Boundary interpolation for contractive-valued functions on circular domains in $\mathbb{C}^n$

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Abstract. We consider a boundary interpolation problem for operator-valued functions defined on a class of bounded complete circular domains in $\mathbb{C}^n$ (including as particular cases, Cartan domains of types I, II and III) which satisfy the von Neumann's inequality. The solvability criterion is obtained and the set of all solutions is parametrized in terms of a family of Redheffer linear fractional transformations.

1. Introduction

The classical Schur class $\mathcal{S}$ consisting of complex-valued analytic functions mapping the unit disk $\mathbb{D}$ into the closed unit disk $\overline{\mathbb{D}}$ has been a source of much study and inspiration and has served as a proving ground for new methods for over a century now, beginning with the seminal work of Schur (for the original paper of Schur and a survey of some of the impact and applications in signal processing, see [28]). A major development has been the interpolation and realization theories for Schur functions and more recently, for their operator-valued analogues. The operator-valued Schur class $\mathcal{S}(\mathcal{E},\mathcal{E}_*)$ consisting of analytic functions S on the unit disk with values $S(z)$ equal to contraction operators between two Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$ has played a prominent role in both engineering and operator-theoretic applications (see e.g. [20, 21, 33, 29, 34]). We mention in particular that any such function $S(z)$ can be realized in the form

$$S(z) = D + zC(I - zA)^{-1}B \tag{1.1}$$

where the connecting operator (or *colligation*)

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{H} \\ \mathcal{E}_* \end{bmatrix}$$

is unitary, and where $\mathcal{H}$ is some auxiliary Hilbert space (the *internal space* for the colligation). It is also well known that the Schur class of functions satisfies a von

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Neumann inequality: *if $S \in \mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ and T is a contraction operator on a Hilbert space $\mathcal{H}'$, then $\|F(rT)\| \leq 1$ for all $r < 1$.* Here $S(rT)$ can be defined, e.g., by

$$S(rT) = \sum_{n=0}^{\infty} r^n F_n \otimes T \in \mathcal{L}(\mathcal{E} \otimes \mathcal{H}', \mathcal{E}_* \otimes \mathcal{H}') \quad \text{if} \quad F(z) = \sum_{n=0}^{\infty} F_n z^n.$$

A related result is the Nevanlinna-Pick interpolation theorem, here stated for the scalar case for simplicity:

Given points $z_1, \dots, z_n \in \mathbb{D}$ and $w_1, \dots, w_n \in \mathbb{C}$, there exists $S \in \mathcal{S}$ with $S(z_j) = w_j$ for $j = 1, \dots, n$ if and only if the associated Pick matrix $\Lambda = \left[\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}}\right]_{i,j=1}^{n}$ is positive semidefinite.

By the classical Fatou's lemma, the boundary radial limits $S(t) = \lim_{r \to 1} S(rt)$ exist at almost every point t on the unit circle $\mathbb{T}$ (for operator-valued functions such limits exist almost everywhere in the strong sense; see [33, Chapter 5]) and do not exceed one in modulus (are contractive in the operator-valued case). Another classical result by Julia-Carathéodory asserts that the boundedness of the quantity $d(z) = \frac{1-|S(z)|^2}{1-|z|^2}$ in some radial neighborhood of a point $t \in \mathbb{T}$ guarantees the existence of the limit $\lim_{r \to 1} S(rt)$ (which is unimodular) and $\lim_{r \to 1} d(rt)$ (which is nonnegative). The related interpolation result (stated again for the scalar case) is:

Given points $z_1, \dots, z_n \in \mathbb{T}$ and numbers $w_1, \dots, w_n$ and $\gamma_1, \dots, \gamma_n$ such that

$$|w_j| = 1 \quad \text{and} \quad \gamma_j \geq 0 \quad \text{for} \quad i = 1, \dots, n, \tag{1.2}$$

there exists $S \in \mathcal{S}$ with

$$\lim_{r \to 1} S(rz_j) = w_j \quad \text{and} \quad \lim_{r \to 1} \frac{1 - |S(rz_j)|^2}{1 - |r|^2} \leq \gamma_j \quad \text{for} \quad i = 1, \dots, n \tag{1.3}$$

if and only if the associated Pick matrix $\Lambda = [\Lambda_{ij}]_{i,j=1}^n$ with the entries

$$\Lambda_{ij} = \begin{cases} \dfrac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}}, & i \neq j \\ \gamma_i, & i = j, \end{cases}$$

is positive semidefinite.

Note that assumptions (1.2) are not restrictive as follows from the Julia-Carathéodory theorem.

The boundary Nevanlinna–Pick interpolation problem was worked out using quite different approaches: an indefinite modification of the Sz.-Nagy–Koranyi method [9], the method of fundamental matrix inequalities [31], the recursive Schur algorithm [25], the Grassmannian approach [14], via realization theory [13], and via unitary extensions of partially defined isometries [30]. Note also that a similar problem with equality sign in the second series of conditions in (1.3) was considered in [37, 27, 19].

In fact most of the above papers handled a more general tangential version of the boundary Nevanlinna–Pick problem for matrix and operator valued Schur functions. The corresponding result is:

Given Hilbert spaces $\mathcal{E}$, $\mathcal{E}_*$ and $\mathcal{E}_L$, given points $z_1, \dots, z_n \in \mathbb{T}$ and operators $a_1, \dots, a_n \in \mathcal{L}(\mathcal{E}_L, \mathcal{E}_*)$, $c_1, \dots, c_n \in \mathcal{L}(\mathcal{E}_L, \mathcal{E})$, $\gamma_1, \dots, \gamma_n \in \mathcal{L}(\mathcal{E}_L)$ such that

$$a_j^* a_j = c_j^* c_j \quad \text{and} \quad \gamma_j \geq 0 \quad \text{for} \quad i = 1, \dots, n, \tag{1.4}$$

there exists $S \in \mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ with

$$\lim_{r \to 1} S(rz_j)^* a_j = c_j \tag{1.5}$$

and

$$\lim_{r \to 1} a_j^* \frac{I_{\mathcal{E}_*} - S(rz_j)S(rz_j)^*}{1 - |r|^2} a_j \leq \gamma_j \quad \text{for} \quad i = 1, \dots, n \tag{1.6}$$

if and only if the associated Pick matrix $\Lambda = [\Lambda_{ij}]_{i,j=1}^n$ with the entries

$$\Lambda_{ij} = \begin{cases} \dfrac{a_i^* a_j - c_i^* c_j}{1 - z_i \overline{z_j}}, & i \neq j \\ \gamma_i, & i = j, \end{cases}$$

is positive semidefinite.

Interpolation theory for Schur functions has been extended recently to multivariable settings in several ways. Parametrizations of the set of all solutions of nonboundary Nevanlinna–Pick problems were obtained in [16] for the polydisk $\mathbb{D}^n \subset \mathbb{C}^n$ setting and in [17] for the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ setting. Note that solvability criteria for these problems were established earlier in [2, 1, 15, 3] for the case of the polydisk, in [36, 4, 32, 24, 35, 18] for the unit ball setting (including more general reproducing kernel Hilbert space and noncommutative Toeplitz-operator settings), and in [38] for Cartesian products of unit balls of arbitrary dimensions.

In [12] we considered nonboundary Nevanlinna–Pick interpolation problem for a class of contractive-valued functions analytic on a more general class of domains introduced in [5] and which we now recall.

We start with a polynomial $p \times q$ matrix-valued function

$$\mathbf{P}(z) = \begin{bmatrix} \mathbf{p}_{11}(z) & \dots & \mathbf{p}_{1q}(z) \\ \vdots & & \vdots \\ \mathbf{p}_{p1}(z) & \dots & \mathbf{p}_{pq}(z) \end{bmatrix} : \mathbb{C}^n \to \mathbb{C}^{p \times q} \tag{1.7}$$

and we define the domain $\mathcal{D}_{\mathbf{P}} \in \mathbb{C}^n$ by

$$\mathcal{D}_{\mathbf{P}} = \{ z \in \mathbb{C}^n : \|\mathbf{P}(z)\|_{\mathbb{C}^{p \times q}} < 1 \}.$$

Let $\mathcal{E}$ and $\mathcal{E}_*$ be two separable Hilbert spaces and $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ be the space of bounded linear operators from $\mathcal{E}$ into $\mathcal{E}_*$. We denote by $\mathcal{SA}_{\mathcal{D}_{\mathbf{P}}}(\mathcal{E}, \mathcal{E}_*)$ the Schur–Agler class of $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued functions $S(z) = S(z_1, \dots, z_n)$ which are analytic on $\mathcal{D}_{\mathbf{P}}$ and such that

$$\|S(T_1, \dots, T_n)\| \leq 1$$

for any collection of n commuting operators $(T_1, \dots, T_n)$ on a Hilbert space $\mathcal{K}$, subject to

$$\|\mathbf{P}(T_1, \dots, T_n)\| < 1.$$

Domains $\mathcal{D}_{\mathbf{P}}$ and classes $\mathcal{SA}_{\mathcal{D}_{\mathbf{P}}}(\mathcal{E}, \mathcal{E}_*)$ (for $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$) have been introduced in [5]. It was shown that the Taylor joint spectrum of the commuting n-tuple $(T_1, \dots, T_n)$ is contained in $\mathcal{D}_{\mathbf{P}}$ whenever $\|\mathbf{P}(T_1, \dots, T_n)\| < 1$, and hence $S(T_1, \dots, T_n)$ is well defined for any $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued function S which is analytic on $\mathcal{D}_{\mathbf{P}}$ by the Taylor functional calculus (see [23]). Operator-valued Schur-Agler classes were studied in [12]. The proof of the following theorem can be found in [12] and in [5] (for the case of scalar-valued functions); we state only the parts needed for our purposes here. In the statement and in the sequel, we often abbreviate expressions of the sort $\mathbf{P}(z) \otimes I_{\mathcal{H}}$ to simply $\mathbf{P}(z)$ without comment.

