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A boundary Nevanlinna–Pick problem for a class of analytic matrix-valued functions in the unit ball ☆

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Abstract

We solve a tangential boundary interpolation problem with a finite number of interpolating points for a multivariable analogue of the Schur class. The description of all solutions is parametrized in terms of a linear fractional transformation whose entries are given explicitly in terms of the interpolation data. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

In this paper, we consider a boundary Nevanlinna–Pick interpolation problem for a multivariable analogue $\mathscr{S}_d^{p\times q}$ of the Schur class. This class consists, by definition, of all $\mathbb{C}^{p\times q}$-valued functions S analytic in the unit ball

$$\mathbb{B}^d = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d \colon \sum_{j=1}^{d} |z_j|^2 < 1 \right\}$$

of $\mathbb{C}^d$ and such that the kernel

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$$K_S(z,w) = \frac{I_p - S(z)S(w)^*}{1 - \langle z, w\rangle} \tag{1.1}$$

is positive on $\mathbb{B}^d$. The latter means that

$$\sum_{j,\ell=1}^{n} c_j^* K_S(z^{(j)}, z^{(\ell)}) c_\ell \geqslant 0$$

for every choice of an integer n, of vectors $c_1, \ldots, c_n \in \mathbb{C}^p$ and of points $z^{(1)}, \ldots, z^{(n)} \in \mathbb{B}^d$ or equivalently, that the Hermitian block matrix with ℓjth entry $K_S(z^{(j)}, z^{(\ell)})$ is positive semidefinite. This property will be denoted by $K_S(z,w) \succeq 0$. Note that positivity of K_S on $\mathbb{B}^d$ characterizes $S(z)$ as a contractive multiplier on the Arveson space [2].

For $d = 1$, the class $\mathscr{S}_1^{p\times q}$ coincides with the classical Schur class (which consists of functions analytic and contractive-valued on the unit disk $\mathbb{D} = \mathbb{B}^1$). However, a $\mathbb{C}^{p\times q}$-valued function S, analytic and contractive-valued in $\mathbb{B}^d$, is characterized by the positive kernel

$$\widetilde{K}_S(z,w) = \frac{I_p - S(z)S(w)^*}{(1 - \langle z, w\rangle)^d}$$

on $\mathbb{B}^d$, which is equal to K_S only for $d = 1$. If $d \geqslant 2$, the Schur class $\mathscr{S}_d^{p\times q}$ is contained properly in the set of analytic contractive-valued functions on $\mathbb{B}^d$.

Points in $\mathbb{C}^d$ will be denoted by $z = (z_1, \ldots, z_d)$, where $z_j \in \mathbb{C}$. In (1.1) and throughout the paper

$$\langle z, w\rangle = \sum_{j=1}^{d} z_j \bar{w}_j \quad (z, w \in \mathbb{C}^d)$$

stands for the standard inner product in $\mathbb{C}^d$. The unit sphere $\partial(B^d)$ will be denoted by $\mathbb{S}^d$.

The Nevanlinna–Pick interpolation problem (to find necessary and sufficient conditions which insure the existence of a function $S \in \mathscr{S}_d^{p\times q}$ taking prescribed values at prescribed points in $\mathbb{B}^d$) has been considered in [1,16,19]. It was shown that (similarly to the one variable case [17,18]) the problem has a solution if and only if the *Pick matrix* of the problem is positive semidefinite. The complete description of the set of all solutions of the tangential Nevanlinna–Pick problem was first obtained in [7]. It was shown that every solution of the problem corresponds to a unitary extension of a partially defined isometric operator, which led to a parametrization of all solutions in terms of a Redheffer linear fractional transformation. It turns out that similar ideas lead to a description of all solutions for much more general bitangential interpolation problem [4]. Another approach, based on the method of fundamental matrix inequalities, has been suggested in [8] for a general tangential interpolation problem. In this paper, we consider a Nevanlinna–Pick type problem when the interpolating points are on the unit sphere $\mathbb{S}^d$ and the prescribed values of contractive multipliers are replaced by radial boundary limits. For the one variable case such a problem (as well as more general problems involving boundary derivatives of higher orders) was considered in [3,5,6,10,11,14].

The boundary Nevanlinna–Pick problem in the class $\mathscr{S}_d^{p\times q}$ can be formulated as follows:

Problem 1.1. Given m distinct points $\beta^{(1)} = (\beta_1^{(1)}, \dots, \beta_d^{(1)}), \dots, \beta_m = (\beta_1^{(m)}, \dots, \beta_d^{(m)})$ on the unit sphere $\mathbb{S}^d$, given vectors $\xi_j \in \mathbb{C}^p$, $\eta_j \in \mathbb{C}^q$ and given numbers γ_j, find necessary and sufficient conditions which insure the existence of a function $S \in \mathscr{S}_d^{p\times q}$ which has prescribed radial boundary limits

$$\lim_{r\to 1} S(r\beta^{(j)})^*\xi_j = \eta_j \quad (j = 1, \dots, m) \tag{1.2}$$

and prescribed upper bounds for radial angular derivatives

$$\lim_{r\to 1} \xi_j^* \frac{I_p - S(r\beta^{(j)})S(r\beta^{(j)})^*}{1 - r^2} \xi_j \leqslant \gamma_j \quad (j = 1, \dots, m). \tag{1.3}$$

The following result which can be considered as a multivariable matrix analogue of the classical Julia–Carathéodory theorem will be useful for the subsequent analysis.

Theorem 1.2. *Let $S \in \mathscr{S}^{p\times q}$, $\beta \in \mathbb{S}^d$ and $\xi \in \mathbb{C}^p$. Then:*
I. *The following three statements are equivalent:*
(1) *S is subject to*

$$\sup_{0\leqslant r<1} \xi^* \frac{I_p - S(r\beta)S(r\beta)^*}{1 - r^2} \xi < \infty.$$

(2) *The radial limit*

$$L_1 := \lim_{r\to 1} \xi^* \frac{I_p - S(r\beta)S(r\beta)^*}{1 - r^2} \xi$$

exists.
(3) *The radial limit*

$$\lim_{r\to 1} S(r\beta)^*\xi = \eta \tag{1.4}$$

exists and serves to define the vector $\eta \in \mathbb{C}^q$. Furthermore,

$$\lim_{r\to 1} S(r\beta)\eta = \xi, \quad \xi^*\xi = \eta^*\eta, \tag{1.5}$$

and the radial limit

$$L_2 = \lim_{r\to 1} \frac{\xi^*\xi - \xi^*S(r\beta)\eta}{1 - r}$$

exists.

II. *If any of the preceding three statements holds true, then* $L_1 = L_2$.
III. *Any two of the three equalities in* (1.4) *and* (1.5) *imply the third.*

Proof. For the proof of all the statements for the single-variable case ($d = 1$) see [12, Lemma 8.3 and Theorem 8.5]. For the case $d \geqslant 2$, let us introduce the slice-function

$$S_\beta(\zeta) := S(\zeta\beta) \quad (\zeta \in \mathbb{D}),$$

which clearly belongs to the classical Schur class $\mathscr{S}_1^{p\times q}$. Applying one-variable results to this function and returning to the original function S, we obtain all the desired assertions. $\square$

Note that tangential analogues of Julia–Carathéodory theorem (including boundary derivatives of higher orders) were considered also in [13] and [10, Section 8]. Multivariable analogues of Julia–Carathéodory theorem can be found in [20, Section 8.5] (although the results from [20] are related to holomorphic maps from $\mathbb{B}^{d_1}$ into $\mathbb{B}^{d_2}$ rather than to functions from the class $\mathscr{S}_d^{p\times q}$, they suggest a different formulation of a boundary Nevanlinna–Pick problem with the radial limits in (1.2) and (1.3) replaced by weak K-limits; such a problem is planned to be considered elsewhere).

Conditions (1.2) are called *left-sided* interpolation conditions for S. It follows from Theorem 1.2 that if the limits in (1.2) exist and equal η_j, then the necessary condition for the limits in (1.3) to exist and to be finite is

$$\xi_j^*\xi_j = \eta_j^*\eta_j \quad (j = 1, \dots, m). \tag{1.6}$$

Now it follows from the third assertion in Theorem 1.2 that S satisfies also right-sided interpolation conditions

$$\lim_{r\to 1} S(r\beta^{(j)})\eta_j = \xi_j \quad (j = 1, \dots, m).$$

Thus, Problem 1.1 is in fact a two-sided interpolation problem and conditions (1.6) are necessary for this problem to have a solution. As in the case of one variable, the solvability criterion of Problem 1.1 can be formulated in terms of the *Pick matrix P* constructed from the interpolation data.

