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Realization and interpolation for Schur–Agler-class functions on domains with matrix polynomial defining function in $\mathbb{C}^n$

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Abstract

We consider a bitangential interpolation problem for operator-valued functions defined on a general class of domains in $\mathbb{C}^n$ (including as particular cases, Cartan domains of types I–III) which satisfy a type of von Neumann inequality associated with the domain. We show that any such function has a realization in terms of a unitary colligation and the defining polynomial for the domain. We show how the solution of various classes of bitangential interpolation problems for this class of functions corresponds to a unitary extension of a particular partially defined isometry uniquely specified by the interpolation data. Criteria for existence of solutions are given (1) in terms of positivity of a certain kernel completely determined by the data, or, more generally, (2) by the existence of a positive-kernel solution of a certain generalized Stein equation completely determined by the data.

Keywords: Schur-Agler functions; Polynomial-matrix defining function; Unitary realizations; Nevanlinna-Pick problem

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1. Introduction and statement of main results

The classical Schur class $\mathcal{S}$ consisting of complex-valued analytic functions mapping the unit disk $\mathbb{D}$ into the closed unit disk $\overline{\mathbb{D}}$ has been a source of much study and inspiration for over a century now, beginning with the seminal work of Schur (for the original paper of Schur and a survey of some of the impact and applications in signal processing, see [28]). More recently, the operator-valued version of the Schur-class $\mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ consisting of analytic functions F on the unit disk with values $F(z)$ equal to contraction operators between two Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$ has played a prominent role in both engineering and operator-theoretic applications (see e.g. [21,22,30,40,41]). This class admits various remarkable characterizations some of which are recalled below. The symbol $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ stands for the algebra of bounded linear operators mapping $\mathcal{E}$ into $\mathcal{E}_*$ and we shall shorten $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ to $\mathcal{L}(\mathcal{E})$.

Theorem 1.1. *Let F be an $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued function analytic on $\mathbb{D}$. Then the following are equivalent*:

(1) *F belongs to $\mathcal{S}(\mathcal{E}, \mathcal{E}_*)$, i.e., $\|F(z)\| \leqslant 1$ for every $z \in \mathbb{D}$.*
(2) *F satisfies the von Neumann inequality*: *$\|F(T)\| \leqslant 1$ for any strictly contractive operator T on a Hilbert space $\mathcal{H}'$, where $F(T)$ is defined by*

$$F(T) = \sum_{n=0}^{\infty} F_n \otimes T^n \in \mathcal{L}(\mathcal{E} \otimes \mathcal{H}', \mathcal{E}_* \otimes \mathcal{H}') \quad \textit{if} \;\; F(z) = \sum_{n=0}^{\infty} F_n z^n.$$

(3) *F admits a representation of the form*

$$F(z) = D + zC(I_{\mathcal{H}} - zA)^{-1}B$$

where the connecting operator (*or colligation*)

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{H} \\ \mathcal{E}_* \end{bmatrix} \tag{1.1}$$

is unitary, *and where $\mathcal{H}$ is some auxiliary Hilbert space* (*the* internal space *for the colligation*).
(4) *There exist a Hilbert space $\mathcal{H}$ and an $\mathcal{L}(\mathcal{H}, \mathcal{E}_*)$-valued function $H(z)$ analytic on $\mathbb{D}$ such that*

$$I_{\mathcal{E}_*} - F(z)F(w)^* = (1 - z\bar{w})H(z)H(w)^* \quad (z, w \in \mathbb{D}). \tag{1.2}$$

(5) *There exist a Hilbert space $\mathcal{H}$ and an $\mathcal{L}(\mathcal{E}, \mathcal{H})$-valued function $G(z)$ analytic on $\mathbb{D}$ such that*

$$I_{\mathcal{E}} - F(z)^*F(w) = (1 - \bar{z}w)G(z)^*G(w) \quad (z, w \in \mathbb{D}). \tag{1.3}$$

(6) *There exist a Hilbert space $\mathcal{H}$ and operator-valued functions H and G as above such that*

$$\begin{bmatrix} I_{\mathcal{E}_*} - F(z)F(w)^* & F(z) - F(w') \\ F(z')^* - F(w)^* & I_{\mathcal{E}} - F(z')^*F(w') \end{bmatrix} = \begin{bmatrix} (1 - z\bar{w})H(z)H(w)^* & (z - w')H(z)G(w') \\ (\overline{z'} - \bar{w})G(z')^*H(w)^* & (1 - \overline{z'}w')G(z')^*G(w') \end{bmatrix} \tag{1.4}$$

for every $z, z', w, w' \in \mathbb{D}$.

Recall that an operator-valued function $K : \Omega \times \Omega \mapsto \mathcal{L}(\mathcal{E})$ is called a *positive kernel* if

$$\sum_{i,j=1}^{r} \langle K(\omega_i, \omega_j)h_j, h_i \rangle_{\mathcal{H}} \geqslant 0 \tag{1.5}$$

for every choice of a positive integer r and of $\omega_1, \ldots, \omega_r \in \Omega$ and $h_1, \ldots, h_r \in \mathcal{H}$. By one of the original results of Aronszajn [9], an equivalent condition is that there exists a function $X : \Omega \mapsto \mathcal{L}(\mathcal{H}', \mathcal{H})$ (for some auxiliary Hilbert space $\mathcal{H}'$) so that K has the factorization

$$K(\omega_1, \omega_2) = X(\omega_1)X(\omega_2)^*.$$

Remark 1.2. The following three kernels

$$\mathbb{K}_{\mathrm{L}}(z, w) = H(z)H(w)^*, \quad \mathbb{K}_{\mathrm{R}}(z, w) = G(z)^*G(w)$$

and

$$\mathbb{K}(z, z', w, w') = \begin{bmatrix} H(z) \\ G(z')^* \end{bmatrix} [H(w)^* \quad G(w')]$$

are clearly positive on $\mathbb{D}^2$ and $\mathbb{D}^4$, respectively. Furthermore, they are uniquely recovered from Eqs. (1.2)–(1.5) in terms of F as follows:

$$\mathbb{K}_{\mathrm{L}}(z, w) = \frac{I_{\mathcal{E}_*} - F(z)F(w)^*}{1 - z\bar{w}}, \quad \mathbb{K}_{\mathrm{R}}(z, w) = \frac{I_{\mathcal{E}} - F(z)^*F(w)}{1 - \bar{z}w} \tag{1.6}$$

and

$$\mathbb{K}(z, z', w, w') = \begin{bmatrix} \dfrac{I_{\mathcal{E}_*} - F(z)F(w)^*}{1 - z\bar{w}} & \dfrac{F(z) - F(w')}{z - w'} \\ \dfrac{F(z')^* - F(w)^*}{\overline{z'} - \bar{w}} & \dfrac{I_{\mathcal{E}} - F(z')^*F(w')}{1 - \overline{z'}w'} \end{bmatrix}, \tag{1.7}$$

which allows us to reformulate the above statements (4)–(6) in Theorem 1.1 in more familiar terms as: *the kernels* $\mathbb{K}_{\mathrm{L}}$ (*respectively*, $\mathbb{K}_{\mathrm{R}}$ *and* $\mathbb{K}$) *are positive on* $\mathbb{D}^2$ ($\mathbb{D}^2$ *and* $\mathbb{D}^4$, *respectively*).

Remark 1.3. The significance of the characterization of the Schur class in terms of positivity of the kernel $\mathbb{K}_{\mathrm{L}}$ for interpolation theory is that it gives the necessity part in the Nevanlinna–Pick type interpolation theorem: *given points* $z_1, \dots, z_n \in \mathbb{D}$ *and* $w_1, \dots, w_n \in \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$, *there exists* $F \in \mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ *with* $F(z_j) = w_j$ *for* $j = 1, \dots, n$ *if and only if the associated Pick operator* $\Lambda = \left[\frac{I_{\mathcal{E}_*} - w_i w_j^*}{1 - z_i \bar{z}_j}\right]$ *is positive semidefinite.*

Positivity of $\mathbb{K}_{\mathrm{L}}$ also leads to the solvability criterion of a more general *left tangential* interpolation problem, while the kernels $\mathbb{K}_{\mathrm{R}}$ and $\mathbb{K}$ provide necessary and sufficient conditions for solvability of *a right tangential* and *bitangential* interpolation problems, respectively; see [14] for more detail.

It is easily checked that each one of statements (2)–(6) in Theorem 1.1 implies (1). The nontrivial (and remarkable) fact is that 1 ⇒ (2)–(6). The situation changes in the following more general setting: let $\mathbf{Q}(z)$ be a polynomial, let $\mathcal{D}_{\mathbf{Q}} \subset \mathbb{C}$ be the domain defined as

$$\mathcal{D}_{\mathbf{Q}} = \{z \in \mathbb{C}^n : |\mathbf{Q}(z)| < 1\}$$

and let us consider the class $\mathcal{S}_{\mathcal{D}_{\mathbf{Q}}}(\mathcal{E}, \mathcal{E}_*)$ of $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued functions S analytic on $\mathcal{D}_{\mathbf{Q}}$ and such that

$$||S(z)|| \leqslant 1 \quad \text{for every } z \in \mathcal{D}_{\mathbf{Q}} \tag{1.8}$$

and the class $\mathcal{S}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ of $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued functions S analytic on $\mathcal{D}_{\mathbf{Q}}$ and satisfying the following von Neumann type inequality:

$$||S(T)|| \leqslant 1 \quad \text{whenever } T \in \mathcal{L}(\mathcal{H}) \quad \text{and} \quad ||\mathbf{Q}(T)|| < 1. \tag{1.9}$$

In particular, in (1.9) we may use $\mathcal{H} = \mathbb{C}$ and $T = z$ where z is a point in $\mathcal{D}_{\mathbf{Q}}$ to see that (1.8) holds and therefore, that $\mathcal{S}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*) \subset \mathcal{S}_{\mathcal{D}_{\mathbf{Q}}}(\mathcal{E}, \mathcal{E}_*)$. In general, this inclusion is proper: for example, letting $\mathbf{Q}(z) = z^2$ we get $\mathcal{D}_{\mathbf{Q}} = \mathbb{D}$ and therefore, $\mathcal{S}_{\mathcal{D}_{\mathbf{Q}}}(\mathcal{E}, \mathcal{E}_*)$ coincides with the classical Schur class. On the other hand the Schur function $S(z) = z$ does not satisfy property (1.9): the operator $T = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^2$ satisfies

$$||\mathbf{Q}(T)|| = ||T^2|| = ||0|| = 0 < 1, \quad \text{while } ||S(T)|| = ||T|| = 2 > 1.$$

It is also clear that the class $\mathcal{S}_{\mathcal{D}_{\mathbf{Q}}}(\mathcal{E}, \mathcal{E}_*)$ depends on the domain $\mathcal{D}_{\mathbf{Q}}$ rather than on $\mathbf{Q}$, whereas $\mathcal{S}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ depends just on $\mathbf{Q}$.