Theorem 1.1. *Let S be a $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued function. The following statements are equivalent:*

1. *S belongs to $\mathcal{SA}_{\mathcal{D}_{\mathbf{P}}}(\mathcal{E}, \mathcal{E}_*)$.*
2. *There exist an auxiliary Hilbert space $\mathcal{H}$ and an analytic function*

$$H(z) = \begin{bmatrix} H_1(z) & \dots & H_p(z) \end{bmatrix} \tag{1.8}$$

defined on $\mathcal{D}_{\mathbf{P}}$ with values in $\mathcal{L}(\mathbb{C}^p \otimes \mathcal{H}, \mathcal{E}_*)$ so that

$$I_{\mathcal{E}_*} - S(z)S(w)^* = H(z)\left(I_{\mathbb{C}^p\otimes\mathcal{H}} - \mathbf{P}(z)\mathbf{P}(w)^*\right)H(w)^*, \tag{1.9}$$

or equivalently, there exists a positive kernel

$$\mathbb{K} = \begin{bmatrix} \mathbb{K}_{1,1} & \dots & \mathbb{K}_{1,p} \\ \vdots & & \vdots \\ \mathbb{K}_{p,1} & \dots & \mathbb{K}_{p,p} \end{bmatrix} : \Omega \times \Omega \mapsto \mathcal{L}(\mathbb{C}^p \otimes \mathcal{E}_*) \tag{1.10}$$

such that

$$I_{\mathcal{E}_*} - S(z)S(w)^* = \sum_{k=1}^{p} \mathbb{K}_{k,k}(z,w) - \sum_{k=1}^{p}\sum_{i,\ell=1}^{q} \mathbf{p}_{ik}(z)\overline{\mathbf{p}_{\ell k}(w)}\mathbb{K}_{i,\ell}(z,w)$$
$$\text{for all } z, w \in \mathcal{D}_{\mathbf{P}}. \tag{1.11}$$

3. *There is a unitary operator*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C}^p \otimes \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^q \otimes \mathcal{H} \\ \mathcal{E}_* \end{bmatrix} \tag{1.12}$$

such that

$$S(z) = D + C\left(I_{\mathbb{C}^p\otimes\mathcal{H}} - \mathbf{P}(z)A\right)^{-1}\mathbf{P}(z)B. \tag{1.13}$$

For S of the form (1.13) *it holds that*

$$I_{\mathcal{E}_*} - S(z)S(w)^* = C\left(I - \mathbf{P}(z)A\right)^{-1}\left(I - \mathbf{P}(z)\mathbf{P}(w)^*\right)\left(I - A^*\mathbf{P}(w)^*\right)^{-1}C^* \tag{1.14}$$

and therefore, the representation (1.9) *is valid for*

$$H(z) = C\left(I - \mathbf{P}(z)A\right)^{-1}. \tag{1.15}$$

The representation (1.13) is called *a unitary realization* of $S \in \mathcal{SA}_{\mathcal{D}_{\mathbf{P}}}(\mathcal{E}, \mathcal{E}_*)$. For the case of the unit disk ($\mathbf{P}(z) = z$ and $\mathcal{D}_{\mathbf{P}} = \mathbb{D}$), the formula (1.13) presents a unitary realization for a clasdscial Schur function S, i.e., coincides with the realization formula (1.1) for the classical case.

Domains $\mathcal{D}_{\mathbf{P}}$ and classes $\mathcal{SA}_{\mathcal{D}_{\mathbf{P}}}(\mathcal{E}, \mathcal{E}_*)$ enable us to consider in a unified way a wide variety of domains in $\mathbb{C}^n$ (we refer to [5, 12] for examples). In [12] we considered the bitangential Nevanlinna-Pick interpolation problem for Schur-Agler functions with all interpolation nodes inside $\mathcal{D}_{\mathbf{P}}$. Ambrozie and Eschmeier in [6] showed how to obtain results on interpolation as an application of a commutant lifting theorem for this setting. In the present paper we shall study the boundary Nevanlinna-Pick problem when prescribed (directional) values of an unknown interpolant are replaced by preassigned (directional) values of radial boundary limits. The techniques and results parallel those appearing in [11] for the special case of the polydisk and in [10] for the special case of the ball; the general case considered here, however, does present new features which must be understood before arriving at a complete solution. The foundation for the present method is the realization formula (1.13) for Schur-Agler functions in Theorem 1.1.

We assume that $\mathcal{D}_{\mathbf{P}}$ is bounded and completely circular, i.e., that *for every* $\zeta \in \mathbb{C}$, $|\zeta| \le 1$, *and* $z \in \mathcal{D}_{\mathbf{P}}$, *it holds that*

$$\zeta z = (\zeta z_1, \dots, \zeta z_n) \in \mathcal{D}_{\mathbf{P}}.$$

We assume furthermore that the distinguished boundary of $\mathcal{D}_{\mathbf{P}}$ defined as

$$\partial\mathcal{D}_{\mathbf{P}} = \{z \in \mathbb{C}^n \,:\, \mathbf{P}(z)\mathbf{P}(z)^* = I_p\},$$

is nonempty. The latter condition implies immediately that $p \le q$, which will be assumed throughout the paper.
In particular, Cartan domains of the first three types (more precisely, such domains with $p \le q$) and their direct products and intersections are included in this scheme.

Let Ω be a subset of $\partial\mathcal{D}_{\mathbf{P}}$. The data set for the interpolation problem is as follows. We are given an auxiliary Hilbert space $\mathcal{E}_L$ and three operator-valued functions

$$\begin{aligned} &\mathbf{a}\colon \Omega \mapsto \mathcal{L}(\mathcal{E}_L, \mathcal{E}_*), \qquad \mathbf{c}\colon \Omega \mapsto \mathcal{L}(\mathcal{E}_L, \mathcal{E}), \\ &\Psi\colon \Omega \mapsto \mathcal{L}(\mathcal{E}_L) \text{ with } \Psi(\xi) \ge 0 \text{ for } \xi \in \Omega. \end{aligned} \tag{1.16}$$

The interpolation problem to be studied here is the following:

Problem 1.2. *Find all functions* $S \in \mathcal{SA}_{\mathcal{D}_{\mathbf{P}}}(\mathcal{E}, \mathcal{E}_*)$ *such that*

$$\lim_{r\to 1} S(r\xi)^*\mathbf{a}(\xi) = \mathbf{c}(\xi) \quad \textit{for all } \xi \in \Omega \tag{1.17}$$

and

$$\lim_{r\to 1} \mathbf{a}(\xi)^* \, \frac{I_{\mathcal{E}_*} - S(r\xi)S(r\xi)^*}{1 - r^2} \, \mathbf{a}(\xi) \le \Psi(\xi) \quad \textit{for all } \xi \in \Omega, \tag{1.18}$$

where the limits in (1.17) *and* (1.18) *are understood in the strong and in the weak sense, respectively.*

Note that in case when Ω consists of finitely many points $z_1, \dots, z_n$, conditions (1.17) and (1.18) take the form (1.5) and (1.6), respectively, with $a_j = \mathbf{a}(z_j)$, $c_j = \mathbf{c}(z_j)$ and $\gamma_j = \Psi(z_j)$.

Condition (1.17) is called the *left-sided* interpolation condition for S. It follows by a multivariable operator-valued analogue of the classical Julia-Carathéodory theorem (see Lemma 2.1 below) that if the limit in (1.17) exists and equals $\mathbf{c}(\xi)$, then a necessary condition for the limits in (1.18) to exist and to be finite is

$$\mathbf{a}(\xi)^*\mathbf{a}(\xi) = \mathbf{c}(\xi)^*\mathbf{c}(\xi) \qquad (\xi \in \Omega). \tag{1.19}$$

It follows again by (the third assertion of) Lemma 2.1, that S satisfies also the right-sided interpolation condition

$$\lim_{r \to 1} S(r\xi)\mathbf{c}(\xi) = \mathbf{a}(\xi) \qquad (\xi \in \Omega).$$

Thus, Problem 1.2 is in fact a two-sided interpolation problem and conditions (1.19) are necessary for this problem to have a solution.

The paper is organized as follows. Upon completion of the present Introduction, in Section 2 we develop the Julia-Carathéodory theory for operator-valued functions on $\mathcal{D}_{\mathbf{P}}$ which we need for our purposes here, formulate the solution criterion (the existence of a positive kernel satisfying a Stein equation together with an inequality constraint) for existence of solutions to Problem 1.2 and derive the necessity part of this condition. In Section 3, under the assumption that this necessary condition holds, we show that solutions of Problem 1.2 are equal to characteristic functions of unitary colligations obtained as unitary extensions of a certain partial isometry constructed from any positive kernel meeting the conditions of the existence criterion. In Section 4 we adapt the techniques of Arov-Grossman to obtain the linear-fractional parametrization for the set of all characteristic functions of colligations equal to a unitary extension of one such partial isometry. When the partial isometry is taken to be that constructed from a particular positive kernel satisfying the conditions of the solution criterion, we have a parametrization for a particular subset of the set of all solutions of Problem 1.2. Taking the union of these over all positive kernels meeting the conditions of the solution criterion then gives us the set of all solutions. In particular, from this description of the set of all solutions we arrive at the sufficiency of the existence criterion.

2. The solvability criterion

In this section we establish the solvability criterion of Problem 1.2. We start with some auxiliary results. The assumption that $\mathcal{D}_{\mathbf{P}}$ is a bounded completely circular domain allows us to translate easily the classical Julia-Carathéodory theorem from the unit disk $\mathbb{D}$ to $\mathcal{D}_{\mathbf{P}}$ using slice–functions.

Lemma 2.1. *Let* $S \in \mathcal{SA}_{\mathcal{D}_{\mathbf{P}}}(\mathcal{E}, \mathcal{E}_*)$, $\beta \in \partial\mathcal{D}_{\mathbf{P}}$, $\mathbf{x} \in \mathcal{E}_*$ *and let* H_j $(j = 1, \dots, p)$ *be* $\mathcal{L}(\mathcal{H}, \mathcal{E}_*)$*-valued functions from the representation* (1.9). *Then:*
I. The following three statements are equivalent:

1. *S is subject to* $\mathbf{L} := \sup_{0 \le r < 1} \mathbf{x}^* \dfrac{I_{\mathcal{E}_*} - S(r\beta)S(r\beta)^*}{1 - r^2} \mathbf{x} < \infty$.
2. *The radial limit* $L := \lim_{r \to 1} \mathbf{x}^* \dfrac{I_{\mathcal{E}_*} - S(r\beta)S(r\beta)^*}{1 - r^2} \mathbf{x}$ *exists.*
3. *The radial limit*

$$\lim_{r \to 1} S(r\beta)^* \mathbf{x} = \mathbf{y} \tag{2.1}$$

exists in the strong sense and serves to define the vector $\mathbf{y} \in \mathcal{E}$. *Furthermore,*

$$\lim_{r \to 1} S(r\beta)\mathbf{y} = \mathbf{x}, \qquad \|\mathbf{y}\|_{\mathcal{E}} = \|\mathbf{x}\|_{\mathcal{E}_*}, \tag{2.2}$$

(the limit is understood in the weak sense) and the radial limit

$$\widetilde{L} = \lim_{r \to 1} \frac{\mathbf{y}^*\mathbf{y} - \mathbf{x}^* S(r\beta)\mathbf{y}}{1 - r} \tag{2.3}$$

exists.