Theorem 1.3. *Problem* 1.1 *has a solution if and only if equalities* (1.6) *hold and the matrix*

$$P = \left(P_{j\ell}\right)_{j,\ell=1}^m, \quad \text{where } P_{j\ell} = \begin{cases} \dfrac{\xi_j^*\xi_\ell - \eta_j^*\eta_\ell}{1 - \langle \beta^{(j)}, \beta^{(\ell)} \rangle}, & j \neq \ell, \\ \gamma_j, & j = \ell, \end{cases} \tag{1.7}$$

is positive semidefinite or equivalently, if and only if the matrix P defined in (1.7) *is a positive semidefinite solution of the generalized Stein equation*

$$P - A_1^*PA_1 - \cdots - A_d^*PA_d = C^*JC, \tag{1.8}$$

where

$$J=\begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad C=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}=\begin{bmatrix} \xi_1 & \cdots & \xi_m \\ \eta_1 & \cdots & \eta_m \end{bmatrix} \tag{1.9}$$

and

$$A_j=\begin{bmatrix} \overline{\beta}_j^{(1)} & & 0 \\ & \ddots & \\ 0 & & \overline{\beta}_j^{(m)} \end{bmatrix} \quad (j=1,\ldots,d). \tag{1.10}$$

The proof will be given in Section 4. Here, we note only that P defined as in (1.7) satisfies the Stein equation (1.8) if and only if conditions (1.6) hold true. Indeed, by (1.9) and (1.10), all the nondiagonal entries of P are uniquely determined from (1.9) and are the same as in (1.7). The *jj*th entry in the matrix on the left-hand side of (1.9) is equal to

$$\gamma_j-\bar{\beta}_1^{(j)}\gamma_j\beta_1^{(j)}-\cdots-\bar{\beta}_d^{(j)}\gamma_j\beta_d^{(j)}=(1-\langle\beta^{(j)},\beta^{(j)}\rangle)\gamma_j=0,$$

whereas the *jj* entry in the matrix on the right-hand side equals to $\xi_j^*\xi_j-\eta_j^*\eta_j$. Thus, conditions (1.6) are equivalent to (1.9).

Note also that conditions (1.6) are necessary and sufficient for the existence of a function $S\in\mathscr{S}_d^{p\times q}$ which satisfies interpolation conditions (1.2) and has finite radial derivatives

$$\lim_{r\to 1}\xi_j^*\frac{I_p-S(r\beta^{(j)})S(r\beta^{(j)})^*}{1-r^2}\xi_j<\infty \quad (j=1,\ldots,m).$$

This result follows immediately from Theorem 1.3, since one can always choose diagonal entries $\gamma_1,\ldots,\gamma_m$ in (1.7) so that P will be positive definite.

The following theorem (which will be proved in Section 4) gives a parametrization of all solutions to Problem 1.1.

Theorem 1.4. *Let P be a positive definite solution of the Stein equation* (1.7), *let* $\operatorname{rank} P=n\leqslant m$ *and let*

$$\nu=\operatorname{rank}(P+C_2^*C_2)-\operatorname{rank} P. \tag{1.11}$$

Then there exists a rational $(p+q)\times(nd+p+q)$ matrix-valued function $\Phi(z)$,

$$\Phi=\begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}:\begin{pmatrix} \mathbb{C}^{nd+p} \\ \mathbb{C}^q \end{pmatrix}\to\begin{pmatrix} \mathbb{C}^p \\ \mathbb{C}^q \end{pmatrix},$$

which defines a map via the linear fractional transformation

$$S(z)=\big(\Phi_{11}(z)\mathscr{E}(z)+\Phi_{12}(z)\big)\big(\Phi_{21}(z)\mathscr{E}(z)+\Phi_{22}(z)\big)^{-1} \tag{1.12}$$

from the set of all functions $\mathscr{E} \in \mathscr{S}_d^{(nd+p)\times q}$ *of the form*

$$\mathscr{E}(z) = \begin{pmatrix} \widehat{\mathscr{E}}(z) & 0 \\ 0 & I_\nu \end{pmatrix}, \quad \widehat{\mathscr{E}} \in \mathscr{S}_d^{(nd+p-\nu)\times(q-\nu)}, \tag{1.13}$$

onto the set of all solutions $S(z)$ *of Problem* 1.1.

In other words, S is a solution of Problem 1.1 if and only if it admits a representation (1.12) for some parameter $\mathscr{E}$ of the form (1.13).

Problem 1.1 is termed *nondegenerate* if the corresponding Pick matrix is positive definite and *degenerate* if P is positive semidefinite.

The paper is organized as follows. Section 2 characterizes the set of all solutions of Problem 1.1 in terms of a positive kernel, Section 3 presents a description of the all solutions of a nondegenerate Problem 1.1 in terms of a linear fractional transformation, and Section 4 treats the degenerate case.

2. Fundamental matrix inequality

In this section, we characterize all the solutions S of Problem 1.1 in terms of a certain positive kernel. This characterization develops Potapov's method of the fundamental matrix inequality (which characterizes the solutions of an interpolation problem in terms of a related fundamental matrix inequality; see e.g. [15]). In [9] a general boundary interpolation problem was considered for matrix-valued Schur functions of the unit disk (i.e., in the class $S_1^{p\times q}$) involving prescribed angular derivatives of higher orders. Here we present a very special case of [9, Theorem 3.1] which is needed for the subsequent analysis (see also [9, Section 8] for more details).

Theorem 2.1. *Let S be a $\mathbb{C}^{p\times q}$-valued function analytic in the open unit disk $\mathbb{D}$. Then S belongs to the class $S_1^{p\times q}$ and meets the interpolation conditions*

$$\lim_{r\to 1} S(r)^*\xi = \eta \quad \textit{and} \quad \lim_{r\to 1} \xi^* \frac{I_p - S(r)S(r)^*}{1-r^2}\xi \leqslant \gamma$$

if and only if the following kernel is positive on the unit disk $\mathbb{D}$:

$$\begin{bmatrix} \gamma & \dfrac{\xi^* - \eta^* S(\omega)^*}{1-\bar{\omega}} \\[2ex] \dfrac{\xi - S(\zeta)\eta}{1-\zeta} & \dfrac{I_p - S(\zeta)S(\omega)^*}{1-\zeta\bar{\omega}} \end{bmatrix} \succeq 0 \quad (\zeta, \omega \in \mathbb{D}).$$

Remark 2.2. Let $A_1, \ldots, A_d$ be matrices defined in (1.10) and let G be the function given by

$$G(z) = I_m - z_1 A_1 - \cdots - z_d A_d. \tag{2.1}$$

By (1.10),

$$G(z) = \begin{bmatrix} 1 - \langle z, \beta^{(1)} \rangle & & 0 \\ & \ddots & \\ 0 & & 1 - \langle z, \beta^{(m)} \rangle \end{bmatrix} \tag{2.2}$$

and therefore, it is invertible at every point $z \in \mathbb{C}^d$ for which $\langle z, \beta^{(j)} \rangle \neq 1$ for $j = 1, \ldots, m$. In particular, G is invertible at every point $z \in \overline{\mathbb{B}}^d \backslash \{\beta^{(1)}, \ldots, \beta^{(m)}\}$.

Theorem 2.3. *Let S be a $\mathbb{C}^{p \times q}$-valued function analytic in $\mathbb{B}^d$ and let P, C, A_j and G be defined by* (1.7), (1.9), (1.10) *and* (2.1), *respectively. Then S is a solution to Problem* 1.1 *if and only if the following kernel is positive on $\mathbb{B}^d$:*

$$\mathbf{S}(z, w) := \begin{bmatrix} P & G(w)^{-*}(C_1^* - C_2^* S(w)^*) \\ (C_1 - S(z)C_2)G(z)^{-1} & K_S(z, w) \end{bmatrix} \succeq 0. \tag{2.3}$$

Proof. Let S belong to $\mathscr{S}_d^{p \times q}$ and satisfy (1.2). Fix a number $r \in (0, 1)$ and m points

$$w^{(j)} = r\beta^{(j)}, \quad j = 1, \ldots, m, \tag{2.4}$$

in $\mathbb{B}^d$. Since the kernel K_S is positive on $\mathbb{B}^d$, it follows that

$$\mathbf{K}_r(z, w) := \begin{bmatrix} \mathbb{K}_r & \Psi_r(w)^* \\ \Psi_r(z) & K_S(z, w) \end{bmatrix} \succeq 0 \quad (z, w \in \mathbb{B}^d),$$

where

$$\begin{aligned} \mathbb{K}_r &= \left[K_S(w^{(j)}, w^{(\ell)}) \right]_{j,\ell=1}^m, \\ \Psi_r(z) &= \left[K_S(z, w^{(1)}) \quad \cdots \quad K_S(z, w^{(m)}) \right] \end{aligned} \tag{2.5}$$

(the dependence of $\mathbb{K}_r$ and Ψ_r on r is conditioned by (2.4)). Let

$$T = \mathrm{diag}[\xi_1, \ldots, \xi_m]. \tag{2.6}$$

Then clearly,

$$\begin{bmatrix} T^* \mathbb{K}_r T & T^* \Psi_r(w)^* \\ \Psi_r(z) T & K_S(z, w) \end{bmatrix} = \begin{bmatrix} T^* & 0 \\ 0 & I_p \end{bmatrix} \mathbf{K}_r(z, w) \begin{bmatrix} T & 0 \\ 0 & I_p \end{bmatrix} \succeq 0 \quad (z, w \in \mathbb{B}^d). \tag{2.7}$$