Since in the case when $\mathbf{Q}(z) = z$, the classes $\mathcal{S}_{\mathcal{D}_{\mathbf{Q}}}(\mathcal{E}, \mathcal{E}_*)$ and $\mathcal{S}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ both reduce to $\mathcal{S}(\mathcal{E}, \mathcal{E}_*)$, each of them can be considered as a certain generalization of the classical Schur class. Furthermore, upon taking advantage of characterizations (3)–(6), one can define the class of operator-valued functions admitting representations of the form

$$S(z) = D + \mathbf{Q}(z)C(I_{\mathscr{H}} - \mathbf{Q}(z)A)^{-1}B, \tag{1.10}$$

where the connecting operator U is the same as in (1.1), as well as the classes of functions for which one of the kernels

$$\mathbb{K}_{\mathrm{L}}(z,w) = \frac{I_{\mathscr{E}_*} - S(z)S(w)^*}{1 - \mathbf{Q}(z)\overline{\mathbf{Q}(w)}}, \quad \mathbb{K}_{\mathrm{R}}(z,w) = \frac{I_{\mathscr{E}} - S(z)^*S(w)}{1 - \overline{\mathbf{Q}(z)}\mathbf{Q}(w)}$$

or

$$\mathbb{K}(z,z',w,w') = \begin{bmatrix} \dfrac{I_{\mathscr{E}_*} - S(z)S(w)^*}{1 - \mathbf{Q}(z)\overline{\mathbf{Q}(w)}} & \dfrac{S(z) - S(w')}{\mathbf{Q}(z) - \mathbf{Q}(w')} \\ \dfrac{S(z')^* - S(w)^*}{\overline{\mathbf{Q}(z')} - \overline{\mathbf{Q}(w)}} & \dfrac{I_{\mathscr{E}} - S(z')^*S(w')}{1 - \overline{\mathbf{Q}(z')}\mathbf{Q}(w')} \end{bmatrix},$$

is positive on $\mathscr{D}^2_{\mathbf{Q}}$ or on $\mathscr{D}^4_{\mathbf{Q}}$, respectively. However, no new generalizations of the classical Schur class arise in this way: all the resulting classes coincide with $\mathscr{S}_{\mathbf{Q}}(\mathscr{E},\mathscr{E}_*)$. In other words, the functions F analytic on $\mathscr{D}_{\mathbf{Q}}$ and satisfying the von Neumann inequality (1.9) are precisely those admitting unitary realizations of the form (1.10) and/or such that the any one of the three associated kernels $\mathbb{K}_{\mathrm{L}}$, $\mathbb{K}_{\mathrm{R}}$ and $\mathbb{K}$ are positive. It is even more remarkable that a similar result can be established in the following multivariable context: we start with a $p \times q$ matrix-valued polynomial in n complex variables

$$\mathbf{Q}(z) = \begin{bmatrix} \mathbf{q}_{11}(z) & \dots & \mathbf{q}_{1q}(z) \\ \vdots & & \vdots \\ \mathbf{q}_{p1}(z) & \dots & \mathbf{q}_{pq}(z) \end{bmatrix} \in \mathbb{C}^{p\times q} \quad \text{for } z \in \mathbb{C}^n \tag{1.11}$$

and we define the domain $\mathscr{D}_{\mathbf{Q}} \in \mathbb{C}^n$ by

$$\mathscr{D}_{\mathbf{Q}} = \{z \in \mathbb{C}^n : ||\mathbf{Q}(z)||_{\mathbb{C}^{p\times q}} < 1\}, \tag{1.12}$$

or equivalently, in terms of real scalar polynomials, as

$$\mathscr{D}_{\mathbf{Q}} = \{z \in \mathbb{C}^n : \rho_\ell(z) > 0 \quad \text{for } \ell = 1, \dots, q\}, \tag{1.13}$$

where we have set

$$\rho_\ell(z) = \det\left[\delta_{i,j} - \sum_{k=1}^{p} \overline{\mathbf{q}_{ki}(z)}\mathbf{q}_{kj}(z)\right]_{i,j=1}^{\ell} \qquad \text{for } \ell = 1, \dots, q$$

and where $\delta_{i,j}$ stands for the Kronecker symbol equal to 1 for $i = j$ and 0 for $i \neq j$. Special choices of

$$\mathbf{Q}(z) = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} \quad \text{and} \quad \mathbf{Q}(z) = [z_1 \quad z_2 \quad \dots \quad z_n] \tag{1.14}$$

lead to the unit polydisk $\mathscr{D}_{\mathbf{Q}} = \mathbb{D}^n$ and the unit ball $\mathscr{D}_{\mathbf{Q}} = \mathbb{B}^n$ of $\mathbb{C}^n$, respectively. The classical Cartan domains of the first three types and their Cartesian products and intersections also can be obtained upon a suitable choice of $\mathbf{Q}$; we refer to [7,8,48] for more examples.

Now we recall the *Schur–Agler class* $\mathscr{SA}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*)$. By definition, the class $\mathscr{SA}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*)$ consists of $\mathscr{L}(\mathscr{E}, \mathscr{E}_*)$-valued functions $S(z) = S(z_1, \dots, z_n)$ analytic on $\mathscr{D}_{\mathbf{Q}}$ such that

$$\|S(T_1, \dots, T_n)\| \leqslant 1$$

for any collection of n commuting operators $(T_1, \dots, T_n)$ on a Hilbert space $\mathscr{K}$, subject to

$$\|\mathbf{Q}(T_1, \dots, T_n)\| < 1.$$

By Ambrozie and Timotin [8, Lemma 1], the Taylor joint spectrum of the commuting n-tuple $(T_1, \dots, T_n)$ is contained in $\mathscr{D}_{\mathbf{Q}}$ whenever $\|\mathbf{Q}(T_1, \dots, T_n)\| < 1$, and hence $S(T_1, \dots, T_n)$ is well defined for any $\mathscr{L}(\mathscr{E}, \mathscr{E}_*)$-valued function S which is analytic on $\mathscr{D}_{\mathbf{Q}}$ by the Taylor functional calculus (see [24]). Upon using $\mathscr{K} = \mathbb{C}$ and $T_j = z_j$ for $j = 1, \dots, n$ where $(z_1, \dots, z_n)$ is a point in $\mathscr{D}_{\mathbf{Q}}$ we conclude that any $\mathscr{L}(\mathscr{E}, \mathscr{E}_*)$ function is contractive valued on $\mathscr{D}_{\mathbf{Q}}$, and thus the class $\mathscr{SA}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*)$ is a subclass of the class $\mathscr{S}_{\mathscr{D}_{\mathbf{Q}}}(\mathscr{E}, \mathscr{E}_*)$ of contractive valued functions analytic on $\mathscr{D}_{\mathbf{Q}}$. As we have already seen in Remark 1.2, this subclass in general is proper.

The classes $\mathscr{SA}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*)$ for the two generic cases (1.14) have been known for a while. The polydisk setting was first presented by Agler [2] and then extended to the operator valued case in [15,18]; see also [3]. The Schur–Agler functions on the unit ball appeared in [26] in connection with a von Neumann's inequality, later in [1,38,45] in connection with complete Nevanlinna–Pick kernels, in [10,44] in connection with the theory of commutative unitary dilations of commutative row contractions, and in [29,39] in connection with Beurling–Lax representations of Beurling–Lax type for invariant subspaces; for a thorough account of the operator-valued case, see [19]. The general setting introduced above unifies these two generic settings and moreover covers some other interesting cases. The general domains $\mathscr{D}_{\mathbf{Q}}$ and classes $\mathscr{SA}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*)$ (for $\mathscr{E} = \mathscr{E}_* = \mathbb{C}$) were already introduced in [8]. It was pointed out there that classical Cartan domains of the first three types together with their Cartesian products and intersections are domains of type $\mathscr{D}_{\mathbf{Q}}$. Indeed, the choice of

$$n = pq \quad \text{and} \quad \mathbf{q}_{ij}(z) = z_{(i-1)q+j} \quad (i = 1, \ldots, p; j = 1, \ldots, q)$$

leads to $\mathscr{D}_{\mathbf{Q}}$ being a Cartan domain of type I. Choosing

$$p = q, \quad n = \frac{p(p+1)}{2}, \quad \mathbf{q}_{ij}(z) = \mathbf{q}_{ji}(z) \quad \text{and} \quad \mathbf{q}_{ij}(z) = z_{\frac{i(i-1)}{2}+j} \quad (1 \leqslant j \leqslant i \leqslant p)$$

we get $\mathscr{D}_{\mathbf{Q}}$ to be a Cartan domain of type II. In the case when

$$p = q, \quad n = \frac{p(p-1)}{2}, \quad \mathbf{q}_{ij}(z) = -\mathbf{q}_{ji}(z) \quad \text{and} \quad \mathbf{q}_{ij}(z) = z_{\frac{i(i-1)}{2}+j} \quad (1 \leqslant j < i \leqslant p)$$

$\mathscr{D}_{\mathbf{Q}}$ turns out to be a Cartan domain of type III.

Remark 1.4. The referee suggested another possible notion of Schur–Agler class associated with a domain $\mathscr{D} \subset \mathbb{C}^n$. Let us say that an $\mathscr{L}(\mathscr{E}, \mathscr{E}_*)$-valued function S belongs to the Schur–Agler class $\mathscr{S}\mathscr{A}'_{\mathscr{D}}(\mathscr{E}, \mathscr{E}_*)$ if $||S(T)|| \leqslant 1$ for all operator-tuples $T = (T_1, \ldots, T_n)$ for which $\mathscr{D}$ is a spectral set for T. Let us suppose that there is a matrix polynomial $\mathbf{Q}$ so that $\Omega = \mathscr{D}_{\mathbf{Q}}$, i.e., so that

$$\mathscr{D} = \{z \in \mathbb{C}^n : ||\mathbf{Q}(z)|| < 1\}.$$

Then, if $T = (T_1, \ldots, T_n)$ is a commuting operator-tuple for which $\mathscr{D}$ is a complete spectral set, in particular it follows that

$$||\mathbf{Q}(T)|| \leqslant \sup_{z \in \partial \mathscr{D}} ||\mathbf{Q}(z)|| = 1.$$

It then follows from the definitions that $\mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*) \subset \mathscr{S}\mathscr{A}'_{\mathscr{D}}(\mathscr{E}, \mathscr{E}_*)$. We leave any other connections between $\mathscr{S}\mathscr{A}_{\mathscr{D}}(\mathscr{E}, \mathscr{E}_*)$ and $\mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*)$ for discussion on another occasion.