II. Any two of the three equalities in (2.1) *and* (2.2) *imply the third.*
III. If any of the three statements in part I holds true, then the radial limits

$$T_j = \lim_{r \to 1} H_j(r\beta)^* \mathbf{x} \quad (j = 1, \dots, p) \tag{2.4}$$

exist in the strong sense and the block operator

$$T = \begin{bmatrix} T_1 \\ \vdots \\ T_p \end{bmatrix} \tag{2.5}$$

satisfies

$$T^* \Lambda_{\mathbf{P}}(\beta) T = L = \widetilde{L} \le \mathbf{L}, \tag{2.6}$$

where $\Lambda_{\mathbf{P}}(\beta) \in \mathcal{L}(\mathbb{C}^p \otimes \mathcal{H})$ *is defined via the limit*

$$\Lambda_{\mathbf{P}}(\beta) = \lim_{r \to 1} \frac{I_p - \mathbf{P}(\beta r)\mathbf{P}(\beta r)^*}{1 - r^2}. \tag{2.7}$$

Proof: For the proof of all the statements for the single-variable case ($d = 1$) and $\mathbf{P}(z) = z$ (i.e., for the case of the unit disk) see [10, Lemma 2.3] (all the statements but those related to T_j's and for finite dimensional $\mathcal{E}$ and $\mathcal{E}_*$ are contained in [26, Lemma 8.3, Lemma 8.4, and Theorem 8.5]). Note that in this case $\Lambda_{\mathbf{P}}(\beta) = I_p$. These proofs rely on the theory colligations and realization for Schur-class functions; for a more classical treatment (including for the case of the ball) and a survey of the various proofs of the Julia-Carathéodory theorem, we refer to Section 2.3 of [22].

For the general case, let us introduce the slice-function

$$S_\beta(\zeta) := S(\beta\zeta), \qquad (\zeta \in \mathbb{D}), \tag{2.8}$$

which clearly belongs to the classical Schur class $\mathcal{S}(\mathcal{E},\, \mathcal{E}_*)$ (indeed, it follows from (1.9) that S takes contractive values at each point of $\mathcal{D}_{\mathbf{P}}$ and in particular at each point $z \in \mathcal{D}_{\mathbf{P}}$ of the form $z = \zeta\beta$ $(\zeta \in \mathbb{D})$).

The first two statements of the lemma concern the boundary behavior of the function S_β near a boundary point $\zeta = 1$. Applying one-variable results to the slice-function S_β and returning to the original function S, we obtain the two first assertions. It is clear that the vectors $\mathbf{x} \in \mathcal{E}_*$ and $\mathbf{y} \in \mathcal{E}$ can be replaced by bounded linear operators from an auxiliary Hilbert space $\mathcal{E}_L$ into $\mathcal{E}_*$ and $\mathcal{E}$, respectively, with the limits then taken in the strong sense.

To prove the third statement, we first show that the limit in (2.7) exists. To this end we introduce the slice function

$$\mathbf{P}_\beta(\zeta) = \mathbf{P}(\beta\zeta), \qquad (\zeta \in \mathbb{D}). \tag{2.9}$$

Since $\mathcal{D}_{\mathbf{P}}$ is a bounded completely circular domain, it holds for every point $\beta \in \partial\mathcal{D}_{\mathbf{P}}$ and $\zeta \in \mathbb{D}$, that

$$\mathbf{P}_\beta(\zeta)\mathbf{P}_\beta(\zeta)^* = \mathbf{P}(\zeta\beta)\mathbf{P}(\zeta\beta)^* \le I_p$$

and thus, $\mathbf{P}_\beta$ belongs to the classical Schur class $\mathcal{S}(\mathbb{C}^q,\, \mathbb{C}^p)$. Since it takes a coisometric value at $\zeta = 1$, one can apply the one-variable result (statements I and II) of the lemma to $\mathbf{x} = I_p$ and $\mathbf{y} = S(\beta)^*$ to conclude that

$$\lim_{r\to 1} \frac{I_p - \mathbf{P}_\beta(r)\mathbf{P}_\beta(r)^*}{1-r^2} = \lim_{r\to 1} \frac{\mathbf{P}_\beta(1) - \mathbf{P}_\beta(r)}{1-r}\mathbf{P}_\beta(1)^* \tag{2.10}$$

and since $\mathbf{P}_\beta$ is a polynomial, the second limit in (2.10) exists and equals $\mathbf{P}_\beta'(1)\mathbf{P}_\beta(1)^*$. Therefore, the limit in (2.7) exists.

Since $\mathbf{P}_\beta(\zeta)$ is a Schur function, there is an auxiliary Hilbert space $\mathcal{G}_\beta$ and a unitary operator

$$\mathbf{U}_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \; : \; \begin{bmatrix} \mathcal{G}_\beta \\ \mathbb{C}^q \end{bmatrix} \to \begin{bmatrix} \mathcal{G}_\beta \\ \mathbb{C}^p \end{bmatrix} \tag{2.11}$$

such that

$$\mathbf{P}_\beta(\zeta) = D_1 + \zeta C_1 \left(I_{\mathcal{G}_\beta} - \zeta A_1\right)^{-1} B_1. \tag{2.12}$$

On the other hand, upon setting $z = \zeta\beta$ in (1.13) we get

$$S_\beta(\zeta) = D + C\left(I_{\mathbb{C}^p\otimes\mathcal{H}} - \mathbf{P}_\beta(\zeta)A\right)^{-1}\mathbf{P}_\beta(\zeta)B \tag{2.13}$$

(in this realization we have in fact $\mathbf{P}_\beta(\zeta)\otimes I_{\mathcal{H}}$, which (by the above convention) is still denoted by $\mathbf{P}_\beta(\zeta)$) and similarly, we conclude from (1.9) that

$$\begin{aligned} L &= \lim_{r\to 1} \mathbf{x}^* \frac{I_{\mathcal{E}_*} - S(r\beta)S(r\beta)^*}{1-r^2}\mathbf{x} \\ &= \lim_{r\to 1} \mathbf{x}^* H(r\beta)\frac{I_{\mathbb{C}^p\otimes\mathcal{H}} - \mathbf{P}_\beta(r)\mathbf{P}_\beta(r)^*}{1-r^2}H(r\beta)^*\mathbf{x}. \end{aligned} \tag{2.14}$$

Now we combine realization formulas (2.12) and (2.13) to get a unitary realization for the Schur function S_β:

$$S_\beta(\zeta) = D_2 + \zeta C_2 \left(I - \zeta A_2\right)^{-1} B_2, \tag{2.15}$$

where the operator

$$\mathbf{U}_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \ : \ \begin{bmatrix} \mathcal{G}_\beta \\ \mathcal{E} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{G}_\beta \\ \mathcal{E}_* \end{bmatrix} \tag{2.16}$$

given by

$$\begin{aligned} A_2 &= A_1 + B_1(I - AD_1)^{-1}AC_1, && (2.17)\\ B_2 &= B_1(I - AD_1)^{-1}B, && (2.18)\\ C_2 &= C(I - D_1A)^{-1}C_1, \\ D_2 &= D + C(I - D_1A)^{-1}D_1B, \end{aligned}$$

is unitary. Indeed, by (1.13) and (2.12),

$$\begin{aligned} H(\zeta\beta) &= C(I - \mathbf{P}_\beta(\zeta)A)^{-1} \\ &= C(I - D_1A - \zeta C_1 \left(I - \zeta A_1\right)^{-1} B_1A)^{-1} \\ &= C(I - D_1A)^{-1} \left[I - \zeta C_1(I - \zeta A_1)^{-1}B_1A(I - D_1A)^{-1}\right]^{-1} \\ &= C(I - D_1A)^{-1} + \zeta C_2(I - \zeta A_2)^{-1}B_1A(I - D_1A)^{-1}, && (2.19) \end{aligned}$$

and taking into account that

$$\zeta(I - \zeta A_2)^{-1}B_1A(I - D_1A)^{-1}C_1(I - \zeta A_1)^{-1} = (I - \zeta A_2)^{-1} - (I - \zeta A_1)^{-1},$$

we obtain

$$\begin{aligned} S_\beta(\zeta) &= D + H(\zeta\beta)\mathbf{P}_\beta(\zeta)B \\ &= D + \left[C(I - D_1A)^{-1} + \zeta C_2(I - \zeta A_2)^{-1}B_1A(I - D_1A)^{-1}\right] \\ &\qquad \times \left[D_1 + \zeta C_1 \left(I - \zeta A_1\right)^{-1} B_1\right] B \\ &= D + C(I - D_1A)^{-1}D_1B + \zeta C_2 \left(I - \zeta A_1\right)^{-1} B_1B \\ &\quad + \zeta C_2(I - \zeta A_2)^{-1}B_1A(I - D_1A)^{-1}D_1B \\ &\quad + \zeta C_2 \left[(I - \zeta A_2)^{-1} - (I - \zeta A_1)^{-1}\right] B_1B \\ &= D_2 + \zeta C_2(I - \zeta A_2)^{-1}B_1 \left[I + A(I - D_1A)^{-1}D_1\right] B \\ &= D_2 + \zeta C_2(I - \zeta A_2)^{-1}B_1(I - AD_1)^{-1}B \\ &= D_2 + \zeta C_2(I - \zeta A_2)^{-1}B_2. \end{aligned}$$

The verification of the fact that the operator $\mathbf{U}_2$ is unitary is straightforward; it involves the explicit formulas for the blocks A_2, B_2, C_2, D_2 and the fact that the operators $\mathbf{U}$ and $\mathbf{U}_1$ are unitary. Alternatively, one can pursue the following higher level path which perhaps is more informative.

The starting point is the observation that the colligation

$$\mathbf{U}_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} : \begin{bmatrix} \mathcal{G}_\beta \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{G}_\beta \\ \mathcal{E}_* \end{bmatrix}$$

arises as the feedback coupling of $\mathbf{U}_1 \otimes (I_{\mathbb{C}} \oplus I_{\mathcal{H}})$ with $\mathbf{U}$:

$$\mathbf{U}_2 \colon \begin{bmatrix} x_1 \\ u \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{x}_1 \\ y \end{bmatrix} \text{ if and only if}$$

$$\mathbf{U} \colon \begin{bmatrix} x \\ u \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{x} \\ y \end{bmatrix} \text{ and } \mathbf{U}_1 \otimes (I_{\mathbb{C}} \otimes I_{\mathcal{H}}) \colon \begin{bmatrix} x_1 \\ \widetilde{x} \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{x}_1 \\ x \end{bmatrix}.$$

By the general principle that a feedback coupling of 2×2 block unitary operators is again unitary, we see that the colligation $\mathbf{U}_2$ is unitary.