By (1.2) and (1.7),

$$\begin{aligned} \lim_{r \to 1} \xi_\ell^* K_S(w^{(\ell)}, w^{(j)}) \xi_j &= \lim_{r \to 1} \xi_\ell^* \frac{I_p - S(r\beta^{(\ell)}) S(r\beta^{(j)})^*}{1 - \langle r\beta^{(\ell)}, r\beta^{(j)} \rangle} \xi_j \\ &= \frac{\xi_\ell^* \xi_j - \eta_\ell^* \eta_j}{1 - \langle \beta^{(\ell)}, \beta^{(j)} \rangle} \quad (j \neq \ell), \end{aligned} \tag{2.8}$$

$$
\begin{aligned}
\lim_{r\to 1} \xi_j^* K_S(w^{(j)}, w^{(j)})\xi_j &= \lim_{r\to 1} \xi_j^* \frac{I_p - S(r\beta^{(j)})S(r\beta^{(j)})^*}{1-r^2}\xi_j \\
&= \widetilde{\gamma}_j \leqslant \gamma_j, \qquad (2.9)\\
\lim_{r\to 1} K_S(z, w^{(j)})\xi_j &= \lim_{r\to 1} \frac{I_p - S(z)S(r\beta^{(j)})^*}{1-\langle z, r\beta^{(j)}\rangle}\xi_j \\
&= \frac{\xi_j - S(z)\eta_j}{1-\langle z, \beta^{(j)}\rangle} \quad (\ell, j = 1, \ldots, m), \qquad (2.10)
\end{aligned}
$$

which imply, on account of partitionings (2.5) and (2.6), the existence of the following radial limits:

$$
\widetilde{P} := \lim_{r\to 1} T^*\mathbb{K}_r T \quad \text{and} \quad \Psi(z) = \lim_{r\to 1} \Psi_r(z)T. \tag{2.11}
$$

Moreover, it follows from (2.8) and (2.9) that

$$
\widetilde{P} = \left(\widetilde{P}_{j\ell}\right)_{j,\ell=1}^m, \quad \text{where } \widetilde{P}_{j\ell} = \begin{cases} \dfrac{\xi_j^*\xi_\ell - \eta_j^*\eta_\ell}{1-\langle \beta_j, \beta_\ell\rangle}, & j \neq \ell, \\ \widetilde{\gamma}_j, & j = \ell, \end{cases} \tag{2.12}
$$

whereas (1.9), (2.2) and (2.10) lead to

$$
\begin{aligned}
\Psi(z) &= \begin{bmatrix} \dfrac{\xi_1 - S(z)\eta_1}{1-\langle z, \beta^{(1)}\rangle} & \cdots & \dfrac{\xi_m - S(z)\eta_m}{1-\langle z, \beta^{(m)}\rangle} \end{bmatrix} \\
&= (C_1 - S(z)C_2)\,(I_n - z_1A_1 - \cdots - z_dA_d)^{-1} \\
&= (C_1 - S(z)C_2)\,G(z)^{-1}. \qquad (2.13)
\end{aligned}
$$

Thus, taking the limit in (2.7) as $r \to 1$ we obtain, on account of (2.11)–(2.13),

$$
\begin{aligned}
&\lim_{r\to} \begin{bmatrix} T^*\mathbb{K}_r T & T^*\Psi_r(w)^* \\ \Psi_r(z)T & K_S(z,w) \end{bmatrix} \\
&\quad = \begin{bmatrix} \widetilde{P} & G(w)^{-*}(C_1^* - C_2^*S(w)^*) \\ (C_1 - S(z)C_2)G(z)^{-1} & K_S(z,w) \end{bmatrix} \succeq 0. \qquad (2.14)
\end{aligned}
$$

Comparing (2.12) and (1.7) we conclude that $\widetilde{P} \leqslant P$ and thus, (2.14) implies (2.3).

Conversely, let S be a $\mathbb{C}^{p\times q}$-valued function analytic in $\mathbb{B}^d$ for which the kernel (2.3) is positive on $\mathbb{B}^d$. Then, in particular, the kernel $K_S(z,w)$ is positive on $\mathbb{B}^d$ and thus, S belongs to $\mathscr{S}_d^{p\times q}$. The positivity of the kernel (2.3) implies also that the following kernels are positive on $\mathbb{B}^d$:

$$
\begin{bmatrix} \gamma_j & \dfrac{\xi_j^* - \eta_j^*S(w)^*}{1-\langle \beta^{(j)}, w\rangle} \\ \dfrac{\xi_j - S(z)\eta_j}{1-\langle z, \beta^{(j)}\rangle} & \dfrac{I_p - S(z)S(w)^*}{1-\langle z, w\rangle} \end{bmatrix} \succeq 0 \quad (j = 1, \ldots, m). \tag{2.15}
$$

Let us introduce the slice-functions

$$S_j(\zeta) = S(\zeta\beta^{(j)}) \quad (\zeta \in \mathbb{D}) \tag{2.16}$$

for $j = 1, \ldots, m$, which are analytic and contractive-valued on the open unit disk $\mathbb{D}$. Setting $z = \zeta\beta^{(j)}$ and $w = \omega\beta^{(j)}$ in (2.15), we obtain then that

$$\begin{bmatrix} \gamma_j & \dfrac{\xi_j^* - \eta_j^* S_j(\omega)^*}{1 - \bar{\omega}} \\ \dfrac{\xi_j - S_j(\zeta)\eta_j}{1 - \zeta} & \dfrac{I_p - S_j(\zeta)S_j(\omega)^*}{1 - \zeta\bar{\omega}} \end{bmatrix} \succeq 0 \quad (j = 1, \ldots, m).$$

By Theorem 2.1, S_j satisfies the interpolation conditions

$$\lim_{r \to 1} S_j(r)^* \xi_j = \eta_j$$

and

$$\lim_{r \to 1} \xi_j^* \frac{I_p - S_j(r)S_j(r)^*}{1 - r^2} \xi_j \leqslant \gamma_j \quad (j = 1, \ldots, m),$$

which immediately imply (1.2) and (1.3), by the definition (2.16) of S_j. $\square$

3. Description of all solutions in the nondegenerate case

By Theorem 2.3, the set of all solutions of Problem 1.1 coincides with the set of all functions $S \in \mathscr{S}_d^{p\times q}$ such that the kernel (2.3) is positive. In this section, we parametrize this set under the assumption that P is positive definite.

Let J be the signature matrix as in (1.9), let

$$\mathbf{J} = \begin{bmatrix} I_{md} & 0 \\ 0 & J \end{bmatrix}$$

and let Θ be a $\mathbb{C}^{(md+p)\times q}$-valued function analytic on $\mathbb{B}^d$. We say that Θ is $(J, \mathbf{J})$-*inner* in $\mathbb{B}^d$ if

$$\Theta(z)\mathbf{J}\Theta(z)^* \leqslant J \quad (z \in \mathbb{B}^d) \tag{3.1}$$

and

$$\Theta(z)\mathbf{J}\Theta(z)^* = J \tag{3.2}$$

at every point z on $\mathbb{S}^d$ at which Θ is analytic. The next lemma provides an example of a $(J, \mathbf{J})$-inner function.

Lemma 3.1. *Let P be a positive solution of the Stein equation* (1.8). *Then the function*

$$\begin{aligned} \Theta(z) = {} & \begin{bmatrix} 0 & \cdots & 0 & I_{p+q} \end{bmatrix} + CG(z)^{-1}P^{-1} \\ & \times \begin{bmatrix} (z_1 I_m - A_1^*)P^{1/2} & \cdots & (z_d I_m - A_d^*)P^{1/2} & -C^*J \end{bmatrix} \end{aligned} \tag{3.3}$$

is $(J, \mathbf{J})$*-inner in* $\mathbb{B}^d$, *analytic at every point* $z \in \overline{\mathbb{B}^d}$ *except for* $\beta^{(1)}, \ldots, \beta^{(m)}$ *and moreover*,

$$\frac{J - \Theta(z)\mathbf{J}\Theta(w)^*}{1 - \langle z, w\rangle} = CG(z)^{-1}P^{-1}G(w)^{-*}C^* \qquad (z, w \in \overline{\mathbb{B}^d}\backslash\{\beta^{(1)}, \ldots, \beta^{(m)}\}). \tag{3.4}$$

Proof. By definition (3.3), Θ is analytic at every point z at which $G(z)$ is invertible and thus, by Remark 2.2, it is analytic at every point $z \in \overline{\mathbb{B}^d}\backslash\{\beta^{(1)}, \ldots, \beta^{(m)}\}$. The proof of (3.4) is quite straightforward (see e.g. [8]) and relies on the identity (1.8) rather than on special structures of matrices P, C_1, C_2 and A_j. Relations (3.1) and (3.2) follow immediately from (3.4). □

Lemma 3.2. *Let a* $(J, \mathbf{J})$*-inner function* Θ *be analytic on an open set* $\mathscr{U} \subset \mathbb{B}^d$ *and continuous on* $\overline{\mathscr{U}}$. *Furthermore, let*

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} : \begin{bmatrix} \mathbb{C}^{md+p} \\ \mathbb{C}^q \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^p \\ \mathbb{C}^q \end{bmatrix} \tag{3.5}$$

be the partition of Θ *into four blocks of the indicated sizes. Then for every choice of* $\mathscr{E} \in \mathscr{S}^{(md+p)\times q}$, *the function* $(\Theta_{21}\mathscr{E} + \Theta_{22})^{-1}$ *is uniformly bounded in norm on* $\mathscr{U}$.