The present paper extends the work of [8] to the operator-valued case. The following theorem gives several equivalent characterizations of when a given $\mathscr{L}(\mathscr{E}, \mathscr{E}_*)$-valued function F defined on a subset Ω of $\mathscr{D}_{\mathbf{Q}}$ extends to a function S defined on all of $\mathscr{D}_{\mathbf{Q}}$ in the class $\mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*)$; in particular, taking $\Omega = \mathscr{D}_{\mathbf{Q}}$ in the statement of the theorem gives several equivalent characterizations for the class $\mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*)$ itself. We shall provide a complete proof of the following result in Section 3; the scalar-valued case (where $\mathscr{E} = \mathscr{E}_* = \mathbb{C}$) can be found in [8] in a somewhat different form.

Theorem 1.5. *Let $\mathbf{Q}$ be a $p \times q$-matrix valued polynomial as above. Suppose that F is an $\mathscr{L}(\mathscr{E}, \mathscr{E}_*)$-valued function defined on a subset Ω of $\mathscr{D}_{\mathbf{Q}}$. The following statements are equivalent*:

(1) *There is a function $S \in \mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*)$ such that $S|_{\Omega} = F$.*

(2) *There exist an auxiliary Hilbert space* $\mathcal{H}$ *and a function*

$$H(z) = [H_1(z) \quad \dots \quad H_p(z)] \tag{1.15}$$

defined on Ω *with values in* $\mathcal{L}(\mathbb{C}^p \otimes \mathcal{H}, \mathcal{E}_*)$ *so that*

$$I_{\mathcal{E}_*} - F(z)F(w)^* = H(z)(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)\mathbf{Q}(w)^*)H(w)^* \tag{1.16}$$

for all $z, w \in \Omega$.

(3) *There exist an auxiliary Hilbert space* $\mathcal{H}$ *and a function*

$$G(z) = \begin{bmatrix} G_1(z) \\ \vdots \\ G_q(z) \end{bmatrix} \tag{1.17}$$

defined on Ω *with values in* $\mathcal{L}(\mathbb{C}^q \otimes \mathcal{H}, \mathcal{E})$ *so that*

$$I_{\mathcal{E}} - F(z)^*F(w) = G(z)^*(I_{\mathbb{C}^q \otimes \mathcal{H}} - \mathbf{Q}(z)^*\mathbf{Q}(w))G(w) \tag{1.18}$$

for all $z, w \in \Omega$.

(4) *There exist an auxiliary Hilbert space* $\mathcal{H}$ *and functions* $H(z)$ *and* $G(z)$ *as in* (1.15) *and* (1.17), *so that*

$$\begin{aligned} &\begin{bmatrix} I_{\mathcal{E}_*} \\ F(z')^* \end{bmatrix} [I_{\mathcal{E}_*} \quad F(w')] - \begin{bmatrix} F(z) \\ I_{\mathcal{E}} \end{bmatrix} [F(w)^* \quad I_{\mathcal{E}}] \\ &\quad = \begin{bmatrix} H(z) \\ G(z')^*\mathbf{Q}(z')^* \end{bmatrix} [H(w)^* \quad \mathbf{Q}(w')G(w')] \\ &\quad - \begin{bmatrix} H(z)\mathbf{Q}(z) \\ G(z')^* \end{bmatrix} [\mathbf{Q}(w)^*H(w)^* \quad G(w')] \end{aligned} \tag{1.19}$$

for all $z, z', w, w' \in \Omega$.

(5) *There is a unitary operator*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C}^p \otimes \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^q \otimes \mathcal{H} \\ \mathcal{E}_* \end{bmatrix} \tag{1.20}$$

such that

$$F(z) = D + C(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)B \quad \text{for all } z \in \Omega. \tag{1.21}$$

Moreover, if F *is of the form* (1.21), *then one extension* S *of* F *to an element of* $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ *is given by*

$$S(z) = D + C(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)B \quad \text{for all } z \in \mathcal{D}_{\mathbf{Q}} \tag{1.22}$$

and it holds that

$$I_{\mathscr{E}_*} - S(z)S(w)^* = C(I - \mathbf{Q}(z)A)^{-1}(I - \mathbf{Q}(z)\mathbf{Q}(w)^*)(I - A^*\mathbf{Q}(w)^*)^{-1}C^*, \tag{1.23}$$

$$S(z) - S(w) = C(I - \mathbf{Q}(z)A)^{-1}(\mathbf{Q}(z) - \mathbf{Q}(w))(I - A\mathbf{Q}(w))^{-1}B, \tag{1.24}$$

$$I_{\mathscr{E}} - S(z)^*S(w) = B^*(I - \mathbf{Q}(z)^*A^*)^{-1}(I - \mathbf{Q}(z)^*\mathbf{Q}(w))(I - A\mathbf{Q}(w))^{-1}B. \tag{1.25}$$

Hence representations (1.16), (1.18) *and* (1.19) *are valid on all of* $\mathscr{D}_{\mathbf{Q}}$ (*with* S *in place of* F) *with*

$$H(z) = C(I - \mathbf{Q}(z)A)^{-1} \quad \textit{and} \quad G(z) = (I - A\mathbf{Q}(z))^{-1}B. \tag{1.26}$$

Remark 1.6. Similarly to Remark 1.2, statements (2)–(4) in Theorem 1.5 can be reformulated in terms of positive kernels as follows.

Statements (2)–(4) in Theorem 1.5 are equivalent, respectively, to statements (2′)–(4′) below:

(2′) There exists a positive kernel

$$\mathbb{K}_{\mathrm{L}} = \begin{bmatrix} \mathbb{K}_{\mathrm{L};1,1} & \cdots & \mathbb{K}_{\mathrm{L};1,p} \\ \vdots & & \vdots \\ \mathbb{K}_{\mathrm{L};p,1} & \cdots & \mathbb{K}_{\mathrm{L};p,p} \end{bmatrix} : \Omega \times \Omega \mapsto \mathscr{L}(\mathbb{C}^p \otimes \mathscr{E}_*) \tag{1.27}$$

such that

$$I_{\mathscr{E}_*} - F(z)F(w)^* = \sum_{k=1}^{p} \mathbb{K}_{\mathrm{L};k,k}(z,w) - \sum_{k=1}^{p} \sum_{i,\ell=1}^{q} \mathbf{q}_{ik}(z)\overline{\mathbf{q}_{\ell k}(w)}\mathbb{K}_{\mathrm{L};i,\ell}(z,w) \quad \text{for all } z,w \in \Omega. \tag{1.28}$$

(3′) There exists a positive kernel

$$\mathbb{K}_{\mathrm{R}} = \begin{bmatrix} \mathbb{K}_{\mathrm{R};1,1} & \cdots & \mathbb{K}_{\mathrm{R};1,q} \\ \vdots & & \vdots \\ \mathbb{K}_{\mathrm{R};q,1} & \cdots & \mathbb{K}_{\mathrm{R};q,q} \end{bmatrix} : \Omega \times \Omega \mapsto \mathscr{L}(\mathbb{C}^q \otimes \mathscr{E}) \tag{1.29}$$

so that

$$I_{\mathscr{E}} - F(z)^*F(w) = \sum_{k=1}^{q} \mathbb{K}_{\mathrm{R};k,k}(z,w) - \sum_{k=1}^{q} \sum_{i,\ell=1}^{p} \overline{\mathbf{q}_{ki}(z)}\mathbf{q}_{k\ell}(w)\mathbb{K}_{\mathrm{R};i,\ell}(z,w) \quad \text{for all } z,w \in \Omega. \tag{1.30}$$

(4′) There exist kernels $\mathbb{K}_{\rm L}$ (as in (1.27)), $\mathbb{K}_{\rm R}$ (as in (1.29)) and

$$\mathbb{K}_{\rm LR} = \begin{bmatrix} \mathbb{K}_{{\rm LR};1,1} & \cdots & \mathbb{K}_{{\rm LR};1,q} \\ \vdots & & \vdots \\ \mathbb{K}_{{\rm LR};p,1} & \cdots & \mathbb{K}_{p,q} \end{bmatrix} : \Omega \times \Omega \mapsto \mathscr{L}(\mathbb{C}^p \otimes \mathscr{E}, \mathbb{C}^q \otimes \mathscr{E}_*)$$

satisfying identities (1.28), (1.30) and

$$F(z) - F(w) = \sum_{i=1}^{p} \sum_{\ell=1}^{q} (\mathbf{q}_{i\ell}(z) - \mathbf{q}_{i\ell}(w))\mathbb{K}_{{\rm LR};i,\ell}(z,w) \quad (zw \in \Omega), \tag{1.31}$$

respectively, and such that the kernel

$$\begin{aligned} ((z,z'),(w,w')) &\mapsto \mathbb{K} \\ &:= \begin{bmatrix} \mathbb{K}_{\rm L}(z,w) & \mathbb{K}_{\rm LR}(z,w') \\ \mathbb{K}_{\rm LR}(w,z')^* & \mathbb{K}_{\rm R}(z',w') \end{bmatrix} \in \mathscr{L}((\mathbb{C}^p \otimes \mathscr{E}_*) \oplus (\mathbb{C}^q \otimes \mathscr{E})) \end{aligned} \tag{1.32}$$

is positive on $(\Omega \times \Omega) \times (\Omega \times \Omega)$.

Indeed, the equivalence between (1.16) and (1.28) (i.e., between statements (2) and (2′)) can be seen by using the formula

$$\mathbb{K}_{\rm L}(z,w) = \begin{bmatrix} H_1(z) \\ \vdots \\ H_p(z) \end{bmatrix} [H_1(w)^* \quad \cdots \quad H_p(w)^*] \tag{1.33}$$

to establish a correspondence between positive kernels of the form (1.27) and functions $H(z)$ of the form (1.15). Given that $\mathbb{K}_{\rm L}$ and H are related in this way, we then compute

$$\begin{aligned} &\sum_{k=1}^{p} \mathbb{K}_{{\rm L};k,k}(z,w) - \sum_{k=1}^{p} \sum_{i,\ell=1}^{q} \mathbf{q}_{ik}(z)\overline{\mathbf{q}_{\ell k}(w)}\mathbb{K}_{{\rm L};i,\ell}(z,w) \\ &\quad = \sum_{k=1}^{p} H_k(z)H_k(w)^* - \sum_{k=1}^{p} \sum_{i,\ell=1}^{q} \mathbf{q}_{ik}(z)\overline{\mathbf{q}_{\ell k}(w)}H_i(z)H_\ell(w)^* \\ &\quad = H(z)(I_{\mathbb{C}^p \otimes \mathscr{H}} - \mathbf{Q}(z)\mathbf{Q}(w)^*)H(w)^* \end{aligned} \tag{1.34}$$

and the equivalence between (1.16) and (1.28) follows. The equivalence between (1.18) and (1.30) (that is, between statements (3) and (3′)) follows in a similar way by using the formula

$$\mathbb{K}_{\rm R}(z,w) = \begin{bmatrix} G_1(z)^* \\ \vdots \\ G_q(z)^* \end{bmatrix} [G_1(w) \quad \cdots \quad G_q(w)] \tag{1.35}$$

to establish a correspondence between positive kernels $\mathbb{K}_{\mathrm{R}}(z,w)$ of the form (1.29) and functions $G(z)$ of the form (1.17). Similarly, the equivalence between (1.19) and (1.31) in part (4) of the statement of the Theorem follows by using (1.33), (1.35) and

$$\mathbb{K}_{\mathrm{LR}}(z,w) = \begin{bmatrix} H_1(z) \\ \vdots \\ H_p(z) \end{bmatrix} [G_1(w) \quad \cdots \quad G_q(w)],$$

that is, using the formula

$$\mathbb{K}(z,z',w,w') = \mathbb{T}(z,z')^* \mathbb{T}(w,w') \tag{1.36}$$

where we have set

$$\mathbb{T}(w,w') = [H_1(w)^* \quad \cdots \quad H_p(w)^* \quad G_1(w') \quad \cdots \quad G_q(w')], \tag{1.37}$$

to establish a correspondence between positive kernels of the form (1.32) and analytic operator functions $H(z)$ and $G(z)$ as in (1.15) and (1.17).