It follows from the unitarity of the realization (2.15) that

$$\frac{I_{\mathcal{E}_*} - S_\beta(\zeta) S_\beta(\omega)^*}{1 - \zeta\overline{\omega}} = C_2 (I - \zeta A_2)^{-1} (I - \overline{\omega} A_2^*)^{-1} C_2^*$$

and by the one-variable result, there exists the strong limit

$$\widetilde{T} := \lim_{r \to 1} (I - r A_2^*)^{-1} C_2^* \mathbf{x}. \tag{2.20}$$

Setting $\zeta = r$ in (2.19) and letting $r \to 1$, we come to the existence of the following strong limit

$$T := \lim_{r \to 1} H(r\beta)^* \mathbf{x} = (I - A^* D_1^*)^{-1} \left(C^* \mathbf{x} + A^* B_1^* \widetilde{T} \right). \tag{2.21}$$

On account of the block structure (1.8) of H, the strong limits T_j's in (2.4) exist. It remains to show that T satisfies relations (2.6). But now this follows from (2.14), (2.7) and (2.21):

$$L = \lim_{r \to 1} \mathbf{x}^* H(r\beta) \frac{I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{P}_\beta(r) \mathbf{P}_\beta(r)^*}{1 - r^2} H(r\beta)^* \mathbf{x} = T^* \Lambda_{\mathbf{P}}(\beta) T,$$

which completes the proof. □

To state the solvability criterion of Problem 1.2 we introduce some notations. Let

$$E_1 = \begin{bmatrix} I_{\mathcal{E}_L} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ I_{\mathcal{E}_L} \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad E_p = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{\mathcal{E}_L} \end{bmatrix} \in \mathcal{L}(\mathcal{E}_L, \mathbb{C}^p \otimes \mathcal{E}_L), \tag{2.22}$$

let $N_1, \ldots, N_q : \Omega \to \mathcal{L}(\mathcal{E}, \mathbb{C}^p \otimes \mathcal{E}_L)$ be the functions given by

$$N_k(\xi) = \begin{bmatrix} \overline{\mathbf{p}_{1,k}(\xi)} I_{\mathcal{E}_L} \\ \vdots \\ \overline{\mathbf{p}_{p,k}(\xi)} I_{\mathcal{E}_L} \end{bmatrix} \qquad (k = 1, \ldots, q) \tag{2.23}$$

and let $\Lambda_{\mathbf{P}}:\ \Omega\to\mathbb{C}^{p\times p}$ be the function defined via the limit

$$\Lambda_{\mathbf{P}}(\xi)=[\Lambda_{ij}(\xi)]_{i,j=1}^{p}=\lim_{r\to 1}\frac{I_p-\mathbf{P}(r\xi)\mathbf{P}(r\xi)^*}{1-r^2}. \tag{2.24}$$

By the preceding analysis, $\Lambda_{\mathbf{P}}$ is well defined at each point $\xi\in\partial\mathcal{D}_{\mathbf{P}}$.

Theorem 2.2. *Problem 1.2 has a solution if and only if there exist p^2 $\mathcal{L}(\mathcal{E})$-valued kernels $K_{ij}(\xi,\mu)$ on $\Omega\times\Omega$ subject to the Stein identity*

$$\sum_{j=1}^{p}K_{jj}(\xi,\mu)-\sum_{k=1}^{q}\left(\sum_{i,\ell=1}^{p}\mathbf{p}_{ik}(\xi)\overline{\mathbf{p}_{k\ell}(\mu)}K_{i\ell}(\xi,\mu)\right)=\mathbf{a}(\xi)^*\mathbf{a}(\mu)-\mathbf{c}(\xi)^*\mathbf{c}(\mu) \tag{2.25}$$

and to the constraint

$$\sum_{i,j=1}^{p}\Lambda_{ij}(\xi)K_{ij}(\xi,\xi)\le\Psi(\xi) \tag{2.26}$$

for every $\xi,\mu\in\Omega$ and such that the kernel

$$K(\xi,\mu)=\begin{bmatrix}K_{11}(\xi,\mu) & \dots & K_{1p}(\xi,\mu)\\ \vdots & & \vdots\\ K_{p1}(\xi,\mu) & \dots & K_{pp}(\xi,\mu)\end{bmatrix} \tag{2.27}$$

is positive on Ω:

$$K(\xi,\mu)\succeq 0. \tag{2.28}$$

Taking advantage of operators (2.22) and functions (2.23) and (2.24), one can rewrite relation (2.25) as

$$\sum_{k=1}^{p}E_k^*K(\xi,\mu)E_k-\sum_{k=1}^{q}N_k(\xi)^*K(\xi,\mu)N_k(\mu)=\mathbf{a}(\xi)^*\mathbf{a}(\mu)-\mathbf{c}(\xi)^*\mathbf{c}(\mu). \tag{2.29}$$

Proof: Here we check the necessity of conditions (2.25)–(2.28). The proof of the sufficiency part is postponed until Section 4 where it will be obtained as a consequence of slightly stronger results. Let S be a solution of Problem 1.2, that is, let S be an element of $\mathcal{SA}_{\mathcal{D}_{\mathbf{P}}}(\mathcal{E},\,\mathcal{E}_*)$ which satisfies the interpolation conditions (1.17) and (1.18). Since S belongs to $\mathcal{SA}_{\mathcal{D}_{\mathbf{P}}}(\mathcal{E},\,\mathcal{E}_*)$, the identity (1.9) holds for some $\mathcal{L}(\mathbb{C}^p\otimes\mathcal{H},\mathcal{E}_*)$-valued function H which is analytic on $\mathcal{D}_{\mathbf{P}}$. Let H be decomposed into blocks $H_1,\dots,H_p$ as in (1.8) and let $T_j(\xi)$ stand for the following strong limit

$$T_j(\xi):=\lim_{r\to 1}H_j(r\xi)^*\mathbf{a}(\xi)\quad(j=1,\dots,p;\ \xi\in\Omega), \tag{2.30}$$

which exists at every point $\xi \in \Omega$, by Lemma 2.1. Then the block operator-valued function

$$T(\xi) := \begin{bmatrix} T_1(\xi) \\ \vdots \\ T_p(\xi) \end{bmatrix} = \lim_{r\to 1} H(r\xi)^* \mathbf{a}(\xi) \quad (\xi \in \Omega) \tag{2.31}$$

satisfies (again by Lemma 2.1)

$$T(\xi)^* \Lambda_{\mathbf{P}}(\xi) T(\xi) = L(\xi) := \lim_{r\to 1} \mathbf{a}(\xi)^* \frac{I_{\mathcal{E}_*} - S(r\xi)S(r\xi)^*}{1-r^2} \mathbf{a}(\xi) \tag{2.32}$$

for each $\xi \in \Omega$. Let

$$K_{ij}(\xi, \mu) = T_i(\xi)^* T_j(\mu) \quad (i, j = 1, \dots, p;\ \xi, \mu \in \Omega). \tag{2.33}$$

Then the kernel $K(\xi, \mu)$ defined in (2.27) admits a representation

$$K(\xi, \mu) = \begin{bmatrix} T_1(\xi)^* \\ \vdots \\ T_p(\xi)^* \end{bmatrix} \begin{bmatrix} T_1(\mu) & \dots & T_p(\mu) \end{bmatrix} \tag{2.34}$$

and is clearly positive on Ω. Furthermore, condition (2.26) holds by (1.18), (2.32) and (2.34):

$$\begin{aligned} \sum_{i,j=1}^{p} \Lambda_{ij}(\xi) K_{ij}(\xi, \xi) &= T(\xi)^* \Lambda_{\mathbf{P}}(\xi) T(\xi) \\ &= \lim_{r\to 1} \mathbf{a}(\xi)^* \frac{I_{\mathcal{E}_*} - S(r\xi)S(r\xi)^*}{1-r^2} \mathbf{a}(\xi) \le \Psi(\xi). \end{aligned}$$

Finally, setting $z = r\xi$ and $w = r\mu$ $(\xi \neq \mu)$ in (1.9) and multiplying both sides in the resulting identity by $\mathbf{a}(\xi)^*$ on the left and by $\mathbf{a}(\mu)$ on the right, we get

$$\mathbf{a}(\xi)^* \left(I_{\mathcal{E}_*} - S(r\xi)S(r\mu)^*\right) \mathbf{a}(\mu) = \mathbf{a}(\xi)^* H(r\xi)(I - \mathbf{P}(r\xi)\mathbf{P}(r\mu)^*) H(r\mu)^* \mathbf{a}(\mu). \tag{2.35}$$

Making use of interpolation condition (1.17), we get

$$\lim_{r\to 1} \mathbf{a}(\xi)^* \left(I_{\mathcal{E}_*} - S(r\xi)S(r\mu)^*\right) \mathbf{a}(\mu) = \mathbf{a}(\xi)^* \mathbf{a}(\mu) - \mathbf{c}(\xi)^* \mathbf{c}(\mu),$$

whereas relations (2.31), (2.34) together with (1.13) and partitionings (1.7) and (1.8) lead to

$$\begin{aligned}
&\lim_{r\to 1} \mathbf{a}(\xi)^* H(r\xi))\,(I_{\mathbb{C}^p\otimes\mathcal{H}} - \mathbf{P}(r\xi)\mathbf{P}(r\mu)^*)\,H(r\mu)^*\mathbf{a}(\mu)\\
&= \lim_{r\to 1}\sum_{j=1}^{p} \mathbf{a}(\xi)^* H_j(r\xi) H_j(r\mu)^*\mathbf{a}(\mu)\\
&\quad - \lim_{r\to 1}\sum_{i,\ell=1}^{p} \mathbf{a}(\xi)^* H_i(r\xi)\left(\sum_{k=1}^{q} \mathbf{p}_{ik}(r\xi)\overline{\mathbf{p}_{k\ell}(r\mu)}\right) H_\ell(r\mu)^*\mathbf{a}(\mu)\\
&= \sum_{j=1}^{p} T_j(\xi)^* T_j(\mu) - \sum_{i,\ell=1}^{p}\left(\sum_{k=1}^{q} \mathbf{p}_{ik}(\xi)\overline{\mathbf{p}_{k\ell}(\mu)}\right) T_i(\xi)^* T_\ell(\mu)\\
&= \sum_{j=1}^{p} K_{jj}(\xi,\mu) - \sum_{k=1}^{q}\left(\sum_{i,\ell=1}^{p} \mathbf{p}_{ik}(\xi)\overline{\mathbf{p}_{k\ell}(\mu)} K_{i\ell}(\xi,\mu)\right).
\end{aligned}$$

Taking limits as $r \to 1$ on both sides in (2.35) and making use of the last two equalities, we get (2.25). □

3. Solutions to the interpolation problem and unitary extensions

In this section we analyze the structure of the set of solutions of Problem 1.2 under the assumption that the necessary conditions (2.25)–(2.28) hold. We therefore assume throughout this section: *we are given an interpolation data set* $(\mathbf{a}, \mathbf{c}, \Psi)$ *as in* (1.16) *and there exist an* $\mathcal{L}(\mathcal{E}_L)$*-valued positive kernel* $K(\xi,\mu)$ *on* $\Omega\times\Omega$ *satisfying the necessary conditions* (2.25)–(2.28) *in Theorem 2.2.*

We define a $\mathbf{P}$-colligation as a quadruple

$$\mathcal{C} = \{\mathcal{H},\ \mathcal{E},\ \mathcal{E}_*,\ \mathbf{U}\} \tag{3.1}$$

consisting of three Hilbert spaces $\mathcal{H}$ (*the state space*), $\mathcal{E}$ (*the input space*) and $\mathcal{E}_*$ (*the output space*), together with a connecting operator