Proof. Substituting the partitioning (3.5) of Θ together with conformal partitionings of J and $\mathbf{J}$ into (3.1) and (3.2) and comparing the 22 blocks we conclude that

$$\Theta_{21}(z)\Theta_{21}(z)^* - \Theta_{22}(z)\Theta_{22}(z)^* \leqslant -I_q \quad (z \in \overline{\mathscr{U}}).$$

The last inequality implies that

$$\Theta_{22}\Theta_{22}^* \geqslant I_q + \Theta_{21}\Theta_{21}^* \quad \text{and} \quad \Theta_{22}^{-1}\Theta_{21}\Theta_{21}^*\Theta_{22}^{-*} \leqslant I_q - \Theta_{22}^{-1}\Theta_{22}^{-*}.$$

Therefore, the functions Θ_{22}, Θ_{22}^{-1} and $\Theta_{22}^{-1}\Theta_{21}$ are, respectively, invertible, continuous and strictly contractive at every point $z \in \overline{\mathscr{U}}$. Let

$$\max_{z\in\overline{\mathscr{U}}} \|\Theta_{22}^{-1}(z)\| = M \quad \text{and} \quad \max_{z\in\overline{\mathscr{U}}} \|\Theta_{22}^{-1}(z)\Theta_{21}(z)\| = \delta < 1.$$

Since $\mathscr{E}$ takes contractive values on $\mathbb{B}^d$, it follows that for every $z \in \mathscr{U}$,

$$\begin{aligned} \|(\Theta_{21}(z)\mathscr{E}(z) + \Theta_{22}(z))^{-1}\| &\leqslant \|\Theta_{21}(z)^{-1}\| \cdot \|(I_q + \Theta_{22}^{-1}(z)\Theta_{21}(z)\mathscr{E}(z))^{-1}\| \\ &\leqslant M\|(I_q + \Theta_{22}^{-1}(z)\Theta_{21}(z)\mathscr{E}(z))^{-1}\| \\ &\leqslant M \max_{U\in\mathbb{C}^{q\times q},\ \|U\|\leqslant\delta} \|(I_q + U)^{-1}\| \\ &\leqslant \frac{M}{1-\delta}, \end{aligned}$$

which completes the proof. □

Corollary 3.3. *Let Θ be analytic at a point $\beta \in \mathbb{S}^d$. Then*

$$\sup_{0\leqslant r<1} \| (\Theta_{21}(r\beta)\mathscr{E}(r\beta) + \Theta_{22}(r\beta))^{-1} \| < \infty$$

for every choice of $\mathscr{E} \in \mathscr{S}^{(md+p)\times q}$.

Proof. If Θ is analytic at $\beta \in \mathbb{S}^d$, then $\mathscr{U} \in \mathbb{B}^d$ can be chosen so that $\overline{\mathscr{U}}$ will contain the complex segment connecting β with the origin. Now the desired bound follows from the previous lemma. □

The following theorem gives a parametrization of all solutions of Problem 1.1 in the case when the Pick matrix P is positive definite.

Theorem 3.4. *Let P be a positive solution of the Stein equation* (1.8), *let Θ be the* $(J, \mathbf{J})$*-inner function given in* (3.3) *and partitioned into four blocks as in* (3.5). *Then all solutions S of Problem* 1.1 *are described by the linear fractional transformation*

$$S(z) = (\Theta_{11}(z)\mathscr{E}(z) + \Theta_{12}(z))\,(\Theta_{21}(z)\mathscr{E}(z) + \Theta_{22}(z))^{-1}\,, \tag{3.6}$$

when the parameter $\mathscr{E}$ varies on the set $\mathscr{S}^{(md+p)\times q}$.

Proof. It follows from Lemma 3.2 that for every $\mathscr{E} \in \mathscr{S}^{(md+p)\times q}$, the matrix $(\Theta_{21}(z)\mathscr{E}(z) + \Theta_{22}(z))$ is invertible at every point $z \in \mathbb{B}^d$ and thus, the transformation (3.6) is well defined. By Theorem 2.3, it suffices to describe the set of all solutions S to the inequality (2.3). Since $P > 0$, (2.3) is equivalent to

$$\begin{aligned} &\frac{I_p - S(z)S(w)^*}{1 - \langle z, w\rangle} - (C_1 - S(z)C_2)G(z)^{-1}P^{-1}G(\omega)^{-*} \\ &\times (C_1^* - C_2^* S(w)^*) \succeq 0, \end{aligned}$$

which in turn can be written as

$$\begin{bmatrix} I_p & -S(z) \end{bmatrix} \left\{ \frac{J}{1 - \langle z, w\rangle} - CG(z)^{-1}P^{-1}G(w)^{-*}C \right\} \begin{bmatrix} I_p \\ -S(w)^* \end{bmatrix} \succeq 0.$$

Taking advantage of (3.4), we rewrite the last inequality as

$$\begin{bmatrix} I_p & -S(z) \end{bmatrix} \frac{\Theta(z)\mathbf{J}\Theta(w)^*}{1 - \langle z, w\rangle} \begin{bmatrix} I_p \\ -S(w)^* \end{bmatrix} \succeq 0 \quad (z, w \in \mathbb{B}^d). \tag{3.7}$$

It was shown in [8] that the set of all solutions S satisfying (3.7) (for a $(J, \mathbf{J})$-inner function Θ not necessarily being of the form (3.3)) is parametrized by formula (3.6). □

4. Degenerate case

In this section, we consider degenerate Problem 1.1, i.e., the case when the Pick matrix P is positive semidefinite. We shall show that the set of all solutions still can be parametrized by a linear fractional transformation, with parameters $\mathscr{E}$ being of a special form. To be more precise, let the matrix P given in (1.7) be positive semidefinite, let rank $P = n \leqslant m$ and let the interpolating points $\beta^{(j)}$ be arranged so that the upper left $n \times n$ block of P is positive definite. Thus,

$$\begin{aligned} &P = \begin{bmatrix} P_1 & P_2^* \\ P_2 & P_3 \end{bmatrix}, \quad \text{where } P_1 \in \mathbb{C}^{n\times n},\ P_1 > 0, \\ &P_3 = P_2 P_1^{-1} P_2^*. \end{aligned} \tag{4.1}$$

Let

$$\begin{aligned} &C_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \quad C_2 = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}, \\ &A_j = \begin{bmatrix} A_{j1} & 0 \\ 0 & A_{j2} \end{bmatrix} \quad (j = 1, \ldots, d) \end{aligned}$$

be block decompositions of matrices C_1, C_2 and A_j conformal with (4.2), so that

$$\begin{aligned} &\begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} = \begin{bmatrix} \xi_1 & \cdots & \xi_n \\ \eta_1 & \cdots & \eta_n \end{bmatrix}, \quad \begin{bmatrix} C_{12} \\ C_{22} \end{bmatrix} = \begin{bmatrix} \xi_{n+1} & \cdots & \xi_m \\ \eta_{m+1} & \cdots & \eta_m \end{bmatrix}, \\ &A_{j1} = \begin{bmatrix} \overline{\beta}_j^{(1)} & & 0 \\ & \ddots & \\ 0 & & \overline{\beta}_j^{(n)} \end{bmatrix}, \quad A_{j2} = \begin{bmatrix} \overline{\beta}_j^{(n+1)} & & 0 \\ & \ddots & \\ 0 & & \overline{\beta}_j^{(m)} \end{bmatrix} \\ &\quad (j = 1, \ldots, d), \end{aligned}$$

and let

$$G_1(z) = I_n - z_1 A_{11} - \cdots - z_d A_{d1}.$$

It is easily seen that P_1 is the Pick matrix of the following truncated (and nondegenerate, since $P_1 > 0$) interpolation problem:

Find all functions $S \in \mathscr{S}_d^{p\times q}$ satisfying interpolation conditions (1.2) *and* (1.3) *for $j = 1, \ldots, n$*:

$$\begin{aligned} &\lim_{r\to 1} S(r\beta^{(j)})^* \xi_j = \eta_j \quad \textit{and} \quad \lim_{r\to 1} \xi_j^* \frac{I_p - S(r\beta^{(j)}) S(r\beta^{(j)})^*}{1 - r^2} \xi_j \leqslant \gamma_j \\ &\quad (j = 1, \ldots, n). \end{aligned} \tag{4.2}$$

Remark 4.1. By Theorem 3.4, the set of all functions $S \in \mathscr{S}_d^{p\times q}$ satisfying (4.2) is parametrized by the formula

$$S(z) = (\widetilde{\Theta}_{11}(z)\mathscr{E}(z) + \widetilde{\Theta}_{12}(z)) \left(\widetilde{\Theta}_{21}(z)\mathscr{E}(z) + \widetilde{\Theta}_{22}(z)\right)^{-1}, \tag{4.3}$$

with the parameter $\mathscr{E}$ varying on the set $\mathscr{S}^{(nd+p)\times q}$ and the transfer function

$$\widetilde{\Theta} = \begin{bmatrix} \widetilde{\Theta}_{11} & \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{21} & \widetilde{\Theta}_{22} \end{bmatrix} : \begin{bmatrix} \mathbb{C}^{nd+p} \\ \mathbb{C}^{q} \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^{p} \\ \mathbb{C}^{q} \end{bmatrix}$$

given by

$$\begin{aligned} \widetilde{\Theta}(z) = & \begin{bmatrix} 0 & \cdots & 0 & I_{p+q} \end{bmatrix} + \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} G_1(z)^{-1} P_1^{-1} \\ & \times \begin{bmatrix} (z_1 I_n - A_{11}^*) P_1^{1/2} & \cdots & (z_d I_n - A_{d1}^*) P_1^{1/2} & -C_{11}^* & C_{21}^* \end{bmatrix}. \end{aligned} \tag{4.4}$$

It remains to choose among all functions S of the form (4.3) those which satisfy also interpolation conditions (1.2) and (1.3) for $j = n+1, \ldots, m$. We shall show that this can be achieved by an appropriate choice of parameters $\mathscr{E}$ in (4.3).

The function $\widetilde{\Theta}$ has the same structure as Θ given by (3.3) and similarly to (3.4),

$$\begin{aligned} J - \widetilde{\Theta}(z) \widetilde{\mathbf{J}} \widetilde{\Theta}(w)^* = & (1 - \langle z, w \rangle) \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} \\ & \times G_1(z)^{-1} P_1^{-1} G_1(w)^{-*} \begin{bmatrix} C_{11}^* & C_{21}^* \end{bmatrix}, \end{aligned} \tag{4.5}$$

where

$$\widetilde{\mathbf{J}} = \begin{bmatrix} I_{nd} & 0 \\ 0 & J \end{bmatrix}. \tag{4.6}$$

By Remark 2.2, $G_1(z)$ is invertible (and therefore, $\widetilde{\Theta}(z)$ is analytic) at every point from the closed unit ball $\overline{\mathbb{B}}^d$ except for n points $\beta^{(1)}, \ldots, \beta^{(n)}$. In particular, Θ is analytic at the remaining interpolating points $\beta^{(n+1)}, \ldots, \beta^{(m)}$. Since $\langle \beta^{(j)}, \beta^{(j)} \rangle = 1$, it follows from (4.5) that

$$\widetilde{\Theta}(\beta^{(j)}) \widetilde{\mathbf{J}} \widetilde{\Theta}(\beta^{(j)})^* = J \quad (j = n+1, \ldots, m). \tag{4.7}$$

Note also that the jth row $(P_2)_j$ of the matrix P_2 from the decomposition (4.2) can be written in the matrix form as

$$\begin{aligned} (P_2)_j = & \begin{bmatrix} \dfrac{\xi_j^* \xi_1 - \eta_j^* \eta_1}{1 - \langle \beta^{(j)}, \beta^{(1)} \rangle} & \cdots & \dfrac{\xi_j^* \xi_n - \eta_j^* \eta_n}{1 - \langle \beta^{(j)}, \beta^{(n)} \rangle} \end{bmatrix} \\ = & \begin{bmatrix} \xi_j^* & -\eta_j^* \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} G_1(\beta^{(j)})^{-1} \end{aligned} \tag{4.8}$$

and comparing diagonal entries in the equality $P_3 = P_2 P_1^{-1} P_2^*$, we obtain

$$\begin{aligned} \gamma_j = & \begin{bmatrix} \xi_j^* & -\eta_j^* \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} G_1(\beta^{(j)})^{-1} P_1^{-1} G_1(\beta^{(j)})^{-*} \\ & \times \begin{bmatrix} C_{11}^* & C_{21}^* \end{bmatrix} \begin{bmatrix} \xi_j \\ -\eta_j \end{bmatrix} \end{aligned} \tag{4.9}$$

for $j = n+1, \ldots, m$.

Lemma 4.2. *Let* $\widetilde{\xi}_{n+1}, \dots, \widetilde{\xi}_m \in \mathbb{C}^{nd+p}$ *and* $\widetilde{\eta}_{n+1}, \dots, \widetilde{\eta}_m \in \mathbb{C}^q$ *be the vectors given by*

$$\begin{bmatrix} \widetilde{\xi}_j \\ \widetilde{\eta}_j \end{bmatrix} = \widetilde{\mathbf{J}}\widetilde{\Theta}(\beta^{(j)})^* \begin{bmatrix} \xi_j \\ -\eta_j \end{bmatrix} \quad (j = n+1, \dots, m). \tag{4.10}$$

Then

$$\widetilde{\xi}_j^* \widetilde{\xi}_j = \widetilde{\eta}_j^* \widetilde{\eta}_j \quad (j = n+1, \dots, m), \tag{4.11}$$

$$\begin{bmatrix} \widetilde{\eta}_{n+1} & \cdots & \widetilde{\eta}_m \end{bmatrix} = C_2 \begin{bmatrix} -P_1^{-1} P_2^* \\ I_{m-n} \end{bmatrix}, \tag{4.12}$$

$$\operatorname{rank} \begin{bmatrix} \widetilde{\eta}_{n+1} & \cdots & \widetilde{\eta}_m \end{bmatrix} = \operatorname{rank}(P + C_2^* C_2) - \operatorname{rank} P. \tag{4.13}$$

Proof. Equalities (4.11) follow from (4.10), (4.7) and (1.6):

$$\begin{aligned}
\widetilde{\xi}_j^* \widetilde{\xi}_j - \widetilde{\eta}_j^* \widetilde{\eta}_j &= \begin{bmatrix} \widetilde{\xi}_j^* & \widetilde{\eta}_j^* \end{bmatrix} \widetilde{\mathbf{J}} \begin{bmatrix} \widetilde{\xi}_j \\ \widetilde{\eta}_j \end{bmatrix} \\
&= \begin{bmatrix} \xi_j^* & -\eta_j^* \end{bmatrix} \widetilde{\Theta}(\beta^{(j)}) \widetilde{\mathbf{J}} \widetilde{\Theta}(\beta^{(j)})^* \begin{bmatrix} \xi_j \\ -\eta_j \end{bmatrix} \\
&= \begin{bmatrix} \xi_j^* & -\eta_j^* \end{bmatrix} J \begin{bmatrix} \xi_j \\ -\eta_j \end{bmatrix} \\
&= \xi_j^* \xi_j - \eta_j^* \eta_j = 0.
\end{aligned}$$

Next, by (4.10), (4.4) and (4.8),

$$\begin{aligned}
\widetilde{\eta}_j^* &= \begin{bmatrix} \xi_j^* & -\eta_j^* \end{bmatrix} \widetilde{\Theta}(\beta^{(j)}) \begin{bmatrix} 0 \\ -I_q \end{bmatrix} \\
&= \eta_j^* - \begin{bmatrix} \xi_j^* & -\eta_j^* \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} G_1(\beta^{(j)})^{-1} P_1^{-1} C_{21}^* \\
&= \eta_j^* - (P_2)_j P_1^{-1} C_{21}^* \quad (j = n+1, \dots, m).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\begin{bmatrix} \widetilde{\eta}_{n+1}^* \\ \vdots \\ \widetilde{\eta}_m^* \end{bmatrix} &= \begin{bmatrix} \eta_{n+1}^* \\ \vdots \\ \eta_m^* \end{bmatrix} - P_2 P_1^{-1} C_{21}^* \\
&= C_{22}^* - P_2 P_1^{-1} C_{21}^* \\
&= \begin{bmatrix} -P_2 P_1^{-1} & I_{m-n} \end{bmatrix} C_2^*,
\end{aligned}$$

which is equivalent to (4.12). To verify (4.13), we start with an evident equality

$$\begin{aligned}\operatorname{rank}(P+C_2^*C_2)=&\operatorname{rank}\begin{bmatrix} I & 0\\ -P_2P_1^{-1} & I\end{bmatrix}(P+C_2^*C_2)\\ &\times\begin{bmatrix} I & -P_2P_1^{-1}\\ 0 & I\end{bmatrix}.\end{aligned}\tag{4.14}$$