In contrast to the one variable case (see Remark 1.2), however, these kernels $\mathbb{K}_{\mathrm{L}}$, $\mathbb{K}_{\mathrm{R}}$ and $\mathbb{K}$ are not determined by F uniquely (for some exceptional cases, see Examples 2.1 and 2.2).

Theorem 1.5 is a result on full operator-valued interpolation for the class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E},\mathcal{E}_*)$, i.e., for the case where the full value $S(z)$ is specified (as $S(z)=F(z)$ where $F(z)$ is given) at each point z in the set of interpolation nodes Ω. It is also of interest (and of importance in various applications in the classical case—see [14]) to consider tangential interpolation problems for the class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E},\mathcal{E}_*)$. For the left tangential interpolation problem, we assume that we are given data consisting of functions

$$\mathbf{a} : \Omega_{\mathrm{L}} \mapsto \mathcal{L}(\mathcal{E}_{\mathrm{L}},\mathcal{E}_*), \quad \mathbf{c} : \Omega_{\mathrm{L}} \mapsto \mathcal{L}(\mathcal{E}_{\mathrm{L}},\mathcal{E}) \tag{1.38}$$

where $\Omega_{\mathrm{L}} \subset \mathcal{D}_{\mathbf{Q}}$ and where $\mathcal{E}_{\mathrm{L}}$ is a Hilbert space. The problem then is:

Problem 1.7. *Find all functions $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E},\mathcal{E}_*)$ such that S satisfies the interpolation conditions*

$$S(\xi)^* \mathbf{a}(\xi) = \mathbf{c}(\xi) \quad \text{for all } \xi \in \Omega_{\mathrm{L}}. \tag{1.39}$$

For the right tangential interpolation problem, we assume that we are given data consisting of functions

$$\mathbf{b} : \Omega_{\mathrm{R}} \mapsto \mathcal{L}(\mathcal{E}_{\mathrm{R}},\mathcal{E}), \quad \mathbf{d} : \Omega_{\mathrm{R}} \mapsto \mathcal{L}(\mathcal{E}_{\mathrm{R}},\mathcal{E}_*) \tag{1.40}$$

where $\Omega_{\mathrm{R}} \subset \mathcal{D}_{\mathbf{Q}}$ and $\mathcal{E}_{\mathrm{R}}$ is a Hilbert space. The problem then is:

Problem 1.8. *Find all functions $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ such that S satisfies the interpolation conditions*

$$S(\xi)\mathbf{b}(\xi) = \mathbf{d}(\xi) \quad \textit{for all} \ \ \xi \in \Omega_{\mathrm{R}}. \tag{1.41}$$

For the bitangential interpolation problem, we assume that we are given two subsets Ω_{L} and Ω_{R} of $\mathcal{D}_{\mathbf{Q}}$ and data consisting of functions $\mathbf{a}, \mathbf{c}, \mathbf{b}, \mathbf{d}$ as in (1.38) and (1.40). Then the problem is:

Problem 1.9. *Find all functions $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ such that S satisfies the interpolation conditions*

$$S(\xi)^*\mathbf{a}(\xi) = \mathbf{c}(\xi) \quad \text{for all } \xi \in \Omega_{\mathrm{L}}, \quad S(\xi)\mathbf{b}(\xi) = \mathbf{d}(\xi) \quad \textit{for all} \ \ \xi \in \Omega_{\mathrm{R}}. \tag{1.42}$$

Note that the left interpolation problem (Problem 1.7) is the special case of the bitangential problem (Problem 1.9) where $\Omega_{\mathrm{R}} = \emptyset$, and similarly, the right tangential interpolation problem is the special case of the bitangential interpolation problem when $\Omega_{\mathrm{L}} = \emptyset$. Note that the special case of the left interpolation problem where

$$\Omega_{\mathrm{L}} = \Omega, \quad \mathcal{E}_{\mathrm{L}} = \mathcal{E}_*, \quad \mathbf{a}(\xi) = I_{\mathcal{E}_*} \quad \text{and} \quad \mathbf{c}(\xi) = F(\xi)^*,$$

the special case of the right interpolation problem where

$$\Omega_{\mathrm{R}} = \Omega, \quad \mathcal{E}_{\mathrm{R}} = \mathcal{E}, \quad \mathbf{b}(\xi) = I_{\mathcal{E}} \quad \text{and} \quad \mathbf{d}(\xi) = F(\xi),$$

and the special case of the bitangential interpolation problem where $\Omega_{\mathrm{L}} = \Omega_{\mathrm{R}} = \Omega$,

$$\mathcal{E}_{\mathrm{L}} = \mathcal{E}_*, \quad \mathcal{E}_{\mathrm{R}} = \mathcal{E}, \quad \mathbf{a}(\xi) = I_{\mathcal{E}_*}, \quad \mathbf{c}(\xi) = F(\xi)^*, \quad \mathbf{b}(\xi) = I_{\mathcal{E}}, \quad \mathbf{d}(\xi) = F(\xi)$$

are all essentially solved in the various equivalent parts of Theorem 1.5.

Define operators $E_1^{\mathrm{L}}, \dots, E_p^{\mathrm{L}}, E_1^{\mathrm{R}}, \dots, E_q^{\mathrm{R}}$ by

$$E_1^{\mathrm{L}} = \begin{bmatrix} I_{\mathcal{E}_{\mathrm{L}}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, E_2^{\mathrm{L}} = \begin{bmatrix} 0 \\ I_{\mathcal{E}_{\mathrm{L}}} \\ \vdots \\ 0 \end{bmatrix}, \dots, E_p^{\mathrm{L}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{\mathcal{E}_{\mathrm{L}}} \end{bmatrix}, \tag{1.43}$$

$$E_1^{\mathrm{R}} = \begin{bmatrix} I_{\mathcal{E}_{\mathrm{R}}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, E_2^{\mathrm{R}} = \begin{bmatrix} 0 \\ I_{\mathcal{E}_{\mathrm{R}}} \\ \vdots \\ 0 \end{bmatrix}, \dots, E_q^{\mathrm{R}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{\mathcal{E}_{\mathrm{R}}} \end{bmatrix} \tag{1.44}$$

and let

$$\begin{aligned}
&\mathbf{q}_{1\cdot}, \dots, \mathbf{q}_{p\cdot} : \Omega_{\mathrm{R}} \mapsto \mathcal{L}(\mathcal{E}_{\mathrm{R}}, \mathbb{C}^q \otimes \mathcal{E}_{\mathrm{R}}),\\
&\overline{\mathbf{q}}_{\cdot 1}, \dots, \overline{\mathbf{q}}_{\cdot q} : \Omega_{\mathrm{L}} \mapsto \mathcal{L}(\mathcal{E}_{\mathrm{L}}, \mathbb{C}^p \otimes \mathcal{E}_{\mathrm{L}}),\\
&M_1, \dots, M_p : \Omega_{\mathrm{R}} \mapsto \mathcal{L}(\mathcal{E}_{\mathrm{L}} \oplus \mathcal{E}_{\mathrm{R}}, (\mathbb{C}^p \otimes \mathcal{E}_{\mathrm{L}}) \oplus (\mathbb{C}^q \otimes \mathcal{E}_{\mathrm{R}})),\\
&N_1, \dots, N_q : \Omega_{\mathrm{L}} \to \mathcal{L}(\mathcal{E}_{\mathrm{L}} \oplus \mathcal{E}_{\mathrm{R}}, (\mathbb{C}^p \otimes \mathcal{E}_{\mathrm{L}}) \oplus (\mathbb{C}^q \otimes \mathcal{E}_{\mathrm{R}}))
\end{aligned}$$

be the functions given by

$$\mathbf{q}_{j\cdot}(\xi_{\mathrm{R}}) = \begin{bmatrix} \mathbf{q}_{j1}(\xi_{\mathrm{R}}) I_{\mathcal{E}_{\mathrm{R}}} \\ \vdots \\ \mathbf{q}_{jq}(\xi_{\mathrm{R}}) I_{\mathcal{E}_{\mathrm{R}}} \end{bmatrix} \quad \text{for } \xi_{\mathrm{R}} \in \Omega_{\mathrm{R}}, \quad \overline{\mathbf{q}}_{\cdot k}(\xi) = \begin{bmatrix} \overline{\mathbf{q}_{1k}(\xi_{\mathrm{L}})} I_{\mathcal{E}_{\mathrm{L}}} \\ \vdots \\ \overline{\mathbf{q}_{pk}(\xi_{\mathrm{L}})} I_{\mathcal{E}_{\mathrm{L}}} \end{bmatrix} \quad \text{for } \xi_{\mathrm{L}} \in \Omega_{\mathrm{L}}, \tag{1.45}$$

$$\begin{aligned}
M_j(\xi_{\mathrm{R}}) &= \begin{bmatrix} E_j^{\mathrm{L}} & 0 \\ 0 & \mathbf{q}_{j\cdot}(\xi_{\mathrm{R}}) \end{bmatrix} \quad \text{for } \xi_{\mathrm{R}} \in \Omega_{\mathrm{R}},\\
N_k(\xi_{\mathrm{L}}) &= \begin{bmatrix} \overline{\mathbf{q}}_{\cdot k}(\xi_{\mathrm{L}}) & 0 \\ 0 & E_k^{\mathrm{R}} \end{bmatrix} \quad \text{for } \xi_{\mathrm{L}} \in \Omega_{\mathrm{L}},
\end{aligned} \tag{1.46}$$