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}:\quad \begin{bmatrix} \mathbb{C}^p\otimes\mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^q\otimes\mathcal{H} \\ \mathcal{E}_* \end{bmatrix}. \tag{3.2}$$

The colligation is said to be *unitary* if the connecting operator $\mathbf{U}$ is unitary. A colligation

$$\widetilde{\mathcal{C}} = \{\widetilde{\mathcal{H}},\ \mathcal{E},\ \mathcal{E}_*,\ \widetilde{\mathbf{U}}\} \tag{3.3}$$

is said to be *unitarily equivalent* to the colligation $\mathcal{C}$ if there is a unitary operator $\alpha:\ \mathcal{H}\to\widetilde{\mathcal{H}}$ such that

$$\begin{bmatrix} \alpha\otimes I_q & 0 \\ 0 & I_{\mathcal{E}_*} \end{bmatrix}\mathbf{U} = \widetilde{\mathbf{U}}\begin{bmatrix} \alpha\otimes I_p & 0 \\ 0 & I_{\mathcal{E}} \end{bmatrix}$$

The *characteristic function* of the colligation $\mathcal{C}$ is defined as

$$S_{\mathcal{C}}(z) = D + C\left(I_{\mathbb{C}^p\otimes\mathcal{H}} - \mathbf{P}(z)A\right)^{-1}\mathbf{P}(z)B. \tag{3.4}$$

Thus, Theorem 1.1 asserts that a $\mathcal{L}(\mathcal{E},\,\mathcal{E}_*)$-valued function S analytic on $\mathcal{D}_{\mathbf{P}}$ belongs to the class $\mathcal{SA}_{\mathcal{D}_{\mathbf{P}}}(\mathcal{E},\,\mathcal{E}_*)$ if and only it is the characteristic function of some unitary $\mathbf{P}$-colligation $\mathcal{C}$ of the form (3.2).

Remark 3.1. *Unitarily equivalent colligations have the same characteristic function.*

In this section, under the assumption that the necessary conditions (2.25)–(2.28) in Theorem 2.2 hold, we identify a class of unitary colligations, the characteristic functions of which give rise exactly to the set of all solutions of Problem 1.2. These unitary colligations coincide with unitary extensions of certain partial isometry $\mathcal{V}_K$ constructed from some positive kernel meeting the conditions (2.25)–(2.28) for Problem 1.2. We therefore assume that there exist positive kernel solutions K of conditions (2.25)–(2.28) and we let E_j and N_j be defined as in (2.22) and (2.23), respectively.

Let $\mathcal{H}_0$ be the linear space of $\mathcal{E}_L$-valued functions $f(\xi)$ defined on Ω which take nonzero values at at most finitely many points and let $\mathcal{H}_1$ be the linear space of $\mathbb{C}^p\otimes\mathcal{E}_L$-valued functions $\xi\mapsto h(\xi)$ defined on Ω which take nonzero values at at most finitely many points Let $X\in\mathcal{L}(\mathcal{H}_0,\mathcal{E}_*)$ and $Y\in\mathcal{L}(\mathcal{H}_0,\,\mathcal{E})$ be operators defined by

$$Xf = \sum_{\xi\in\Omega}\mathbf{a}(\xi)f(\xi), \qquad Yf = \sum_{\xi\in\Omega}\mathbf{c}(\xi)f(\xi). \tag{3.5}$$

For a fixed choice of positive kernel K meeting conditions (2.25)–(2.28), let $D_K(h,g)$ be the quadratic form on $\mathcal{H}_1\times\mathcal{H}_1$ defined as

$$D_K(h,g) = \sum_{\xi_i,\xi_\ell\in\Omega}\langle K(\xi_i,\xi_\ell)h(\xi_\ell),\; g(\xi_i)\rangle_{\mathbb{C}^p\otimes\mathcal{E}_L}. \tag{3.6}$$

It follows by the definitions of the spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ and by the definitions (2.22) and (2.23) of E_j and N_k, that for every $f\in\mathcal{H}_0$, the functions E_jf and N_kf belong to $\mathcal{H}_1$. Furthermore, it follows from B(2.25) that

$$\sum_{j=1}^{p}D_K(E_jf,\,E_jg) - \sum_{k=1}^{q}D_K(N_kf,\,N_kg) = \langle Xf,\;Xg\rangle_{\mathcal{E}_*} - \langle Yf,\;Yg\rangle_{\mathcal{E}}. \tag{3.7}$$

We say that $h_1\sim h_2$ if and only if $D_K(h_1-h_2,\;y)=0$ for all $y\in\mathcal{H}_0$ and denote $[h]$ the equivalence class of h with respect to the above equivalence. The linear space of equivalence classes endowed with the inner product

$$\langle[h],\;[g]\rangle = D_K(h,\;g) \tag{3.8}$$

is a prehilbert space, whose completion we denote by $\widehat{\mathcal{H}}_K$. Rewriting (3.7) as

$$\sum_{j=1}^{p}\langle [E_j f],\, [E_j g]\rangle_{\widehat{\mathcal{H}}} + \langle Yf,\, Yg\rangle_{\mathcal{E}} = \sum_{k=1}^{q}\langle [N_k f],\, [N_k g]\rangle_{\widehat{\mathcal{H}}} + \langle Xf,\, Xg\rangle_{\mathcal{E}_*},$$

we conclude that the linear map

$$\mathbf{V}_K : \begin{bmatrix} [E_1 f] \\ \vdots \\ [E_p f] \\ Yf \end{bmatrix} \rightarrow \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_q f] \\ Xf \end{bmatrix} \tag{3.9}$$

is an isometry from

$$\mathcal{D}_{\mathbf{V}_K} = \text{Clos}\left\{ \begin{bmatrix} [E_1 f] \\ \vdots \\ [E_p f] \\ Yf \end{bmatrix}, \ f \in \mathcal{H}_1 \right\} \subset \begin{bmatrix} \mathbb{C}^p \otimes \widehat{\mathcal{H}}_K \\ \mathcal{E} \end{bmatrix} \tag{3.10}$$

onto

$$\mathcal{R}_{\mathbf{V}_K} = \text{Clos}\left\{ \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_q f] \\ Xf \end{bmatrix}, \ f \in \mathcal{H}_1 \right\} \subset \begin{bmatrix} \mathbb{C}^q \otimes \widehat{\mathcal{H}}_K \\ \mathcal{E}_* \end{bmatrix}. \tag{3.11}$$

The next two lemmas establish a correspondence between solutions S to Problem 1.2 and unitary extensions of the partially defined isometry $\mathbf{V}_K$ associated with some positive kernel solution K of (2.25)–(2.28) given in (3.6).

Lemma 3.2. *Let S be a solution of Problem 1.2. Then there exists a kernel K satisfying conditions* (2.25)–(2.28) *such that S is a characteristic function of a unitary colligation*

$$\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} : \begin{bmatrix} \mathbb{C}^p \otimes (\widehat{\mathcal{H}}_K \oplus \widetilde{\mathcal{H}}) \\ \mathcal{E} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{C}^q \otimes (\widehat{\mathcal{H}}_K \oplus \widetilde{\mathcal{H}}) \\ \mathcal{E}_* \end{bmatrix}, \tag{3.12}$$

which is an extension of the isometry $\mathbf{V}_K$ given in (3.9).

Proof: Let S be a solution to Problem 1.2. In particular, S belongs to $\mathcal{SA}_{\mathcal{D}_{\mathbf{P}}}(\mathcal{E},\, \mathcal{E}_*)$ and by Theorem 1.1, it is the characteristic function of some unitary colligation $\mathcal{C}$ of the form (3.2). In other words, S admits a unitary realization (1.13) with the state space $\mathcal{H}$ and representation (1.9) holds for the function H defined via (1.15) and decomposed as in (1.8). This function is analytic and $\mathcal{L}(\mathbb{C}^p \otimes \mathcal{H},\, \mathcal{E}_*)$-valued on $\mathcal{D}_{\mathbf{P}}$ and leads to the following representation

$$S(z) = D + H(z)\mathbf{P}(z)B \tag{3.13}$$

of S, which is equivalent to (1.13).

The interpolation conditions (1.17) and (1.18) which are assumed to be satisfied by S, force certain restrictions on the connecting operator $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. By Lemma 2.1, the strong limit (2.31) exists. Substituting (3.13) into (1.17) we get

$$\lim_{r\to 1} \left(D^* + B^*\mathbf{P}(r\xi)^*H(r\xi)^*\right)\mathbf{a}(\xi) = \mathbf{c}(\xi) \quad (\xi \in \Omega),$$

where the limit is understood in the strong sense. It also follows from (1.15) that $C + H(z)Z(z)A = H(z)$ and therefore, that (strongly)

$$C^*\mathbf{a}(\xi) + \lim_{r\to 1} A^*\mathbf{P}(r\xi)^*H(r\xi)^*\mathbf{a}(\xi) = \lim_{r\to 1} H(r\xi)^*\mathbf{a}(\xi).$$

By (2.31), the two last (displayed) equalities are equivalent to

$$D^*\mathbf{a}(\xi) + B^*\mathbf{P}(\xi)^*T(\xi) = \mathbf{c}(\xi) \tag{3.14}$$

and

$$C^*\mathbf{a}(\xi) + A^*\mathbf{P}(\xi)^*T(\xi) = T(\xi), \tag{3.15}$$

respectively, which can be written in matrix form as

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} \mathbf{P}(\xi)^*T(\xi) \\ \mathbf{a}(\xi) \end{bmatrix} = \begin{bmatrix} T(\xi) \\ \mathbf{c}(\xi) \end{bmatrix} \quad (\xi \in \Omega).$$

Since the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is unitary, we conclude from the last equality that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T(\xi) \\ \mathbf{c}(\xi) \end{bmatrix} = \begin{bmatrix} \mathbf{P}(\xi)^*T(\xi) \\ \mathbf{a}(\xi) \end{bmatrix}. \tag{3.16}$$

Let K_{ij} and K be defined as in (2.33) and (2.34), let $\mathbf{V}_K$ be the isometry given in (3.9) and let

$$R(\xi) = \begin{bmatrix} T_1(\xi) & \dots & T_p(\xi) \end{bmatrix} \tag{3.17}$$

so that

$$K(\xi, \mu) = R(\xi)^*R(\mu). \tag{3.18}$$

Define the operator $\mathbf{T}: \mathcal{H}_1 \to \mathcal{H}$ by the rule

$$\mathbf{T}h = \sum_{\xi} R(\xi)h(\xi). \tag{3.19}$$

Upon making subsequent use of (3.8), (3.6), (3.18) and (3.19), we get

$$\begin{aligned}
\langle [h],\, [y] \rangle_{\widehat{\mathcal{H}}_K} = D_K(h,\, y) &= \sum_{\xi_i, \xi_\ell} \langle K(\xi_i, \xi_\ell)h(\xi_\ell),\; y(\xi_i)\rangle_{\mathcal{E}} \\
&= \sum_{\xi_i, \xi_\ell} \langle R(\xi_\ell)h(\xi_\ell),\; R(\xi_i)y(\xi_i)\rangle_{\mathcal{H}} \\
&= \left\langle \sum_{\xi_\ell} R(\xi_\ell)h(\xi_\ell),\; \sum_{\xi_i} R(\xi_i)y(\xi_i) \right\rangle_{\mathcal{H}} \\
&= \langle \mathbf{T}h,\; \mathbf{T}y\rangle_{\mathcal{H}}.
\end{aligned}$$