Setting for short

$$\mathscr{N} := C_2\begin{bmatrix} -P_1^{-1}P_2^* \\ I_{m-n}\end{bmatrix}$$

and making use of equality

$$P\begin{bmatrix} -P_1^{-1}P_2^* \\ I_{m-n}\end{bmatrix}=0,$$

which follows directly from (4.2), we rewrite (4.14) as

$$\operatorname{rank}(P+C_2^*C_2)=\operatorname{rank}\begin{bmatrix} P_1+C_{21}^*C_{21} & C_{21}^*\mathscr{N}\\ \mathscr{N}^*C_{21} & \mathscr{N}^*\mathscr{N}\end{bmatrix}.$$

By the standard Schur complement argument,

$$\begin{aligned}\operatorname{rank}(P+C_2^*C_2)=&\operatorname{rank}(P_1+C_{21}^*C_{21})\\ &+\operatorname{rank}(\mathscr{N}^*(I-C_{21}(P_1+C_{21}^*C_{21})^{-1}C_{21}^*)\mathscr{N}),\end{aligned}$$

which implies (4.13), since

$$\operatorname{rank}(P_1+C_{21}^*C_{21})=\operatorname{rank}P_1=n=\operatorname{rank}P$$

and

$$\begin{aligned}&\operatorname{rank}\left[\mathscr{N}^*(I-C_{21}(P_1+C_{21}^*C_{21})^{-1}C_{21}^*)\mathscr{N}\right]\\ &\quad=\operatorname{rank}\left[\mathscr{N}^*(I+C_{21}P_1^{-1}C_{21}^*)^{-1}\mathscr{N}\right]=\operatorname{rank}\mathscr{N}. \qquad\square\end{aligned}$$

Lemma 4.3. *Let K be any $N\times q$ contractive matrix, let $\xi\in\mathbb{C}^N$, $\eta\in\mathbb{C}^q$ and suppose that $\xi^*\xi=\eta^*\eta$. Then the following three equalities are equivalent*:

$$\xi=K\eta,\quad \eta=K^*\xi,\quad \xi^*\xi=\xi^*K\eta.$$

For the proof see [12, Lemma 0.9].

Lemma 4.4. *Let $\beta\in\mathbb{S}^d$, let $\mathscr{E}\in\mathscr{S}_d^{N\times q}$, let $a(z)$ be a $\mathbb{C}^N$-valued function on $\mathbb{B}^d$ and let*

$$\lim_{r\to 1}a(r\beta)=\xi\quad \textit{and}\quad \lim_{r\to 1}a(r\beta)^*\frac{I_N-\mathscr{E}(r\beta)\mathscr{E}(r\beta)^*}{1-r^2}a(r\beta)=0.\tag{4.15}$$

Then there exist radial boundary limits

$$\eta:=\lim_{r\to 1}\mathscr{E}(r\beta)^*a(r\beta)=\lim_{r\to 1}\mathscr{E}(r\beta)^*\xi\tag{4.16}$$

so that $\xi^*\xi = \eta^*\eta$ *and moreover,*

$$\mathscr{E}(z)\eta \equiv \xi \quad \text{and} \quad \mathscr{E}(z)^*\xi \equiv \eta. \tag{4.17}$$

Proof. Under assumption that the first limit in (4.15) exists, the existence of the second limit in (4.15) implies the existence of the first limit in (4.16) and the equality $\eta^*\eta = \xi^*\xi$. Since $\mathscr{E} \in \mathscr{S}_d^{N\times q}$, it follows from the triangle inequality,

$$\begin{aligned}\|\mathscr{E}(z)^*\xi - \eta\| &\leqslant \|\xi - a(z)\| \cdot \|\mathscr{E}(z)^*\| + \|\mathscr{E}(z)^*a(z) - \eta\| \\ &\leqslant \|\xi - a(z)\| + \|\mathscr{E}(z)a(z) - \eta\|.\end{aligned}$$

Thus $\lim_{r\to 1} \|\mathscr{E}(r\beta)^*\xi - \eta\| = 0$, which completes the proof of (4.16).

Furthermore, since $\mathscr{E}$ belongs to $\mathscr{S}_d^{N\times q}$, the kernel

$$K_{\mathscr{E}}(z, w) = \frac{I_N - \mathscr{E}(z)\mathscr{E}(w)^*}{1 - \langle z, w\rangle}$$

is positive on $\mathbb{B}^d$. In particular, the following block matrix is positive semidefinite:

$$\begin{bmatrix} \dfrac{I_N - \mathscr{E}(w)\mathscr{E}(w)^*}{1 - \langle w, w\rangle} & \dfrac{I_N - \mathscr{E}(w)\mathscr{E}(z)^*}{1 - \langle w, z\rangle} \\ \dfrac{I_N - \mathscr{E}(z)\mathscr{E}(w)^*}{1 - \langle z, w\rangle} & \dfrac{I_N - \mathscr{E}(z)\mathscr{E}(z)^*}{1 - \langle z, z\rangle} \end{bmatrix} \geqslant 0,$$

where z and w are two fixed points in $\mathbb{B}^d$. Multiplying this matrix by

$$\begin{bmatrix} a(w) & 0 \\ 0 & I_N \end{bmatrix}$$

on the right, by its adjoint on the left and setting $w = r\beta$, we get

$$\begin{bmatrix} a(r\beta)^* \dfrac{I_N - \mathscr{E}(r\beta)\mathscr{E}(r\beta)^*}{1 - r^2} a(r\beta) & a(r\beta)^* \dfrac{I_N - \mathscr{E}(r\beta)\mathscr{E}(z)^*}{1 - \langle r\beta, z\rangle} \\ \dfrac{I_N - \mathscr{E}(z)\mathscr{E}(r\beta)^*}{1 - \langle z, r\beta\rangle} a(r\beta) & \dfrac{I_N - \mathscr{E}(z)\mathscr{E}(z)^*}{1 - \langle z, z\rangle} \end{bmatrix} \geqslant 0. \tag{4.18}$$

Taking in (4.18) the limit as r tends to 1 and making use of (4.15) and (4.16), we get

$$\begin{bmatrix} 0 & \dfrac{\xi^* - \eta^*\mathscr{E}(z)^*}{1 - \langle \beta, z\rangle} \\ \dfrac{\xi - \mathscr{E}(z)\eta}{1 - \langle z, \beta\rangle} & \dfrac{I_N - \mathscr{E}(z)\mathscr{E}(z)^*}{1 - \langle z, z\rangle} \end{bmatrix} \geqslant 0.$$

Since $\langle z, \beta\rangle \neq 1$ for every $z \in \mathbb{B}^d$, it follows from the last inequality that $\mathscr{E}(z)\eta \equiv \xi$, which proves the first identity in (4.17). The second identity follows from the first by Lemma 4.3. □

Lemma 4.5. *Let $\widetilde{\xi}_{n+1}, \ldots, \widetilde{\xi}_m \in \mathbb{C}^{nd+p}$ and $\widetilde{\eta}_{n+1}, \ldots, \widetilde{\eta}_m \in \mathbb{C}^q$ be the vectors given in* (4.10) *and let S be of the form* (4.3) *for some parameter $\mathscr{E} \in \mathscr{S}_d^{(nd+p)\times q}$. Then S satisfies interpolation conditions* (1.2) *and* (1.3) *for $j = n+1, \ldots, m$ if and only if $\mathscr{E}$ is subject to*

$$\mathscr{E}(z)\widetilde{\eta}_j \equiv \widetilde{\xi}_j \quad (j = n+1, \ldots, m). \tag{4.19}$$