where $j = 1, \dots, p$ and $k = 1, \dots, q$. Note that, in terms of this notation with $\mathcal{E}_{\mathrm{L}}$ taken to be $\mathcal{E}_*$ and $\mathcal{E}_{\mathrm{R}}$ taken to be $\mathcal{E}$, Eqs. (1.28), (1.30) and (1.31) to be satisfied by the respective kernels $\mathbb{K}_{\mathrm{L}}$, $\mathbb{K}_{\mathrm{R}}$ and $\mathbb{K}_{\mathrm{RL}}$ can be written in a more matricial form as

$$I_{\mathcal{E}_*} - F(\xi_{\mathrm{L}})F(\mu_{\mathrm{L}})^* = \sum_{j=1}^{p} (E_j^{\mathrm{L}})^* \mathbb{K}_{\mathrm{L}}(\xi_{\mathrm{L}}, \mu_{\mathrm{L}}) E_j^{\mathrm{L}} - \sum_{k=1}^{q} \overline{\mathbf{q}}_{\cdot k}(\xi_{\mathrm{L}})^* \mathbb{K}_{\mathrm{L}}(\xi_{\mathrm{L}}, \mu_{\mathrm{L}}) \overline{\mathbf{q}}_{\cdot k}(\mu_{\mathrm{L}}), \tag{1.47}$$

$$I_{\mathcal{E}} - F(\xi_{\mathrm{R}})^* F(\mu_{\mathrm{R}}) = \sum_{k=1}^{q} (E_k^{\mathrm{R}})^* \mathbb{K}_{\mathrm{R}}(\xi_{\mathrm{R}}, \mu_{\mathrm{R}}) E_k^{\mathrm{R}} - \sum_{j=1}^{p} \mathbf{q}_{j\cdot}(\xi_{\mathrm{R}})^* \mathbb{K}_{\mathrm{R}}(\xi_{\mathrm{R}}, \mu_{\mathrm{R}}) \mathbf{q}_{j\cdot}(\mu_{\mathrm{R}}), \tag{1.48}$$

$$F(\xi_{\mathrm{L}}) - F(\mu_{\mathrm{R}}) = \sum_{k=1}^{q} \overline{\mathbf{q}}_{\cdot k}(\xi_{\mathrm{L}})^* \mathbb{K}_{\mathrm{LR}}(\xi_{\mathrm{L}}, \mu_{\mathrm{R}}) E_k^{\mathrm{R}} - \sum_{j=1}^{p} (E_j^{\mathrm{L}})^* \mathbb{K}_{\mathrm{LR}}(\xi_{\mathrm{L}}, \mu_{\mathrm{R}}) \mathbf{q}_{j\cdot}(\mu_{\mathrm{R}}),$$

respectively. Furthermore, the latter three identities all together are equivalent to the following block matrix identity:

$$\begin{bmatrix} I \\ F(\xi_{\rm R})^* \end{bmatrix} [I \quad F(\mu_{\rm R})] - \begin{bmatrix} F(\xi_{\rm L}) \\ I \end{bmatrix} [F(\mu_{\rm L})^* \quad I]$$
$$= \sum_{j=1}^{p} M_j(\xi_{\rm R})^* \mathbb{K}(\xi_{\rm L}, \xi_{\rm R}; \mu_{\rm L}, \mu_{\rm R}) M_j(\mu_{\rm R})$$
$$- \sum_{k=1}^{q} N_k(\xi_{\rm L})^* \mathbb{K}(\xi_{\rm L}, \xi_{\rm R}; \mu_{\rm L}, \mu_{\rm R}) N_k(\xi_{\rm L}), \tag{1.49}$$

where $\mathbb{K}$ is the kernel of the special form (1.32).

Note that if $S = F$ satisfies the interpolation conditions (1.39) on $\Omega_{\rm L} \subset \mathscr{D}_{\mathbf{Q}}$, multiplication of (1.47) on the left by $\mathbf{a}(\xi_{\rm L})^*$ and on the right by $\mathbf{a}(\mu_{\rm L})$ leads to the Stein equation

$$\sum_{j=1}^{p} (E_j^{\rm L})^* K_{\rm L}(\xi_{\rm L}, \mu_{\rm L}) E_j^{\rm L} - \sum_{k=1}^{q} \overline{\mathbf{q}}_{\cdot k}(\xi_{\rm L})^* K_{\rm L}(\xi_{\rm L}, \mu_{\rm L}) \overline{\mathbf{q}}_{\cdot k}(\mu_{\rm L})$$
$$= \mathbf{a}(\xi_{\rm L})^* \mathbf{a}(\mu_{\rm L}) - \mathbf{c}(\xi_{\rm L})^* \mathbf{c}(\mu_{\rm L}) \tag{1.50}$$

with $K_{\rm L}$ equal to the positive kernel on $\Omega_{\rm L} \times \Omega_{\rm L}$ given by

$$K_{\rm L}(\xi_{\rm L}, \mu_{\rm L}) = \mathbf{a}(\xi_{\rm L})^* \mathbb{K}_{\rm L}(\xi_{\rm L}, \mu_{\rm L}) \mathbf{a}(\mu_{\rm L})$$

being satisfied for all $\xi_{\rm L}, \mu_{\rm L} \in \Omega_{\rm L}$. Similarly, if $S = F$ satisfies the interpolation conditions (1.41) on $\Omega_{\rm R} \subset \mathscr{D}_{\mathbf{Q}}$, then multiplication of (1.48) on the left by $\mathbf{b}(\xi_{\rm R})^*$ and on the right by $\mathbf{b}(\mu_{\rm R})$ leads to the Stein equation

$$\sum_{k=1}^{q} (E_k^{\rm R})^* K_{\rm R}(\xi_{\rm R}, \mu_{\rm R}) E_k^{\rm R} - \sum_{j=1}^{p} \mathbf{q}_{j\cdot}(\xi_{\rm R})^* K_{\rm R}(\xi_{\rm R}, \mu_{\rm R}) \mathbf{q}_{j\cdot}(\xi_{\rm R})$$
$$= \mathbf{b}(\xi_{\rm R})^* \mathbf{b}(\mu_{\rm R}) - \mathbf{d}(\xi_{\rm R})^* \mathbf{d}(\mu_{\rm R}) \tag{1.51}$$

with $K_{\rm R}$ equal to the positive kernel on $\Omega_{\rm R} \times \Omega_{\rm R}$ given by

$$K_{\rm R}(\xi_{\rm R}, \mu_{\rm R}) = \mathbf{b}(\xi_{\rm R})^* \mathbb{K}_{\rm R}(\xi_{\rm R}, \mu_{\rm R}) \mathbf{b}(\mu_{\rm R}).$$

being satisfied for all $\xi_{\rm R}$, $\mu_{\rm R} \in \Omega_{\rm R}$. Finally, if $F = S$ satisfies the bitangential interpolation conditions (1.42) for all $\xi_{\rm L} \in \Omega_{\rm L}$ and $\xi_{\rm R} \in \Omega_{\rm R}$, then multiplication of (1.49) on the left by $\begin{bmatrix} \mathbf{a}(\xi_{\rm L})^* & 0 \\ 0 & \mathbf{b}(\xi_{\rm R})^* \end{bmatrix}$ and on the right by $\begin{bmatrix} \mathbf{a}(\mu_{\rm L}) & 0 \\ 0 & \mathbf{b}(\mu_{\rm R}) \end{bmatrix}$ leads to the

Stein equation

$$\sum_{j=1}^{p} M_j(\xi_R)^* K(\xi_L, \xi_R, \mu_L, \mu_R) M_j(\mu_R) - \sum_{k=1}^{q} N_k(\xi_L)^* K(\xi_L, \xi_R, \mu_L, \mu_R) N_k(\mu_L)$$
$$= \begin{bmatrix} \mathbf{a}(\xi_L)^* \\ \mathbf{d}(\xi_R)^* \end{bmatrix} [\mathbf{a}(\mu_L) \quad \mathbf{d}(\mu_R)] - \begin{bmatrix} \mathbf{c}(\xi_L)^* \\ \mathbf{b}(\xi_R)^* \end{bmatrix} [\mathbf{c}(\mu_L) \quad \mathbf{b}(\mu_R)] \tag{1.52}$$

with K equal to the positive kernel on $(\Omega_L \times \Omega_R) \times (\Omega_L \times \Omega_R)$ given by

$$K(\xi_L, \xi_R, \mu_L, \mu_R) = \begin{bmatrix} \mathbf{a}(\xi_L)^* & 0 \\ 0 & \mathbf{b}(\xi_R)^* \end{bmatrix} \mathbb{K}(\xi_L, \xi_R, \mu_L, \mu_R) \begin{bmatrix} \mathbf{a}(\mu_L) & 0 \\ 0 & \mathbf{b}(\mu_R) \end{bmatrix}$$

being satisfied for all $\xi_L, \mu_L \in \Omega_L$ and $\xi_R, \mu_R \in \Omega_R$. We thus have already arrived at the necessity direction for the following solution of the tangential interpolation problems (1.7)–(1.9).

Theorem 1.10. *Suppose that we are given subsets Ω_L and Ω_R of $\mathcal{D}_{\mathbf{Q}}$ and data functions* $\mathbf{a}, \mathbf{c}, \mathbf{b}, \mathbf{d}$ *as in* (1.38) *and* (1.40), *and* E_j^L, E_k^R, $\overline{\mathbf{q}}_{\cdot k}$, $\mathbf{q}_{j\cdot}$, M_j *and* N_k *are defined as in* (1.43)–(1.46). *Then*:

(1) *The left tangential interpolation problem* (*Problem* 1.7) *has a solution if and only if there is a positive kernel*

$$K_L : \Omega_L \times \Omega_L \mapsto \mathcal{L}(\mathbb{C}^p \otimes \mathcal{E}_L)$$

which satisfies the Stein equation (1.50) *for all* $\xi_L, \mu_L \in \Omega_L$.

(2) *The right tangential interpolation problem* (*Problem* 1.8) *has a solution if and only if there is a positive kernel*

$$K_R : \Omega_R \times \Omega_R \mapsto \mathcal{L}(\mathbb{C}^q \otimes \mathcal{E}_R)$$

which satisfies the Stein equation (1.51) *for all* $\xi_R, \mu_R \in \Omega_R$.

(3) *The bitangential interpolation problem* (*Problem* 1.9) *has a solution if and only if there is a positive kernel*

$$K : (\Omega_L \times \Omega_R) \times (\Omega_L \times \Omega_R) \mapsto \mathcal{L}((\mathbb{C}^p \otimes \mathcal{E}_L) \oplus (\mathbb{C}^q \otimes \mathcal{E}_R))$$

of the special form

$$K(\xi_L, \xi_R, \mu_L, \mu_R) = \begin{bmatrix} K_L(\xi_L, \mu_L) & K_{LR}(\xi_L, \mu_R) \\ K_{LR}(\mu_L, \xi_R)^* & K_R(\xi_R, \mu_R) \end{bmatrix} \tag{1.53}$$

which satisfies the Stein equation (1.52) *for every* $\xi_L, \mu_L \in \Omega_L$ *and* $\xi_R, \mu_R \in \Omega_R$.