Therefore, the linear transformation U defined by the rule

$$U\ :\ \ \mathbf{T}f \rightarrow [f] \qquad (f \in \mathcal{H}_0) \tag{3.20}$$

can be extended to the unitary map (which still is denoted by U) from $\overline{\operatorname{Ran}\mathbf{T}}$ onto $\widehat{\mathcal{H}}_K$. Noticing that $\overline{\operatorname{Ran}\mathbf{T}}$ is a subspace of $\mathcal{H}$ and setting

$$\mathcal{N} := \mathcal{H} \ominus \overline{\operatorname{Ran}\mathbf{T}} \quad \text{and} \quad \widetilde{\mathcal{H}} := \widehat{\mathcal{H}}_K \oplus \mathcal{N},$$

we define the unitary map $\widetilde{U}\ :\ \ \mathcal{H} \rightarrow \widetilde{\mathcal{H}}$ by the rule

$$\widetilde{U}\, g = \begin{cases} Ug & \text{for} \quad g \in \overline{\operatorname{Ran}\mathbf{T}} \\ g & \text{for} \quad g \in \mathcal{N}. \end{cases} \tag{3.21}$$

Introducing the operators

$$\widetilde{A} = (\widetilde{U} \otimes I_q)A(\widetilde{U} \otimes I_p)^*, \quad \widetilde{B} = (\widetilde{U} \otimes I_q)B, \quad \widetilde{C} = C(\widetilde{U} \otimes I_p)^*, \quad \widetilde{D} = D$$

we construct the colligation $\widetilde{\mathcal{C}}$ via (3.3) and (3.12). By definition, $\widetilde{\mathcal{C}}$ is unitarily equivalent to the initial colligation $\mathcal{C}$ defined in (3.1). By Remark 3.1, $\widetilde{\mathcal{C}}$ has the same characteristic function as $\mathcal{C}$, that is, $S(z)$. It remains to check that the connecting operator of $\widetilde{\mathcal{C}}$ is an extension of $\mathbf{V}_K$, i.e.,

$$\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} [E_1 f] \\ \vdots \\ [E_p f] \\ Yf \end{bmatrix} = \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_q f] \\ Xf \end{bmatrix}, \qquad f \in \mathcal{H}_0. \tag{3.22}$$

To this end, note that by (3.20), (3.21), it holds for every $f \in \mathcal{H}_0$ that

$$\widetilde{U}^*\left([E_j f]\right) = \mathbf{T}(E_j f) = \sum_{\xi} R(\xi)E_j f(\xi) = \sum_{\xi} T_j(\xi)f(\xi) \quad (j = 1, \ldots, p)$$

and therefore,

$$(\widetilde{U} \otimes I_p)^* \left(\begin{bmatrix} [E_1 f] \\ \vdots \\ [E_p f] \end{bmatrix} \right) = \begin{bmatrix} T_1(\xi) \\ \vdots \\ T_p(\xi) \end{bmatrix} f(\xi) = T(\xi)f(\xi). \tag{3.23}$$

Similarly,

$$\widetilde{U} \left(\sum_{\xi} R(\xi)N_k(\xi)f(\xi) \right) = \widetilde{U}\mathbf{T}(N_k f) = [N_k f] \quad (k = 1, \ldots, q)$$

and since $R(\xi)N_k(\xi)$ is equal to the k-th block row of $\mathbf{P}(\xi)T(\xi)$ (which is clear from the definitions (3.17), (2.32) and (2.23) of R, T and N_k), it follows that

$$(\widetilde{U} \otimes I_q)\left(\mathbf{P}(\xi)^* T(\xi)f(\xi)\right) = \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_p f] \end{bmatrix}. \tag{3.24}$$

Thus, by (3.16), (3.23) and (3.24),

$$\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} [E_1 f] \\ \vdots \\ [E_p f] \\ Yf \end{bmatrix} = \begin{bmatrix} \widetilde{U} \otimes I_q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} (\widetilde{U} \otimes I_p)^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} [E_1 f] \\ \vdots \\ [E_p f] \\ Yf \end{bmatrix}$$

$$= \begin{bmatrix} \widetilde{U} \otimes I_q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \left(\sum_{\xi} \begin{bmatrix} T(\xi) \\ \mathbf{c}(\xi) \end{bmatrix} f(\xi) \right)$$

$$= \begin{bmatrix} \widetilde{U} \otimes I_q & 0 \\ 0 & I \end{bmatrix} \left(\sum_{\xi} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T(\xi) \\ \mathbf{c}(\xi) \end{bmatrix} f(\xi) \right)$$

$$(3.25) \qquad = \begin{bmatrix} \widetilde{U} \otimes I_q & 0 \\ 0 & I \end{bmatrix} \left(\sum_{\xi} \begin{bmatrix} \mathbf{P}(\xi)^* T(\xi) \\ \mathbf{a}(\xi) \end{bmatrix} f(\xi) \right) = \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_q f] \\ Xf \end{bmatrix},$$

which proves (3.22) and completes the proof of the lemma. □

The converse statement will be proved in Theorem 3.4 below. We start with some auxiliary results.

Lemma 3.3. *Let $A: \mathbb{C}^p \otimes \mathcal{H} \to \mathbb{C}^q \otimes \mathcal{H}$ be a contraction, and let β be a point in $\partial \mathcal{D}_{\mathbf{P}}$. Then the following limit*

$$(3.26) \qquad \Upsilon_\beta := \lim_{r \to 1} (1-r)(I - A\mathbf{P}(r\beta))^{-1}$$

exists in the strong sense and satisfies

$$(3.27) \qquad \Upsilon_\beta \ge 0, \quad A\mathbf{P}(\beta)\Upsilon_\beta = \Upsilon_\beta \quad \text{and} \quad \Upsilon_\beta A = \Upsilon_\beta \mathbf{P}(\beta)^*.$$

Furthermore, the following limits also exist in the strong sense:

$$(3.28) \qquad \lim_{r \to 1} (I - A\mathbf{P}(r\beta))^{-1}(I - A\mathbf{P}(\beta)) = I - \Upsilon_\beta \mathbf{P}(\beta)^* W_\beta,$$

$$(3.29) \qquad \lim_{r \to 1} (I - \mathbf{P}(\beta)A)(I - \mathbf{P}_\beta(r)A)^{-1} = I - W_\beta \Upsilon_\beta \mathbf{P}(\beta)^*,$$

where

$$(3.30) \qquad W_\beta = \lim_{r \to 1} \frac{\mathbf{P}(\beta) - \mathbf{P}(r\beta)}{1-r}.$$

Proof: To show the strong convergence of the limit in (3.26), consider a unitary realization (2.12) with the state space $\mathcal{G}_\beta$ of the Schur function $\mathbf{P}_\beta(\zeta)$. By a calculation similar to (2.19),

$$(3.31) \quad (I - A\mathbf{P}_\beta(\zeta))^{-1} = (I - AD_1)^{-1}\left[I + \zeta A C_1 (I - \zeta A_2)^{-1} B_1 (I - AD_1)^{-1}\right],$$

where A_2 is given in (2.17). Since A_2 is a contraction on $\mathcal{G}_\beta$, the limit

$$\lim_{r\to 1}(1-r)\left(I_{\mathcal{G}_\beta}-rA_2\right)^{-1}=P_{\mathrm{Ker}(I-A_2)}$$

converges strongly to the orthogonal projection of $\mathcal{G}_\beta$ onto the eigenspace $\{x\in\mathcal{G}_\beta:\;\; A_2x=x\}$ of fixed vectors of A_2 (for the proof see [10, Section 2]). Then it follows from the two last relations that the limit in (3.26) converges strongly to the operator

$$\Upsilon_\beta=(I-AD_1)^{-1}AC_1P_{\mathrm{Ker}(I-A_2)}B_1(I-AD_1)^{-1}.$$

To show that this operator satisfies conditions (3.27), consider the function

$$\Phi_\beta(\zeta)=(I-A\mathbf{P}_\beta(\zeta))^{-1}(I+A\mathbf{P}_\beta(\zeta)). \tag{3.32}$$

Since

$$\Phi_\beta(\zeta)+\Phi_\beta(\zeta)^*=2(I-A\mathbf{P}_\beta(\zeta))^{-1}\left[I-A\mathbf{P}_\beta(\zeta)\mathbf{P}_\beta(\zeta)^*A^*\right](I-\mathbf{P}_\beta(\zeta)^*A^*)^{-1}\geq 0,$$

the real part of Φ_β is positive semidefinite on $\mathbb{D}$ and therefore, Φ admits a Herglotz representation

$$\Phi_\beta(\zeta)=i\Im\Phi_\beta(0)+\int_{\mathbb{T}}\frac{t+\zeta}{t-\zeta}d\Sigma_\beta(t)$$

with a positive operator-valued measure Σ_β. Then a consequence of the Lebesgue Dominated Convergence Theorem is that

$$\lim_{r\to 1}(1-r)\Phi_\beta(r)=2\Sigma_\beta(\{1\}) \tag{3.33}$$

where $\Sigma_\beta(\{1\})\geq 0$ is the measure assigned by Σ_β at the point $t=1$. Therefore,

$$\lim_{r\to 1}(1-r)\frac{\Phi_\beta(r)+I}{2}=\Sigma_\beta(\{1\}). \tag{3.34}$$

By (3.32),

$$\frac{\Phi_\beta(r)+I}{2}=(I-A\mathbf{P}_\beta(r))^{-1}=(I-A\mathbf{P}(r\beta))^{-1}$$

and upon comparing (3.26) with (3.34) we conclude that the limit in (3.26) exists and Υ_β is given by

$$\Upsilon_\beta=\Sigma_\beta(\{1\})\geq 0. \tag{3.35}$$

From (3.32), (3.33) and (3.35) we see that

$$\lim_{r\to 1}(1-r)(I-A\mathbf{P}(r\beta))^{-1}(I+A\mathbf{P}(r\beta))=2\Upsilon_\beta,$$

which together with (3.26) implies

$$\lim_{r\to 1}(1-r)(I-A\mathbf{P}(r\beta))^{-1}A\mathbf{P}(r\beta)=\lim_{r\to 1}(1-r)A\mathbf{P}(r\beta)(I-A\mathbf{P}(r\beta))^{-1}=\Upsilon_\beta.$$

The limits in the latter relations can be split into products which leads us to

$$A\mathbf{P}(\beta)\Upsilon_\beta=\Upsilon_\beta A\mathbf{P}(\beta)=\Upsilon_\beta.$$

Multiplying the second equality by $\mathbf{P}(\beta)^*$ and taking into account that $\mathbf{P}(\beta)\mathbf{P}(\beta)^* = I$ we come to

$$\Upsilon_\beta A = \Upsilon_\beta \mathbf{P}(\beta)^*,$$

which completes the proof of (3.27).