Proof. A simple manipulation shows that (4.3) is equivalent to

$$\begin{bmatrix} I_p & -S(z) \end{bmatrix} \begin{bmatrix} \widetilde{\Theta}_{11}(z) \\ \widetilde{\Theta}_{21}(z) \end{bmatrix} \mathscr{E}(z) \equiv -\begin{bmatrix} I_p & -S(z) \end{bmatrix} \begin{bmatrix} \widetilde{\Theta}_{12}(z) \\ \widetilde{\Theta}_{22}(z) \end{bmatrix} \tag{4.20}$$

and therefore, on account of (4.5) and (4.6),

$$\begin{aligned}
&\begin{bmatrix} I_p & -S(z) \end{bmatrix} \begin{bmatrix} \widetilde{\Theta}_{11}(z) \\ \widetilde{\Theta}_{21}(z) \end{bmatrix} \frac{I_N - \mathscr{E}(z)\mathscr{E}(z)^*}{1 - \langle z, z\rangle} \begin{bmatrix} \widetilde{\Theta}_{11}(z)^* & \widetilde{\Theta}_{21}(z)^* \end{bmatrix} \begin{bmatrix} I_p \\ -S(z)^* \end{bmatrix} \\
&= \begin{bmatrix} I_p & -S \end{bmatrix} \frac{\begin{bmatrix} \widetilde{\Theta}_{11} \\ \widetilde{\Theta}_{21} \end{bmatrix} \begin{bmatrix} \widetilde{\Theta}_{11}^* & \widetilde{\Theta}_{21}^* \end{bmatrix} - \begin{bmatrix} \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{22} \end{bmatrix} \begin{bmatrix} \widetilde{\Theta}_{12}^* & \widetilde{\Theta}_{22}^* \end{bmatrix}}{1 - \langle z, z\rangle} \begin{bmatrix} I_p \\ -S^* \end{bmatrix} \\
&= \begin{bmatrix} I_p & -S(z) \end{bmatrix} \frac{\widetilde{\Theta}(z)\widetilde{\mathbf{J}}\widetilde{\Theta}(z)^*}{1 - \langle z, z\rangle} \begin{bmatrix} I_p \\ -S(z)^* \end{bmatrix} \\
&= \begin{bmatrix} I_p & -S(z) \end{bmatrix} \left\{ \frac{J}{1 - \langle z, z\rangle} - \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} G(z)^{-1} P^{-1} G(z)^{-*} \begin{bmatrix} C_{11}^* & C_{21}^* \end{bmatrix} \right\} \\
&\quad \times \begin{bmatrix} I_p \\ -S(z)^* \end{bmatrix} \\
&= \frac{I_p - S(z)S(z)^*}{1 - \langle z, z\rangle} - \begin{bmatrix} I_p & -S(z) \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} G_1(z)^{-1} P_1^{-1} G_1(z)^{-*} \\
&\quad \times \begin{bmatrix} C_{11}^* & C_{21}^* \end{bmatrix} \begin{bmatrix} I_p \\ -S(z)^* \end{bmatrix}.
\end{aligned} \tag{4.21}$$

Let us assume that S satisfies interpolation conditions (1.2) and (1.3) for $j = n + 1, \ldots, m$ and set

$$a_j(z) = \begin{bmatrix} \widetilde{\Theta}_{11}(z)^* & \widetilde{\Theta}_{21}(z)^* \end{bmatrix} \begin{bmatrix} I_p \\ -S(z)^* \end{bmatrix} \xi_j. \tag{4.22}$$

It follows from (1.2) and (4.10) that

$$\lim_{r\to 1} a_j(r\beta^{(j)}) = \begin{bmatrix} \widetilde{\Theta}_{11}(\beta^{(j)})^* & \widetilde{\Theta}_{21}(\beta^{(j)})^* \end{bmatrix} \begin{bmatrix} \xi_j \\ -\eta_j \end{bmatrix} = \widetilde{\xi}_j. \tag{4.23}$$

Furthermore, multiplying both sides in (4.20) by ξ_j^* on the left and taking adjoints in the resulting identity, we get

$$\mathscr{E}(z)^* a_j(z) = -\begin{bmatrix} \widetilde{\Theta}_{12}(z)^* & \widetilde{\Theta}_{22}(z)^* \end{bmatrix} \begin{bmatrix} I_p \\ -S(z)^* \end{bmatrix} \xi_j$$

and similarly to the preceding calculation,

$$\begin{aligned} &\lim_{r\to 1} \mathscr{E}(r\beta^{(j)})^* a_j(r\beta^{(j)}) \\ &\quad = -\lim_{r\to 1} \begin{bmatrix} \widetilde{\Theta}_{12}(r\beta^{(j)})^* & \widetilde{\Theta}_{22}(r\beta^{(j)})^* \end{bmatrix} \begin{bmatrix} I_p \\ -S(r\beta^{(j)})^* \end{bmatrix} \xi_j \\ &\quad = -\begin{bmatrix} \widetilde{\Theta}_{12}(\beta^{(j)})^* & \widetilde{\Theta}_{22}(\beta^{(j)})^* \end{bmatrix} \begin{bmatrix} \xi_j \\ -\eta_j \end{bmatrix} \\ &\quad = \widetilde{\eta}_j. \end{aligned} \tag{4.24}$$

Finally, multiplying (4.21) by ξ_j^* on the left and by ξ_j on the right and taking into account (4.22), we get

$$\begin{aligned} &a_j(z)^* \frac{I_N - \mathscr{E}(z)\mathscr{E}(z)^*}{1 - \langle z, z\rangle} a_j(z) \\ &\quad = \xi_j^* \frac{I_p - S(z)S(z)^*}{1 - \langle z, z\rangle} \xi_j - \xi_j^* \begin{bmatrix} I_p & -S(z) \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} G_1(z)^{-1} P_1^{-1} G_1(z)^{-*} \\ &\qquad \times \begin{bmatrix} C_{11}^* & C_{21}^* \end{bmatrix} \begin{bmatrix} I_p \\ -S(z)^* \end{bmatrix} \xi_j \end{aligned}$$

and setting $z = r\beta^{(j)} \to \beta^{(j)}$ in the last equality, we obtain, on account of (4.9),

$$\begin{aligned} &\lim_{r\to 1} a_j(r\beta^{(j)})^* \frac{I_N - \mathscr{E}(r\beta^{(j)})\mathscr{E}(r\beta^{(j)})^*}{1 - r^2} a_j(r\beta^{(j)}) \\ &\quad = \lim_{r\to 1} \xi_j^* \frac{I_p - S(r\beta^{(j)})S(r\beta^{(j)})^*}{1 - r^2} \xi_j - \gamma_j. \end{aligned}$$

Since the left-hand side limit is positive semidefinite and the limit on the right-hand side does not exceed γ_j (by (1.3)), it follows that

$$\lim_{r\to 1} a_j(r\beta^{(j)})^* \frac{I_N - \mathscr{E}(r\beta^{(j)})\mathscr{E}(r\beta^{(j)})^*}{1 - r^2} a_j(r\beta^{(j)}) = 0.$$

The last relation together with (4.23) allows us to apply Lemma 4.4. In the present context, $a = a_j$, $\xi = \widetilde{\xi}_j$ and the vector η defined via radial limits (4.16) equals (by (4.24)) $\widetilde{\eta}_j$. Thus, the first identity in (4.17) leads to (4.19).

Conversely, let S be of the form (4.3) for some parameter $\mathscr{E} \in \mathscr{S}_d^{(nd+p)\times q}$ subject to (4.19). Making use of (4.12) we conclude by Lemma 4.3 that

$$\mathscr{E}(z)^* \widetilde{\xi}_j \equiv \widetilde{\eta}_j \quad (j = n+1, \ldots, m). \tag{4.25}$$

Multiplying (4.3) by ξ_j^* on the left and subtracting η_j^* from both sides, we get

$$\begin{aligned}\xi_j^* S(z) - \eta_j^* = &\left\{\xi_j^*(\widetilde{\Theta}_{11}(z)\mathscr{E}(z) + \widetilde{\Theta}_{12}(z)) - \eta_j^*(\widetilde{\Theta}_{21}(z)\mathscr{E}(z) + \widetilde{\Theta}_{22}(z))\right\}\\ &\times(\widetilde{\Theta}_{21}(z)\mathscr{E}(z) + \widetilde{\Theta}_{22}(z))^{-1}\\ = &\begin{bmatrix}\xi_j^* & -\eta_j^*\end{bmatrix}\widetilde{\Theta}(z)\begin{bmatrix}\mathscr{E}(z)\\ I_q\end{bmatrix}\left(\widetilde{\Theta}_{21}(z)\mathscr{E}(z) + \widetilde{\Theta}_{22}(z)\right)^{-1},\end{aligned}$$

which can be written as

$$\begin{aligned}\xi_j^* S(z) - \eta_j^* = &\begin{bmatrix}\xi_j^* & -\eta_j^*\end{bmatrix}\left[\widetilde{\Theta}(z) - \widetilde{\Theta}(\beta^{(j)})\right]\begin{bmatrix}\mathscr{E}(z)\\ I_q\end{bmatrix}\\ &\times\left(\widetilde{\Theta}_{21}(z)\mathscr{E}(z) + \widetilde{\Theta}_{22}(z)\right)^{-1},\end{aligned} \tag{4.26}$$

since by (4.10) and (4.25),

$$\begin{bmatrix}\xi_j^* & -\eta_j^*\end{bmatrix}\widetilde{\Theta}(\beta^{(j)})\begin{bmatrix}\mathscr{E}(z)\\ I_q\end{bmatrix} = \begin{bmatrix}\widetilde{\xi}_j^* & \widetilde{\eta}_j^*\end{bmatrix}\widetilde{\mathbf{J}}\begin{bmatrix}\mathscr{E}(z)\\ I_q\end{bmatrix} = \widetilde{\xi}_j^*\mathscr{E}(z) - \widetilde{\eta}_j^* \equiv 0.$$