As a corollary we state the result in explicit form for the case of interpolation by a scalar Schur–Agler-class function at finitely many points in $\mathcal{D}_{\mathbf{Q}}$.

Corollary 1.11 (see Ambrozie and Timotin [8]). *Given N points $z^1, \dots, z^N \in \mathcal{D}_{\mathbf{Q}}$ and N complex numbers $w_1, \dots, w_N \in \mathbb{C}$, there exists a function f in the scalar-valued Schur–Agler class $\mathcal{SA}_{\mathcal{D}_{\mathbf{Q}}} := \mathcal{SA}_{\mathcal{D}_{\mathbf{Q}}}(\mathbb{C}, \mathbb{C})$ if and only if there exists a positive semidefinite $pN \times pN$ matrix*

$$\Gamma = [\gamma_{k,\ell;k',\ell'}]_{k,k'=1,\dots,p;\ \ell,\ell'=1,\dots,N}$$

such that

$$1 - w_\ell \overline{w_{\ell'}} = \sum_{k=1}^{p} \gamma_{k,\ell;k',\ell'} - \sum_{i=1}^{q} \sum_{k,k'=1}^{p} [p_{k,i}(z^\ell)\overline{p_{k',i}(z^{\ell'})}\gamma_{k,\ell;k',\ell'}]. \tag{1.54}$$

Remark 1.12. We mention that the recent preprint [7] of Ambrozie and Eschmeier derives a commutant lifting theorem for the class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ (see [15] for the special case of the polydisk $\mathcal{D}_{\mathbf{Q}} = \mathbb{D}^d$ and [19] for the case of the ball $\mathcal{D}_{\mathbf{Q}} = \mathbb{B}^d$), and obtains Corollary 1.11 (actually a version with higher multiplicity interpolation conditions) as an application. It remains to work out if the general interpolation Problem 1.9 considered here can be obtained as an application of such a commutant lifting theorem.

Let us represent the block matrix entries K_{L}, K_{LR}, K_{R} in the block matrix form (1.53) for K explicitly as

$$K_{\mathrm{L}}(\xi,\mu) = \begin{bmatrix} \Psi_{11}(\xi,\mu) & \dots & \Psi_{1p}(\xi,\mu) \\ \vdots & & \vdots \\ \Psi_{p1}(\xi,\mu) & \dots & \Psi_{pp}(\xi,\mu) \end{bmatrix}, \tag{1.55}$$

$$K_{\mathrm{LR}}(\xi,\mu) = \begin{bmatrix} \Lambda_{11}(\xi,\mu) & \dots & \Lambda_{1q}(\xi,\mu) \\ \vdots & & \vdots \\ \Lambda_{p1}(\xi,\mu) & \dots & \Lambda_{pq}(\xi,\mu) \end{bmatrix}, \tag{1.56}$$

$$K_{\mathrm{R}}(\xi,\mu) = \begin{bmatrix} \Phi_{11}(\xi,\mu) & \dots & \Phi_{1q}(\xi,\mu) \\ \vdots & & \vdots \\ \Phi_{q1}(\xi,\mu) & \dots & \Phi_{qq}(\xi,\mu) \end{bmatrix} \tag{1.57}$$

with

$$\begin{aligned} &\Psi_{j\ell}(\xi,\mu) \in \mathcal{L}(\mathcal{E}_{\mathrm{L}}) \quad \text{for } (\xi,\mu) \in \Omega_{\mathrm{L}} \times \Omega_{\mathrm{L}} \ \ (j,\ell = 1,\dots,p), \\ &\Lambda_{j\ell}(\xi,\mu) \in \mathcal{L}(\mathcal{E}_{\mathrm{R}}, \mathcal{E}_{\mathrm{L}}) \quad \text{for } (\xi,\mu) \in \Omega_{\mathrm{L}} \times \Omega_{\mathrm{R}} \ \ (j = 1,\dots,p;\ \ell = 1,\dots,q), \\ &\Phi_{j\ell}(\xi,\mu) \in \mathcal{L}(\mathcal{E}_{\mathrm{R}}) \quad \text{for } (\xi,\mu) \in \Omega_{\mathrm{R}} \times \Omega_{\mathrm{R}} \ \ (j,\ell = 1,\dots,q). \end{aligned} \tag{1.58}$$

It turns out that for every positive kernel K satisfying conditions of part (3) of Theorem 1.10, there is a solution S of the bitangential interpolation Problem 1.9 such that, for some choice of associated functions $H_j(z)$ and $G_j(z)$ in representations (1.16) and (1.18), it holds that

$$\mathbf{a}(\xi)^* H_j(\xi) H_\ell(\mu)^* \mathbf{a}(\mu) = \Psi_{j\ell}(\xi,\mu) \quad (\xi,\mu \in \Omega_{\mathrm{L}};\ j,\ell = 1,\dots,p), \tag{1.59}$$

$$\mathbf{a}(\xi)^* H_j(\xi) G_\ell(\mu) \mathbf{b}(\mu) = \Lambda_{j\ell}(\xi,\mu) \quad (\xi \in \Omega_{\mathrm{L}}, \mu \in \Omega_{\mathrm{R}};\ j = 1,\dots,p;\ \ell = 1,\dots,q), \tag{1.60}$$

$$\mathbf{b}(\xi)^* G_j(\xi)^* G_\ell(\mu) \mathbf{b}(\mu) = \Phi_{j\ell}(\xi,\mu) \quad (\xi,\mu \in \Omega_{\mathrm{R}};\ j,\ell = 1,\dots,q). \tag{1.61}$$

Furthermore, it turns out that conversely, for every solution S of Problem 1.9 with representations (1.16) and (1.18) (existence of these representations is guaranteed by Theorem 1.5)), the kernel K defined via (1.53)–(1.57) and (1.59)–(1.61) satisfies conditions of Theorem 1.10. Similar remarks apply for the left tangential and right tangential interpolation problems (Problems 1.7 and 1.8). These observations suggest the following modifications of the tangential interpolation Problems 1.7–1.9:

Problem 1.13. *Given two functions* $\mathbf{a}, \mathbf{c}$ *as in* (1.38) *and* p^2 *functions* $\Psi_{j\ell}$ *as in* (1.58), *find all functions* $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ *such that* S *satisfies the left tangential interpolation conditions* (1.39), *and, for some choice of associated functions* $H_j(z)$ *in representation* (1.16) *equalities* (1.59) *hold.*

Problem 1.14. *Given two functions* $\mathbf{b}, \mathbf{d}$ *as in* (1.40) *and* q^2 *functions* $\Phi_{j\ell}$ *as in* (1.58), *find all functions* $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ *such that* S *satisfies the right tangential interpolation conditions* (1.39), *and, for some choice of associated functions* $G_j(z)$ *in representation* (1.18) *equalities* (1.61) *hold.*

Problem 1.15. *Given four functions as in* (1.38) and (1.40) *and* $p^2 + pq + q^2$ *functions* $\Psi_{j\ell}$, $\Lambda_{j\ell}$ *and* $\Phi_{j\ell}$ *as in* (1.58), *find all functions* $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ *such that* S *satisfies the left and right tangential interpolation conditions* (1.39) *and* (1.41), *and, for some choice of associated functions* $H_j(z)$ *and* $G_j(z)$ *in representations* (1.16) *and* (1.18), *equalities* (1.59)–(1.61) *hold true.*

The solutions of Problems 1.13–1.15 are given by the following modification of Theorem 1.10.

Theorem 1.16. *Suppose that we are given subsets* Ω_{L} *and* Ω_{R} *of* $\mathcal{D}_{\mathbf{Q}}$ *and data functions* $\mathbf{a}, \mathbf{c}, \mathbf{b}, \mathbf{d}$ *as in* (1.38) *and* (1.40), *and* E^{L}_j, E^{R}_k, $\overline{\mathbf{q}}_{\cdot k}$, $\mathbf{q}_{j\cdot}$, M_j *and* N_k *are defined as in*

(1.43)–(1.46), *together with* $\Psi_{j\ell}(\xi,\mu)$, $\Lambda_{j\ell}(\xi,\mu)$ *and* $\Phi_{j\ell}(\xi,\mu)$ *as in* (1.58). *Define kernels* $K_{\rm L}(\xi,\mu)$, $K_{\rm LR}(\xi,\mu)$ *and* $K_{\rm R}(\xi,\mu)$ *as in* (1.55)–(1.57). *Then*:

(1) *Problem* 1.13 *has a solution if and only if the kernel* $K_{\rm L}(\xi,\mu)$ *is positive on* $\Omega_{\rm L}\times\Omega_{\rm L}$ *and satisfies the Stein equation* (1.50).
(2) *Problem* 1.14 *has a solution if and only if the kernel* $K_{\rm R}(\xi,\mu)$ *is positive on* $\Omega_{\rm R}\times\Omega_{\rm R}$ *and satisfies the Stein equation* (1.51).
(3) *Problem* 1.15 *has a solution if and only if the kernel* K *of the form* (1.53) *is positive on* $(\Omega_{\rm L}\times\Omega_{\rm R})\times(\Omega_{\rm L}\times\Omega_{\rm R})$ *and satisfies the Stein equation* (1.52).

The paper is organized as follows. After the present Introduction, in Section 2 we present a number of examples of domains $\Omega\subset\mathbb{C}^n$ which can be written in the form $\mathscr{D}_{\mathbf{Q}}$ for an appropriate $\mathbf{Q}$. In Section 3 we give a complete proof of Theorem 1.5. The proof of (1) $\Rightarrow$ (2) relies on a Hahn–Banach separation argument following the ideas in [2,8,15] used to prove closely related results. The proof of (2) $\Rightarrow$ (5) is an adaptation of the "lurking isometry" method which has been used recently as a method of proof in a number of closely related realization and interpolation problems for multivariable functions (see [4,8,11–13,15,19] and also [33–35] for an abstract formalism in the single variable setting). Section 4 gives the reformulation of Theorem 1.10 in terms similar to those in Theorem 1.5 and explains how all the solvability criteria for all the results in the Introduction reduce to the proof of the solvability criterion for the bitangential Problem 1.15 as stated in part (3) of Theorem 1.16. The proof of part (3) of Theorem 1.16 is then completed by an adaptation of the lurking isometry method to the setting where only bitangential interpolation data are given. The final Section 5 applies Theorem 1.10 to obtain a Toeplitz corona theorem for the setting here.

2. Examples of domains $\mathscr{D}_{\mathbf{Q}}$

We present some more examples arising for special choices of the function $\mathbf{Q}$.