Furthermore, on account of (3.26) and the third relation in (3.27),

$$\begin{aligned}\lim_{r\to 1}(I - A\mathbf{P}(r\beta))^{-1}(I - A\mathbf{P}(\beta)) &= I - \lim_{r\to 1}(1-r)(I - A\mathbf{P}(r\beta))^{-1}A\frac{\mathbf{P}(\beta)-\mathbf{P}(r\beta)}{1-r}\\ &= I - \Upsilon_\beta A W_\beta\\ &= I - \Upsilon_\beta \mathbf{P}(\beta)^* W_\beta\end{aligned}$$

and quite similarly,

$$\begin{aligned}\lim_{r\to 1}(I - \mathbf{P}(\beta)A)(I - \mathbf{P}(r\beta)A)^{-1} &= I - \lim_{r\to 1}\frac{\mathbf{P}(\beta)-\mathbf{P}(r\beta)}{1-r}(1-r)(I - A\mathbf{P}(r\beta))^{-1}A\\ &= I - W_\beta \Upsilon_\beta A\\ &= I - W_\beta \Upsilon_\beta \mathbf{P}(\beta)^*,\end{aligned}$$

which completes the proof of the lemma. □

Lemma 3.4. *Let K be a kernel on Ω satisfying conditions* (2.25)–(2.28) *and let $\widetilde{\mathbf{U}}$ of the form* (3.12) *be a unitary extension of the partially defined isometry $\mathbf{V}_K$ given in* (3.9). *Then the characteristic function S of the unitary colligation $\widetilde{\mathcal{C}} = \{\widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}},\ \mathcal{E},\ \mathcal{E}_*,\ \widetilde{\mathbf{U}}\}$,*

$$S(z) = \widetilde{D} + \widetilde{C}\left(I_{\mathbb{C}^p\otimes(\widehat{\mathcal{H}}\oplus\widetilde{\mathcal{H}})} - \mathbf{P}(z)\widetilde{A}\right)^{-1}\mathbf{P}(z)\widetilde{B},$$

is a solution to Problem 1.2.

Proof: We start with a factorization of the form (3.18) for the kernel K

$$K(\xi,\mu) = R(\xi)^*R(\mu) \text{ with } R(\xi) = \begin{bmatrix} T_1(\xi) & \dots & T_p(\xi)\end{bmatrix} \in \mathcal{L}(\mathbb{C}^p \otimes \mathcal{E}_L, \mathcal{H})$$

and then set

$$T(\xi) = \begin{bmatrix} T_1(\xi) \\ \vdots \\ T_d(\xi)\end{bmatrix} \in \mathcal{L}(\mathcal{E}_L, \mathbb{C}^p \otimes \mathcal{H}).$$

By the assumption that $\widetilde{U}$ is a unitary map of the form (3.12) which extends $\mathbf{V}_K$, by reversing the argument in the proof of Lemma 3.2 we see that the operator $\mathbf{U}$ defined by

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D\end{bmatrix} = \begin{bmatrix} (\widetilde{U}\otimes I_q)^* & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D}\end{bmatrix}\begin{bmatrix} \widetilde{U}\otimes I_p & 0 \\ 0 & I\end{bmatrix}$$

satisfies (3.16) (or equivalently, (3.14) and (3.15)), which can be easily seen from (3.25). By Remark 3.1, the colligations $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ defined in (3.1) and (3.3) have the same characteristic functions and thus, S can be taken in the form (1.13).

Let $H(z)$ be defined as in (1.15) and decomposed as in (1.8). We shall use the representation (3.13) of $S(z)$ which is equivalent to (1.13).

Fix $\xi \in \Omega \subseteq \partial\mathcal{D}_{\mathbf{P}}$ and consider the Schur function $\mathbf{P}_{\xi}(\zeta) = \mathbf{P}(\zeta\xi)$, $(\zeta \in \mathbb{D})$. Then it follows from (3.26) and (3.30) that

$$\lim_{r\to 1} \left(\mathbf{P}(\xi) - \mathbf{P}(r\xi)\right)\left(I - A\mathbf{P}(r\xi)\right)^{-1} = W_{\xi}\Upsilon_{\xi}. \tag{3.36}$$

Since the operator $\mathbf{U}$ is unitary, we have in particular,

$$I - AA^* = BB^*. \tag{3.37}$$

Combining the second and the third relations in (3.27) we conclude that

$$\Upsilon_{\xi} = \Upsilon_{\xi}\mathbf{P}(\xi)^*A^* = \Upsilon_{\xi}AA^*. \tag{3.38}$$

It follows now from (3.38) by (3.37) that $\Upsilon_{\xi}BB^* = 0$ and therefore, that

$$\Upsilon_{\xi}B = 0. \tag{3.39}$$

Multiplying both parts in (3.36) by B on the right we and taking into account (3.39) we get

$$\lim_{r\to 1} \left(\mathbf{P}(\xi) - \mathbf{P}(r\xi)\right)\left(I - A\mathbf{P}(r\xi)\right)^{-1}B = 0. \tag{3.40}$$

Using (1.13) and expressions for $D^*\mathbf{a}(\xi)$ and $C^*\mathbf{a}(\xi)$ derived from (3.14) and (3.15), respectively, we get

$$\begin{aligned} S(r\xi)^*\mathbf{a}(\xi) &= D^*\mathbf{a}(\xi) + B^*\mathbf{P}(r\xi)^*\left(I - A^*\mathbf{P}(r\xi)^*\right)^{-1}C^*\mathbf{a}(\xi) \\ &= \mathbf{c}(\xi) - B^*\mathbf{P}(\xi)^*T(\xi) \\ &\quad + B^*\mathbf{P}(r\xi)^*\left(I - A^*\mathbf{P}(r\xi)^*\right)^{-1}\left(I - A^*\mathbf{P}(\xi)^*\right)T(\xi) \\ &= \mathbf{c}(\xi) - B^*\left(I - \mathbf{P}(r\xi)^*A^*\right)^{-1}\left(\mathbf{P}(\xi)^* - \mathbf{P}(r\xi)^*\right)T(\xi). \end{aligned} \tag{3.41}$$

Taking limits in the last identity as r tends to one and taking into account (3.40), we come to (1.17).

It remains to verify (1.18). By (3.41),

$$\mathbf{c}(\xi)^*\mathbf{c}(\xi) - \mathbf{c}(\xi)^*S(r\xi)^*\mathbf{a}(\xi) = \mathbf{c}(\xi)^*B^*\left(I - \mathbf{P}(r\xi)^*A^*\right)^{-1}\left(\mathbf{P}(\xi)^* - \mathbf{P}(r\xi)^*\right)T(\xi). \tag{3.42}$$

It follows from (3.16) that

$$AT(\xi) + B\mathbf{c}(\xi) = \mathbf{P}(\xi)^*T(\xi)$$

and therefore, since $\mathbf{P}(\xi)\mathbf{P}(\xi)^* = I$,

$$\mathbf{c}(\xi)^*B^* = T(\xi)^*(\mathbf{P}(\xi) - A^*) = T(\xi)^*\mathbf{P}(\xi)\left(I - \mathbf{P}(\xi)^*A^*\right).$$

Substituting the latter equality into (3.42) and dividing both parts of the resulting equality by $(1-r)$, we get

$$\begin{aligned} &\frac{\mathbf{c}(\xi)^*\mathbf{c}(\xi) - \mathbf{c}(\xi)^*S(r\xi)^*\mathbf{a}(\xi)}{1-r} \\ &= T(\xi)^*\mathbf{P}(\xi)\left(I - \mathbf{P}(\xi)^*A^*\right)\left(I - \mathbf{P}(r\xi)^*A^*\right)^{-1}\frac{\mathbf{P}(\xi)^* - \mathbf{P}(r\xi)^*}{1-r}T(\xi). \end{aligned}$$

Taking limits in the last identity as $r\to 1$ and using (3.28) we get

$$\lim_{r\to 1}\frac{\mathbf{c}(\xi)^*\mathbf{c}(\xi)-\mathbf{c}(\xi)^*S(r\xi)^*\mathbf{a}(\xi)}{1-r}=T(\xi)^*\mathbf{P}(\xi)\left(I-W_\xi^*\mathbf{P}(\xi)\Upsilon_\xi\right)W_\xi^*T(\xi). \tag{3.43}$$

The operator $\Lambda_{\mathbf{P}}(\xi)\ge 0$ given in (2.7), admits, by (2.10), representations

$$\Lambda_{\mathbf{P}}(\xi)=W_\xi\mathbf{P}(\xi)^*=\mathbf{P}(\xi)W_\xi^*. \tag{3.44}$$

Setting for short

$$U_\xi:=\mathbf{P}(\xi)\Upsilon_\xi W_\xi^* \tag{3.45}$$

and making use of (3.44) we rewrite (3.43) as

$$\lim_{r\to 1}\frac{\mathbf{c}(\xi)^*\mathbf{c}(\xi)-\mathbf{c}(\xi)^*S(r\xi)^*\mathbf{a}(\xi)}{1-r}=T(\xi)^*\Lambda_{\mathbf{P}}(\xi)(I-U_\xi)T(\xi). \tag{3.46}$$

On the other hand, it follows from (1.14) that

$$\begin{aligned}&\mathbf{a}(\xi)^*\frac{I_{\mathcal{E}_*}-S(r\xi)S(r\xi)^*}{1-r^2}\mathbf{a}(\xi)\\ &=\mathbf{a}(\xi)^*C\left(I-\mathbf{P}(r\xi))A\right)^{-1}\frac{I-\mathbf{P}(r\xi)\mathbf{P}(r\xi)^*}{1-r^2}\left(I-A^*\mathbf{P}(r\xi)^*\right)^{-1}C^*\mathbf{a}(\xi).\end{aligned} \tag{3.47}$$

It follows from (3.15) that

$$C^*\mathbf{a}(\xi)=(I-A^*\mathbf{P}(\xi)^*)T(\xi),$$

which being substituted into (3.47), leads to

$$\begin{aligned}\mathbf{a}(\xi)^*\frac{I_{\mathcal{E}_*}-S(r\xi)S(r\xi)^*}{1-r^2}\mathbf{a}(\xi) \quad = \quad & T(\xi)^*(I-\mathbf{P}(\xi)A)\left(I-\mathbf{P}(r\xi))A\right)^{-1}\\ &\times\frac{I-\mathbf{P}(r\xi)\mathbf{P}(r\xi)^*}{1-r^2}\\ &\times\left(I-A^*\mathbf{P}(r\xi)^*\right)^{-1}(I-A^*\mathbf{P}(\xi)^*)T(\xi).\end{aligned}$$

