8.1. *Let T, E, C, P and Θ be given by* (1.4)–(1.6) *and* (1.11)*. Then the function*

$$\widetilde{\Theta}(z) = \left[\widetilde{\Theta}_{ij}(z)\right]_{ij=1}^{2} = I_2 + (z-1)\begin{bmatrix} \frac{E}{C} \end{bmatrix}(I - z\overline{T})^{-1}\overline{P}^{-1}(I - T^t)^{-1}\begin{bmatrix} E^* & -C^t \end{bmatrix} \tag{8.1}$$

satisfies

$$\widetilde{\Theta}(z) = \phi(z)\Theta(z), \quad \textit{where} \quad \phi(z) = \prod_{i=1}^{k}\left(\frac{1-\bar{z}_i}{(1-z\bar{z}_i)(1-z_i)}\right)^{n_i}. \tag{8.2}$$

Proof: Let $\widehat{\Theta}$ be the function given by (1.10). By (2.12),

$$\widehat{\Theta}^{-1} := \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} = \phi\cdot\begin{bmatrix} \Theta_{22} & -\Theta_{12} \\ -\Theta_{21} & \Theta_{11} \end{bmatrix}$$

and thus,

$$\mathbf{T}_\Theta[\mathcal{E}] = \frac{\Phi_{22}\mathcal{E} - \Phi_{12}}{-\Phi_{21}\mathcal{E} + \Phi_{11}}. \tag{8.3}$$

Using explicit formulas for scalar entries in (4.12) and (8.1) we get

$$\begin{aligned} \Phi_{11}(z) &= 1 - (z-1)C(I-T)^{-1}P^{-1}(I-zT^*)^{-1}C^* \\ &= \left(1 - (z-1)C(I-T)^{-1}P^{-1}(I-zT^*)^{-1}C^*\right)^t \\ &= 1 - (z-1)\overline{C}(I-z\overline{T})^{-1}\overline{P}^{-1}(I-T^t)^{-1}C^t = \widetilde{\Theta}_{22}(z) \end{aligned}$$

and quite similarly,

$$\Phi_{12}(z) = -\widetilde{\Theta}_{12}(z), \quad \Phi_{21}(z) = -\widetilde{\Theta}_{21}(z), \quad \Phi_{22}(z) = \widetilde{\Theta}_{11}(z).$$

Substituting the four last relations into (8.3) we conclude that $\mathbf{T}_\Theta[\mathcal{E}] = \mathbf{T}_{\widetilde{\Theta}}[\mathcal{E}]$ for every parameter $\mathcal{E} \in \mathcal{S}_0$. Therefore, $\widetilde{\Theta} = \phi\Theta$ for some scalar valued function ϕ. It remains to show that ϕ is of the form (8.2). To this end, we conclude from (1.10) that

$$\begin{aligned} \widehat{\Theta}_{11}(z) &= 1 - (1-z)C(zI-T)^{-1}P^{-1}(I-T^*)^{-1}C^* \\ &= \det\left(I - (1-z)(zI-T)^{-1}P^{-1}(I-T^*)^{-1}C^*C\right) \\ &= \det\left[(zI-T)^{-1}P^{-1}(I-T^*)^{-1}\right] \\ &\qquad \cdot \det\left[(I-T^*)P(zI-T) - (1-z)C^*C\right] \end{aligned}$$

and on the other hand, we get from (8.1)

$$\begin{aligned} \widetilde{\Theta}_{11}(z) &= 1 + (z-1)E(I-z\overline{T})^{-1}\overline{P}^{-1}(I-T^t)^{-1}E^* \\ &= 1 + (z-1)E(I-T)P^{-1}(I-zT^*)^{-1}(I-T^t)^{-1}E^* \\ &= \det\left(I + (z-1)(I-T)P^{-1}(I-zT^*)^{-1}(I-T^t)^{-1}E^*E\right) \\ &= \det\left[(I-T)^{-1}P^{-1}(I-zT^*)^{-1}\right] \\ &\qquad \cdot \det\left[(I-zT^*)P(I-T) + (1-z)E^*E\right]. \end{aligned}$$

It follows immediately from (1.7) that

$$(I-T^*)P(zI-T) - (1-z)C^*C = (I-zT^*)P(I-T) + (1-z)E^*E$$

and therefore,

$$\frac{\widetilde{\Theta}_{11}(z)}{\widehat{\Theta}_{11}(z)} = \frac{\det\left[(I-T^*)P(zI-T)\right]}{\det\left[(I-zT^*)P(I-T)\right]},$$

which in turn, implies, on account of (1.11),

$$\frac{\widetilde{\Theta}_{11}(z)}{\Theta_{11}(z)} = \prod_{i=1}^{k}\left(\frac{1-\bar{z}_i}{(1-z\bar{z}_i)(1-z_i)}\right)^{n_i} = \phi(z),$$

and completes the proof. □

Since $\phi(z_i) \neq 0$ for $i = 1, \ldots, k$, the transformations (1.12) and

$$\mathcal{E} \to \mathbf{T}_{\widetilde{\Theta}}[\mathcal{E}] \tag{8.4}$$

have the same excluded parameters. Due to a nice realization formula (1.12), the function $\widetilde{\Theta}$ appears in interpolation literature side by side with the function Θ introduced in (1.11). It turns out that in the matrix valued setting (we refer to [6] for details) the description (1.12) suits more to the left sided interpolation problem, whereas the parametrization (8.4) is more appropriate for the right sided problem. Of course, these tangential problems are equivalent in the present scalar valued setting and we could use $\widetilde{\Theta}$ instead of Θ in all preceding considerations. Our choice was caused by the advantage of the fact that Θ is a matrix polynomial (whereas $\widetilde{\Theta}$ is a rational function), which allowed us to make some calculations simpler.

Aknowledgements

The author gratefully aknowledges Prof. L. Rodman and Prof. A. Kheifets for valuable discussions.

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