Example 2.1. Let $p=1$, i.e., let

$$\mathbf{Q}(z)=\begin{bmatrix}\mathbf{q}_1(z) & \dots & \mathbf{q}_q(z)\end{bmatrix}. \tag{2.1}$$

In this case, $\mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E},\mathscr{E}_*)$ can be characterized as the set of all functions S analytic on $\mathscr{D}_{\mathbf{Q}}$ and such that the kernel

$$\frac{I_{\mathscr{E}_*}-S(z)S(w)^*}{1-\mathbf{q}_1(z)\mathbf{q}_1(w)^*-\cdots-\mathbf{q}_q(z)\mathbf{q}_q(w)^*}$$

is positive on $\mathscr{D}_{\mathbf{Q}}$. This case was considered in [19]. A special choice of $\mathbf{Q}$ when $\mathbf{q}_j(z)=z_j$ ($j=1,\dots,q$) leads to the class of Schur multipliers on the unit ball $\mathbb{B}^q$ of $\mathbb{C}^q$. Various interpolation problems in this class were considered in [11,13,19,20,25,43]. If $\mathbf{Q}$ is of the form (2.1), then condition (1.59) is redundant in the formulation of Problems 1.13 and 1.15 (assuming that any solutions exist); indeed by representation (1.16) with $p = 1$ and (1.42),

$$\begin{aligned} K_{\mathrm{L}}(\xi,\mu) &= \mathbf{a}(\xi)^* \frac{I - S(\xi)S(\mu)^*}{1 - \mathbf{q}_1(\xi)\mathbf{q}_1(\mu)^* - \ldots - \mathbf{q}_q(\xi)\mathbf{q}_q(\mu)^*} \mathbf{a}(\mu) \\ &= \frac{\mathbf{a}(\xi)^*\mathbf{a}(\mu) - \mathbf{c}(\xi)^*\mathbf{c}(\mu)}{1 - \mathbf{q}_1(\xi)\mathbf{q}_1(\mu)^* - \ldots - \mathbf{q}_q(\xi)\mathbf{q}_q(\mu)^*}, \end{aligned}$$

from which we see that $K_{\mathrm{L}}(\xi,\mu)$ is completely determined from the other interpolation data. Furthermore, for this case Problems 1.7 and 1.13 are equivalent (assuming that the data set for Problem 1.13 is such that solutions exist).

Example 2.2. Similarly, if $q = 1$, we have

$$\mathbf{Q}(z) = \begin{bmatrix} \mathbf{q}_1(z) \\ \vdots \\ \mathbf{q}_p(z) \end{bmatrix},$$

then condition (1.61) is redundant in the formulation of Problems 1.14 and 1.15 (assuming any solutions exist), since $K_{\mathrm{R}}(\xi,\mu)$ is completely determined from the other interpolation data:

$$\begin{aligned} K_{\mathrm{R}}(\xi,\mu) &= \mathbf{b}(\xi)^* \frac{I - S(\xi)^*S(\mu)}{1 - \mathbf{q}_1(\xi)^*\mathbf{q}_1(\mu) - \ldots - \mathbf{q}_p(\xi)^*\mathbf{q}_p(\mu)} \mathbf{b}(\mu) \\ &= \frac{\mathbf{b}(\xi)^*\mathbf{b}(\mu) - \mathbf{d}(\xi)^*\mathbf{d}(\mu)}{1 - \mathbf{q}_1(\xi)^*\mathbf{q}_1(\mu) - \ldots - \mathbf{q}_p(\xi)^*\,\mathbf{q}_p(\mu)}. \end{aligned}$$

Moreover, Problems 1.8 and 1.14 are equivalent (assuming that the data set for 1.14 is consistent).

Example 2.3. Let $p = q = n = 1$, i.e., let $\mathbf{Q}$ be a scalar valued polynomial of one variable. By the preceding analysis, under the assumption that the interpolation conditions are consistent, conditions (1.59) and (1.61) are redundant. Furthermore, it follows from (1.60), (1.19) and (1.42) that for every choice of $\xi \in \Omega_{\mathrm{L}}$ and $\mu \in \Omega_{\mathrm{R}}$ such that $\mathbf{Q}(\xi) \neq \mathbf{Q}(\mu)$, it holds that

$$K_{\mathrm{LR}}(\xi,\mu) = \mathbf{a}(\xi)^* \frac{S(\xi) - S(\mu)}{\mathbf{Q}(\xi) - \mathbf{Q}(\mu)} \mathbf{b}(\mu) = \frac{\mathbf{a}(\xi)^*\mathbf{d}(\mu) - \mathbf{c}(\xi)^*\mathbf{b}(\mu)}{\mathbf{Q}(\xi) - \mathbf{Q}(\mu)}$$

and thus, $K_{\mathrm{LR}}(\xi,\mu)$ is completely determined at such points by other interpolation conditions and therefore need not be specified. In the case when $\mathbf{Q}(\xi) \neq \mathbf{Q}(\mu)$ for every $\xi \in \Omega_{\mathrm{L}}$ and $\mu \in \Omega_{\mathrm{R}}$, all conditions (1.59)–(1.61) are not needed and Problem 1.9 is equivalent to Problem 1.15 (assuming that Problem 1.15 has any solutions).

If $\mathbf{Q}(z) = z$, then the condition $\mathbf{Q}(\xi) \neq \mathbf{Q}(\mu)$ for every $\xi \in \Omega_{\mathrm{L}}$ and $\mu \in \Omega_{\mathrm{R}}$ is equivalent to $\Omega_{\mathrm{L}} \cap \Omega_{\mathrm{R}} = \emptyset$. In the case when the intersection of Ω_{L} and Ω_{R} is not empty, one should specify K_{LR} on this intersection. The nonredundant interpolation conditions (1.60) for the case $\xi = \mu \in \Omega_{\mathrm{L}} \cap \Omega_{\mathrm{R}}$ can in this case be expressed directly in terms of the interpolant S according to the formula:

$$K_{\mathrm{LR}}(\xi,\xi) = \mathbf{a}(\xi)^* S'(\xi)\mathbf{b}(\xi) \quad \text{for } \xi \in \Omega_{\mathrm{L}} \cap \Omega_{\mathrm{R}}. \tag{2.2}$$

This case is treated in detail (for the case where Ω_{L} and Ω_{R} are assumed to be finite) in the monograph [14]. In particular, there it is shown that the added interpolation condition (2.2) is exactly what is required to make the interpolation conditions (1.39), (1.41) together with (2.2) on S equivalent to S having the *divisor-remainder* (also called *model-matching* in the engineering literature) form

$$S(z) = T_1(z) - T_2(z)Q(z)T_3(z),$$

where T_1, T_2 and T_3 are given operator-valued analytic functions on $\mathbb{D}$ and Q is an arbitrary analytic operator-valued function on $\mathbb{D}$. Here the zero invariant factors of T_2 and T_3 are assumed to be simple for ease of exposition.

Example 2.4. If $p = q$ and $\mathbf{q}_{ij}(z) \equiv 0$ for $i \neq j$, i.e., if

$$\mathbf{Q}(z) = \begin{bmatrix} \mathbf{q}_1(z) & & 0 \\ & \ddots & \\ 0 & & \mathbf{q}_p(z) \end{bmatrix}, \tag{2.3}$$

then $\mathscr{D}_{\mathbf{Q}} = \bigcap_{j=1}^{p} \mathscr{D}_{\mathbf{q}_j}$ and we have only p conditions in each of series (1.59)–(1.61):

$$\mathbf{a}(\xi)^* H_j(\xi) H_j(\mu)^* \mathbf{a}(\mu) = \Psi_j(\xi,\mu) \quad (\xi,\mu \in \Omega_{\mathrm{L}}; j = 1,\dots,p),$$

$$\mathbf{a}(\xi)^* H_j(\xi) G_j(\mu)\mathbf{b}(\mu) = \Lambda_j(\xi,\mu) \quad (\xi \in \Omega_{\mathrm{L}}, \mu \in \Omega_{\mathrm{R}}; j = 1,\dots,p),$$

$$\mathbf{b}(\xi)^* G_j(\xi)^* G_j(\mu)\mathbf{b}(\mu) = \Phi_j(\xi,\mu) \quad (\xi,\mu \in \Omega_{\mathrm{R}};\ j = 1,\dots,q).$$

In particular, if $p = q = n$ and $\mathbf{q}_j(z) = z_j$ ($j = 1,\dots,n$) in (2.3), then $\mathscr{D}_{\mathbf{Q}}$ is the unit polydisk $\mathbb{D}^n$ and in this case, $\mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E},\mathscr{E}_*)$ reduces to the well known Schur–Agler class of the polydisk. Theorem 1.10 for this particular polydisk case can be found in [1,3,12,15,18]. In the paper [16] it is shown that the analogue of condition (2.2) for the polydisk case (required to achieve the equivalence between interpolation and divisor-remainder form in the polydisk setting) is

$$\mathbf{a}(\xi)^* \frac{\partial S}{\partial z_k}(\xi)\mathbf{b}(\xi) = \rho_k(\xi) \quad \text{for all } \xi \in \Omega_{\mathrm{L}} \cap \Omega_{\mathrm{R}} \text{ and } k = 1,\dots,n, \tag{2.4}$$

where ρ_k's are functions on $\Omega_{\mathrm{L}} \cap \Omega_{\mathrm{R}}$ given as part of the interpolation data set. Unlike the situation for the disk case ($n = 1$—see Example 2.3), the relations between conditions (2.4) and (1.59)–(1.61) are not apparent in general; this issue will be taken up on another occasion.

Example 2.5. If

$$\mathbf{Q}(z_{1,1},\dots,z_{d,n_d}) = \begin{bmatrix} \mathbf{Q}_1(z_{1,1},\dots,z_{1,n_1}) & & 0 \\ & \ddots & \\ 0 & & \mathbf{Q}_d(z_{d,1},\dots,z_{d,n_d}) \end{bmatrix},$$

it follows that $\mathscr{D}_{\mathbf{Q}}$ has the Cartesian product decomposition $\mathscr{D}_{\mathbf{Q}} = \mathscr{D}_{\mathbf{Q}_1} \times \cdots \times \mathscr{D}_{\mathbf{Q}_d}$. If furthermore,

$$\mathbf{Q}_j(z_{j,1},\dots,z_{j,n_j}) = [z_{j,1} \quad \dots \quad z_{j,n_j}] \quad (j=1,\dots,d),$$

then $\mathscr{D}_{\mathbf{Q}}$ is the Cartesian product of d unit balls of dimensions $n_1,\dots,n_d$. Nevanlinna–Pick interpolation in the corresponding class $\mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E},\mathscr{E}_*)$ was studied in [48].