Taking limits in the last identity as $r\to 1$ and using (3.29), (2.24) and (3.45) we get

$$\lim_{r\to 1}\mathbf{a}(\xi)^*\frac{I_{\mathcal{E}_*}-S(r\xi)S(r\xi)^*}{1-r^2}\mathbf{a}(\xi)=T(\xi)^*\left(I-U_\xi^*\right)\Lambda_{\mathbf{P}}(\xi)\left(I-U_\xi\right)T(\xi). \tag{3.48}$$

By Lemma 2.1,

$$\lim_{r\to 1}\mathbf{a}(\xi)^*\frac{I_{\mathcal{E}_*}-S(r\xi)S(r\xi)^*}{1-r^2}\mathbf{a}(\xi)=\lim_{r\to 1}\frac{\mathbf{c}(\xi)^*\mathbf{c}(\xi)-\mathbf{c}(\xi)^*S(r\xi)^*\mathbf{a}(\xi)}{1-r},$$

which implies, on account of (3.46) and (3.48), that

$$T(\xi)^*\Lambda_{\mathbf{P}}(\xi)(I-U_\xi)T(\xi)=T(\xi)^*\left(I-U_\xi^*\right)\Lambda_{\mathbf{P}}(\xi)\left(I-U_\xi\right)T(\xi).$$

The last equality implies

$$T(\xi)^*\left(I-U_\xi^*\right)\Lambda_{\mathbf{P}}(\xi)\left(I-U_\xi\right)T(\xi)=T(\xi)^*\left(\Lambda_{\mathbf{P}}(\xi)-U_\xi^*\Lambda_{\mathbf{P}}(\xi)U_\xi\right)T(\xi)$$

and now we conclude from (3.48) and (2.26), that for every $\xi \in \Omega$,

$$\begin{aligned} \lim_{r\to 1} \mathbf{a}(\xi)^* \frac{I_{\mathcal{E}_*} - S(r\sigma(\xi))S(r\sigma(\xi))^*}{1-r^2}\mathbf{a}(\xi) &= T(\xi)^* \left(\Lambda_{\mathbf{P}}(\xi) - U_\xi^* \Lambda_{\mathbf{P}}(\xi) U_\xi\right) T(\xi) \\ &\leq T(\xi)^* \Lambda_{\mathbf{P}}(\xi) T(\xi) \leq \Psi(\xi), \end{aligned}$$

where we used the assumption (2.26) for the last step. This proves (1.18) and completes the proof of the theorem. □

4. The universal unitary colligation associated with the interpolation problem

A general result of Arov and Grossman (see [7], [8]) describes how to parametrize the set of all unitary extensions of a given partially defined isometry $\mathbf{V}$. Their result has been extended to the multivariable case in [16] (for the case of the polydisk), in [17] (for the case of the unit ball) and in [12] for $\mathbf{P}$-colligations. We recall the result from [12] for the reader's convenience.

We assume that we are given an interpolation data set $(\mathbf{a}, \mathbf{c}, \Psi)$ as in (1.16) and a kernel $K(\xi, \mu)$ on $\Omega \times \Omega$ satisfying the conditions (2.25)–(2.28) as in the previous section. Let $\mathbf{V}_K : \mathcal{D}_{\mathbf{V}_K} \to \mathcal{R}_{\mathbf{V}_K}$ with $\mathcal{D}_{\mathbf{V}_K}$ and $\mathcal{R}_{\mathbf{V}_K}$ given as in (3.10) and (3.11) be the isometry associated with K as in (3.9). Introduce the defect spaces

$$\Delta_K = \begin{bmatrix} \mathbb{C}^p \otimes \widehat{\mathcal{H}} \\ \mathcal{E} \end{bmatrix} \ominus \mathcal{D}_{\mathbf{V}_K} \quad \text{and} \quad \Delta_{K*} = \begin{bmatrix} \mathbb{C}^q \otimes \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix} \ominus \mathcal{R}_{\mathbf{V}_K}$$

and let $\widetilde{\Delta}_K$ be another copy of Δ_K and $\widetilde{\Delta}_{K*}$ be another copy of Δ_{K*} with unitary identification maps

$$i_K : \Delta_K \to \widetilde{\Delta}_K \quad \text{and} \quad i_{K*} : \Delta_{K*} \to \widetilde{\Delta}_{K*}. \tag{4.1}$$

Define a unitary operator $\mathbf{U}_{K,0}$ from $\mathcal{D}_{\mathbf{V}_K} \oplus \Delta_K \oplus \widetilde{\Delta}_{K*}$ onto $\mathcal{R}_{\mathbf{V}_K} \oplus \Delta_{K*} \oplus \widetilde{\Delta}_K$ by the rule

$$\mathbf{U}_{K,0} x = \begin{cases} \mathbf{V}_K x, & \text{if } x \in \mathcal{D}_{\mathbf{V}_K} \\ i_K(x) & \text{if } x \in \Delta_K, \\ i_{K*}^{-1}(x) & \text{if } x \in \widetilde{\Delta}_{K*}. \end{cases} \tag{4.2}$$

Identifying $\begin{bmatrix} \mathcal{D}_{\mathbf{V}_K} \\ \Delta_K \end{bmatrix}$ with $\begin{bmatrix} \mathbb{C}^p \otimes \widehat{\mathcal{H}} \\ \mathcal{E} \end{bmatrix}$ and $\begin{bmatrix} \mathcal{R}_{\mathbf{V}_K} \\ \Delta_{K*} \end{bmatrix}$ with $\begin{bmatrix} \mathbb{C}^q \otimes \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix}$, we decompose $\mathbf{U}_{K,0}$ defined by (4.2) according to

$$\mathbf{U}_{K,0} = \begin{bmatrix} U_{K,11} & U_{K,12} & U_{K,13} \\ U_{K,21} & U_{K,22} & U_{K,23} \\ U_{K,31} & U_{K,32} & 0 \end{bmatrix} : \begin{bmatrix} \mathbb{C}^p \otimes \widehat{\mathcal{H}} \\ \mathcal{E} \\ \widetilde{\Delta}_{K*} \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^q \otimes \widehat{\mathcal{H}} \\ \mathcal{E}_* \\ \widetilde{\Delta}_K \end{bmatrix}. \tag{4.3}$$

The $(3,3)$ block in this decomposition is zero, since (by definition (4.2)), for every $x \in \widetilde{\Delta}_{K*}$, the vector $\mathbf{U}_{K,0} x$ belongs to Δ_K, which is a subspace of $\begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix}$ and therefore, is orthogonal to $\widetilde{\Delta}_K$ (in other words $\mathbf{P}_{\widetilde{\Delta}_K}\mathbf{U}_{K,0}|_{\widetilde{\Delta}_{K*}} = 0$, where $\mathbf{P}_{\widetilde{\Delta}_K}$ stands for the orthogonal projection of $\mathcal{R}_{\mathbf{V}_K} \oplus \Delta_{K*} \oplus \widetilde{\Delta}_K$ onto $\widetilde{\Delta}_K$).

The unitary operator $\mathbf{U}_{K,0}$ is the connecting operator of the unitary colligation

$$\mathcal{C}_{K,0} = \left\{\widehat{\mathcal{H}},\ \begin{bmatrix} \mathcal{E} \\ \widetilde{\Delta}_{K*} \end{bmatrix},\ \begin{bmatrix} \mathcal{E}_* \\ \widetilde{\Delta}_K \end{bmatrix},\ \mathbf{U}_{K,0}\right\}, \tag{4.4}$$

which is called *the universal unitary colligation* associated with the kernel K satisfying the necessary conditions (2.25)–(2.28) for existence of solutions of the interpolation problem.

According to (3.4), the characteristic function of the $\mathbf{P}$-colligation $\mathcal{C}_{K,0}$ defined in (4.4) is given by

$$\begin{aligned} \Sigma_K(z) &= \begin{bmatrix} \Sigma_{K,11}(z) & \Sigma_{K,12}(z) \\ \Sigma_{K,21}(z) & \Sigma_{K,22}(z) \end{bmatrix} \\ &= \begin{bmatrix} U_{K,22} & U_{K,23} \\ U_{K,32} & 0 \end{bmatrix} + \begin{bmatrix} U_{K,21} \\ U_{K,31} \end{bmatrix} (I - \mathbf{P}(z)U_{K,11})^{-1}\,\mathbf{P}(z)\begin{bmatrix} U_{K,12} & U_{K,13} \end{bmatrix} \end{aligned} \tag{4.5}$$

and belongs to the class $\mathcal{S}_{\mathbf{P}}(\mathcal{E} \oplus \widetilde{\Delta}_{K*},\ \mathcal{E}_* \oplus \widetilde{\Delta}_K)$, by Theorem 1.1.

Theorem 4.1. *Let $\mathbf{V}_K$ be the isometry defined in* (3.9) *associated with a positive kernel K meeting the necessary conditions* (2.25)–(2.28) *for existence of solutions to the interpolation Problem 1.2, let Σ_K be the function constructed as above from $\mathbf{V}_K$, and let S be a $\mathcal{L}(\mathcal{E},\ \mathcal{E}_*)$-valued function. Then the following are equivalent:*

1. *S is a solution of Problem 1.2.*
2. *There exists a kernel K satisfying conditions* (2.25)–(2.28) *so that S is the characteristic function of a $\mathbf{P}$-colligation $\mathcal{C} = \{\widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}},\ \mathcal{E},\ \mathcal{E}_*,\ \mathbf{U}\}$ with the connecting operator $\mathbf{U}$ being a unitary extension of $\mathbf{V}_K$.*
3. *There exists a positive kernel K satisfying conditions* (2.25)–(2.28) *such that S is of the form*

$$S(z) = \Sigma_{K,11}(z) + \Sigma_{K,12}(z)\left(I_{\widetilde{\Delta}_*} - \mathcal{T}(z)\Sigma_{K,22}(z)\right)^{-1}\mathcal{T}(z)\Sigma_{K,21}(z) \tag{4.6}$$

for a function $\mathcal{T}$ from the class $\mathcal{S}A_{\mathcal{D}_{\mathbf{P}}}(\widetilde{\Delta}_K,\ \widetilde{\Delta}_{K*})$.

Proof: The equivalence $\mathbf{1} \Leftrightarrow \mathbf{2}$ follows by Lemmas 3.2 and 3.3. For the proof of $\mathbf{2} \Leftrightarrow \mathbf{3}$ (for a given fixed choice of K) see [12, Theorem 6.1]. □

As a corollary we obtain the sufficiency part of Theorem 2.2: under assumptions (2.25)–(2.28) the set of all solutions of Problem 1.2 is parametrized by formula (4.6) as one sweeps through all functions $\mathcal{T}$ from the class $\mathcal{S}A_{\mathcal{D}_{\mathbf{P}}}(\widetilde{\Delta}_K,\ \widetilde{\Delta}_{K*})$ and through all positive kernels K meeting the conditions (2.25)–(2.28), and hence in particular is nonempty.

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