Since $\widetilde{\Theta}$ is analytic at $\beta^{(n+1)}, \ldots, \beta^{(m)}$, it follows from Corollary 3.3 that

$$\sup_{0\leqslant r<1}\|(\widetilde{\Theta}_{21}(r\beta^{(j)})\mathscr{E}(r\beta^{(j)}) + \widetilde{\Theta}_{22}(r\beta^{(j)}))^{-1}\| < \infty \quad (j = n+1, \ldots, m).$$

Since moreover, $\widetilde{\Theta}$ is continuous at $z = \beta^{(j)}$, the right-hand side expression in (4.26) tends to zero as $z = r\beta^{(j)} \to \beta^{(j)}$. Therefore, (4.26) implies

$$\lim_{r\to 1}\xi_j^* S(r\beta^{(j)}) = \eta_j^* \quad (j = n+1, \ldots, m). \tag{4.27}$$

Rewriting (4.20) (which is equivalent to (4.3)) as

$$\begin{bmatrix}I_p & -S(z)\end{bmatrix}\ \widetilde{\Theta}(z)\begin{bmatrix}\mathscr{E}(z)\\ I_q\end{bmatrix} \equiv 0,$$

we multiply this identity by $\widetilde{\eta}_j$ on the right and obtain, on account of (4.19),

$$\begin{bmatrix}I_p & -S(z)\end{bmatrix}\ \widetilde{\Theta}(z)\begin{bmatrix}\widetilde{\xi}_j\\ \widetilde{\eta}_j\end{bmatrix} \equiv 0. \tag{4.28}$$

Upon making subsequent use of (4.10), of (4.5) evaluated at $w = \beta^{(j)}$ and of the block structure (1.9) of the matrix J, we get

$$\begin{aligned}&\widetilde{\Theta}(z)\begin{bmatrix}\widetilde{\xi}_j\\ \widetilde{\eta}_j\end{bmatrix}\\ &\quad = \widetilde{\Theta}(z)\widetilde{\mathbf{J}}\widetilde{\Theta}(\beta^{(j)})^*\begin{bmatrix}\xi_j\\ -\eta_j\end{bmatrix}\\ &\quad = \left\{J - (1 - \langle z,\ \beta^{(j)}\rangle)\begin{bmatrix}C_{11}\\ C_{21}\end{bmatrix}G_1(z)^{-1}P_1^{-1}G_1(\beta^{(j)})^{-*}\right.\end{aligned}$$

$$
\begin{aligned}
&\quad \times \begin{bmatrix} C_{11}^* & C_{21}^* \end{bmatrix} \Bigg\} \begin{bmatrix} \xi_j \\ -\eta_j \end{bmatrix} \\
&= \begin{bmatrix} \xi_j \\ \eta_j \end{bmatrix} - (1 - \langle z, \beta^{(j)} \rangle) \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} G_1(z)^{-1} P_1^{-1} G_1(\beta^{(j)})^{-*} \\
&\quad \times \begin{bmatrix} C_{11}^* & C_{21}^* \end{bmatrix} \begin{bmatrix} \xi_j \\ -\eta_j \end{bmatrix},
\end{aligned}
$$

which, being substituted into (4.28), implies

$$
\begin{aligned}
\xi_j - S(z)\eta_j \equiv (1 - \langle z,\ \beta^{(j)} \rangle) \begin{bmatrix} I_p & -S(z) \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} \\
\times G_1(z)^{-1} P_1^{-1} G_1(\beta^{(j)})^{-*} \begin{bmatrix} C_{11}^* & C_{21}^* \end{bmatrix} \begin{bmatrix} \xi_j \\ -\eta_j \end{bmatrix}.
\end{aligned} \tag{4.29}
$$

The right-hand side expression in (4.29) tends to zero as z approaches $\beta^{(j)}$, since $G_1(z)$ is invertible at $\beta^{(j)}$ (for $j = n+1, \ldots, m$), since S of the form (4.3) belongs necessarily to the class $\mathscr{S}_d^{p\times q}$ (and therefore, $\|S(z)\| \leqslant 1$ on $\mathbb{B}^d$), and since $\lim_{z\to\beta^{(j)}} (1 - \langle z,\ \beta^{(j)} \rangle) = 0$. Therefore, (4.29) implies

$$
\lim_{r\to 1} S(\beta^{(j)})\eta_j = \xi_j \quad (j = n+1, \ldots, m). \tag{4.30}
$$

Furthermore, multiplying (4.29) by $\xi_j^*/(1 - \langle z, \beta^{(j)} \rangle)$ on the left and letting $z = r\beta^{(j)} \to \beta^{(j)}$, we come to

$$
\begin{aligned}
\lim_{r\to 1} \frac{\xi_j^*\xi_j - \xi_j^* S(r\beta^{(j)})\eta_j}{1-r} = \lim_{r\to 1} \begin{bmatrix} \xi_j^* & -\xi_j^* S(r\beta^{(j)}) \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} G_1(r\beta^{(j)})^{-1} \\
\times P_1^{-1} G_1(\beta^{(j)})^{-*} \begin{bmatrix} C_{11}^* & C_{21}^* \end{bmatrix} \begin{bmatrix} \xi_j \\ -\eta_j \end{bmatrix},
\end{aligned}
$$

which, in view of (4.27) and (4.9), is equivalent to

$$
\lim_{r\to 1} \frac{\xi_j^*\xi_j - \xi_j^* S(r\beta^{(j)})\eta_j}{1-r} = \gamma_j \quad (j = n+1, \ldots, m). \tag{4.31}
$$

Taking into account (4.27), (4.30) and (4.31) we get, from the two first assertions in Theorem 1.2, that

$$
\lim_{r\to 1} \xi_j^* \frac{I_p - S(r\beta^{(j)})S(r\beta^{(j)})^*}{1-r^2} \xi_j = \gamma_j \quad (j = n+1, \ldots, m).
$$

The last equalities together with (4.30) mean that S satisfies interpolation conditions (1.2) and (1.3) for $j = n+1, \ldots, m$, which completes the proof. □

Proof of Theorem 1.4. By Remark 4.1 and Lemma 4.5, the set of all solutions S of Problem 1.1 are parametrized by formula (4.3), when the parameter $\mathscr{E}$ varies on $\mathscr{S}_d^{(nd+p)\times q}$ and satisfies conditions (4.19). Thus,

$$\mathscr{E}(z)\begin{bmatrix}\widetilde{\eta}_{n+1} & \cdots & \widetilde{\eta}_m\end{bmatrix} = \begin{bmatrix}\widetilde{\xi}_{n+1} & \cdots & \widetilde{\xi}_m\end{bmatrix},$$

which displays the fact that the function $\mathscr{E} \in \mathscr{S}_d^{(nd+p)\times q}$ maps $\mathscr{R}_1 = \operatorname{Ran}[\widetilde{\eta}_{n+1} \ \cdots \ \widetilde{\eta}_m]$ isometrically onto $\mathscr{R}_2 = \operatorname{Ran}[\widetilde{\xi}_{n+1} \ \cdots \ \widetilde{\xi}_m]$ for every $z \in \mathbb{B}^d$. Therefore (see e.g., [12, Lemma 0.13]), $\mathscr{E}$ admits a representation of the form

$$\mathscr{E}(z) = U \begin{bmatrix}\widehat{\mathscr{E}}(z) & 0 \\ 0 & I_\nu\end{bmatrix} V, \tag{4.32}$$

where $U \in \mathbb{C}^{(nd+p)\times(nd+p)}$ and $V \in \mathbb{C}^{q\times q}$ are fixed unitary matrices which depend only on $\mathscr{R}_1$ and $\mathscr{R}_2$ (i.e., only on the interpolation data) and a function $\widehat{\mathscr{E}} \in \mathscr{S}_d^{(nd+p-\nu_1)\times(q-\nu_1)}$, where

$$\nu_1 = \dim\mathscr{R}_1 = \dim \mathscr{R}_2.$$

By (4.13),

$$\dim\mathscr{R}_1 = \operatorname{rank}\begin{bmatrix}\widetilde{\eta}_{n+1} & \cdots & \widetilde{\eta}_m\end{bmatrix} = \operatorname{rank}(P + C_2^*C_2) - \operatorname{rank}P$$

and thus, ν_1 is equal to the integer ν defined via (1.11). Moreover, setting

$$\Phi(z) = \widetilde{\Theta}(z)\begin{pmatrix}U & 0 \\ 0 & V^*\end{pmatrix},$$

it is easily seen that formulas (4.3) and (1.12) with parameters of the form (4.32) and (1.13), respectively, are equivalent. It remains to note that Φ is $(J, \widetilde{\mathbf{J}})$-inner since Θ is $(J, \widetilde{\mathbf{J}})$-inner and the matrices U and V are unitary. $\square$

Proof of Theorem 1.3. The necessity part follows from Theorem 2.3, since the positivity of the kernel (2.3) implies that the Pick matrix P is positive semidefinite. The sufficiency part follows from Theorem 1.4: under the assumption that P is a positive definite solution of the Stein equation (1.8), the set of functions S of the form (1.12) is not empty. $\square$

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