Example 2.6. A number of authors (see [5,6,37,42]) have considered generalizations of Schur-class functions and associated interpolation problems on domains in the complex plane defined by a condition of the form $|a(z)|^2 - |b(z)|^2 > 0$ where $a(z)$ and $b(z)$ are given polynomials. This gives a unified setting which, for example, includes the case of the unit disk ($a(z)=1$, $b(z)=z$) and the upper half plane ($a(z)=z+i$, $b(z)=z-i$). If we allow $\mathbf{Q}(z)=\frac{b(z)}{a(z)}$, this setting fits into our scheme. More generally, one can let

$$\mathbf{Q}(z) = \begin{bmatrix} \frac{z_1-i}{z_1+i} & & 0 \\ & \ddots & \\ & & \frac{z_d-i}{z_d+i} \end{bmatrix}$$

so that $\mathscr{D}_{\mathbf{Q}}$ is a Cartesian product of half planes; this is the setting of recent work of Kalyuzhnyĭ–Verbovetzkiĭ [32].

Example 2.7. If we take

$$\mathbf{Q}(z) = \begin{bmatrix} \frac{z_1-i}{z_1+i} & \frac{z_2\sqrt{2}}{z_1+i} & \dots & \frac{z_d\sqrt{2}}{z_1+i} \end{bmatrix},$$

then the corresponding domain $\mathscr{D}_{\mathbf{Q}}$ is Siegel's domain of the second kind:

$$\{(z_1,\dots,z_d) : \Im z_1 - |z_2|^2 - \cdots - |z_d|^2 > 0\}.$$

Example 2.8. *Projective domains* can be defined as

$$\mathscr{D}_{\mathbf{Q},\mathbf{P}} = \{z \in \mathbb{C}^n : \mathbf{Q}(z)\mathbf{Q}(z)^* < \mathbf{P}(z)\mathbf{P}(z)^*\},$$

where $\mathbf{Q}$ and $\mathbf{P}$ are $p \times q$ and $p \times k$ matrix polynomials. If $p = k = 1$, this notion is equivalent to that in Example 2.6.

Example 2.9. We remark that a common technique for the study of domains Ω in $\mathbb{C}^n$ is through a smooth defining function ρ (see e.g. [36]), i.e., ρ *is a smooth real-valued function defined on* $\mathbb{C}^n$ *with nonvanishing gradient on the boundary* $\partial\Omega$ *of* Ω *such that* $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$. If $q = 1$, we see from (1.13) that our domains $\mathscr{D}_{\mathbf{Q}}$ correspond to the special case where ρ is of the form $\rho(z) = 1 - \sum_{k=1}^{p} |p_k(z)|^2$ for polynomials $p_1(z), \dots, p_p(z)$. On the other hand, when $q > 1$, the domains $\mathscr{D}_{\mathbf{Q}}$ correspond to the intersection of a collection of domains Ω_ℓ with such defining functions ρ_ℓ—allowing nonsmooth boundary as in the case of the polydisk.

3. Characterization of the class $\mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*)$

In this section we present the proof of Theorem 1.5.

For the proof of (1) $\Rightarrow$ (2) in Theorem 1.5, we first need a few preliminaries.

Let $\mathscr{C}$ be the class of all operator-valued functions $\varphi : \Omega \times \Omega \mapsto \mathscr{L}(\mathscr{E}_*)$ having a representation of the form

$$\varphi(z, w) = H(z)(I - \mathbf{Q}(z)\mathbf{Q}(w)^*)H(w)^* \tag{3.1}$$

for some function $H = [H_1 \quad \cdots \quad H_p] : \Omega \mapsto \mathscr{L}(\mathscr{H}' \otimes \mathbb{C}^p, \mathscr{E}_*)$ for some auxiliary Hilbert space $\mathscr{H}'$ for z, w in some subset Ω of $\mathscr{D}_{\mathbf{Q}}$. Sometimes it is convenient to write (3.1) in the equivalent form

$$\varphi(z, w) = \sum_{k=1}^{p} \mathbb{K}_{k,k}(z, w) - \sum_{k=1}^{p} \sum_{k,\ell=1}^{q} \mathbf{q}_{ik}(z)\overline{\mathbf{q}_{\ell k}(w)}\mathbb{K}_{i,\ell}(z, w) \tag{3.2}$$

for a positive kernel

$$\mathbb{K} = \begin{bmatrix} \mathbb{K}_{1,1} & \cdots & \mathbb{K}_{1,p} \\ \vdots & & \vdots \\ \mathbb{K}_{p,1} & \cdots & \mathbb{K}_{p,p} \end{bmatrix} : \Omega \times \Omega \mapsto \mathscr{L}(\mathscr{E}_* \otimes \mathbb{C}^p),$$

this can be seen as in Remark 1.6 via a calculation analogous to (1.34). We shall consider $\mathscr{C}$ as a subset of the linear space $\mathscr{X}$ of all $\mathscr{L}(\mathscr{E}_*)$-valued functions on $\Omega \times \Omega$. It is easy to see that $\mathscr{C}$ is a *cone* in $\mathscr{X}$, i.e., $\mathscr{C}$ is closed under sums and multiplication by nonnegative scalars. We need to establish a few preliminary facts concerning $\mathscr{C}$.

Lemma 3.1. *Fix $\varphi \in \mathscr{C}$. Then, for each $z \in \Omega$ there is a finite constant $M_z < \infty$ so that*

$$||H(z)|| \leqslant M_z$$

for any $H : \Omega \mapsto \mathscr{L}(\mathscr{H}' \otimes \mathbb{C}^p, \mathscr{E}_*)$ giving a representation for φ as in (3.1).

Proof. For $z \in \Omega \subset \mathscr{D}_{\mathbf{Q}}$ fixed, $I - \mathbf{Q}(z)\mathbf{Q}(z)^*$ is invertible by definition of $\mathscr{D}_{\mathbf{Q}}$. Set

$$M_z = (||(I - \mathbf{Q}(z)\mathbf{Q}(z)^*)^{-1}|| \cdot ||\varphi(z,z)||)^{1/2}.$$

Then we compute, for any $H(z)$ giving a representation for φ as in (3.1),

$$\begin{aligned} H(z)H(z)^* &= H(z)(I - \mathbf{Q}(z)\mathbf{Q}(z)^*)^{1/2} \cdot (I - \mathbf{Q}(z)\mathbf{Q}(z)^*)^{-1} \\ &\quad \cdot (I - \mathbf{Q}(z)\mathbf{Q}(z)^*)^{1/2} H(z)^* \\ &\leqslant ||(I - \mathbf{Q}(z)\mathbf{Q}(z)^*)^{-1}|| \cdot H(z)(I - \mathbf{Q}(z)\mathbf{Q}(z)^*)H(z)^* \\ &= ||(I - \mathbf{Q}(z)\mathbf{Q}(z)^*)^{-1}|| \cdot \varphi(z,z) \quad (\text{by (3.1)}) \\ &\leqslant M_z^2 I_{\mathscr{E}_*} \end{aligned}$$

and the lemma follows. □

Lemma 3.2. *Any positive kernel $\varphi : \Omega \times \Omega \mapsto \mathscr{L}(\mathscr{E}_*)$ is in $\mathscr{C}$, i.e., has a representation* (3.1) *for some H.*

Proof. Since $\mathbf{Q}$ is strictly contractive on $\mathscr{D}_{\mathbf{Q}}$, so also is its first row $\mathbf{q}_{1\cdot}(z) = [\mathbf{q}_{11}(z) \; \cdots \; \mathbf{q}_{1q}(z)]$. Hence the scalar function $1 - \mathbf{q}_{1\cdot}(z)\mathbf{q}_{1\cdot}(w)^*$ is invertible on all of $\mathscr{D}_{\mathbf{Q}}$ with inverse given by the convergent geometric series

$$\begin{aligned} (1 - \mathbf{q}_{1\cdot}(z)\mathbf{q}_{1\cdot}(w)^*)^{-1} &= \sum_{\ell=0}^{\infty} (\mathbf{q}_{1\cdot}(z)\mathbf{q}_{1\cdot}(w)^*)^{\ell} \\ &= \sum_{\ell=0}^{\infty} \left(\sum_{k=1}^{q} \mathbf{q}_{1k}(z)\overline{\mathbf{q}_{1k}(w)} \right)^{\ell}. \end{aligned}$$

By Schur's theorem, each term of the infinite series is a positive kernel; hence $(1 - \mathbf{q}_{1\cdot}(z)\mathbf{q}_{1\cdot}(w)^*)^{-1}$ is a positive kernel; choose a function $\eta : \mathscr{H}'' \mapsto \mathbb{C}$ so that we have the factorization

$$(1 - \mathbf{q}_{1\cdot}(z)\mathbf{q}_{1\cdot}(w)^*)^{-1} = \eta(z)\eta(w)^*.$$

Since $\varphi(z,w) : \Omega \times \Omega \mapsto \mathscr{L}(\mathscr{E}_*)$, then φ has a factorization $\varphi(z,w) = H'(z)H'(w)^*$ for some $H' : \Omega \mapsto \mathscr{L}(\mathscr{H}', \mathscr{E}_*)$. Let us now set

$$H(z) = [H'(z) \otimes \eta(z) \quad 0 \quad \cdots \quad 0] : \mathscr{H}' \otimes \mathscr{H}'' \otimes \mathbb{C}^p \mapsto \mathscr{E}_*, \tag{3.3}$$

where we have made the identification $\mathscr{E}_* \cong \mathscr{E}_* \otimes \mathbb{C}$. Then it is straightforward to check that $\varphi(z,w)$ has a representation (3.1) with $H(z)$ as in (3.3), and the lemma follows. $\square$

We shall want to approximate the cone $\mathscr{C}$ by the cone $\mathscr{C}_\varepsilon$ (where $\varepsilon > 0$) given by

$$\mathscr{C}_\varepsilon = \Bigg\{ \varphi : \Omega \times \Omega \mapsto \mathscr{L}(\mathscr{E}_*) : \varphi(z,w)$$
$$= H(z)(I - (1+\varepsilon)^2 \mathbf{Q}(z)\mathbf{Q}(w)^*)H(w)^* + \sum_{j=1}^n (1 - \varepsilon^2 z_j \overline{w_j})\gamma_j(z)\gamma_j(w)^*$$
$$\text{for some } H \in \mathscr{L}(\mathscr{H}' \otimes \mathbb{C}^p, \mathscr{E}_*) \text{ and } \gamma_j : \Omega \mapsto \mathscr{L}(\mathscr{H}_j{}', \mathscr{E}_*) \Bigg\}. \tag{3.4}$$

Sometimes it will be convenient to work with the equivalent representation

$$\varphi(z,w) = \sum_{k=1}^p \mathbb{K}_{\varepsilon;k,k}(z,w) - \sum_{k=1}^p \sum_{i,\ell-1}^q \mathbf{q}_{ik}(z)\overline{\mathbf{q}_{\ell k}(w)}\mathbb{K}_{\varepsilon,i,\ell}(z,w)$$
$$+ \sum_{j=1}^n (1 - \varepsilon^2 z_j \overline{w_j})\Gamma_{j,\varepsilon}(z,w), \tag{3.5}$$

where $\mathbb{K}_\varepsilon = [\mathbb{K}_{\varepsilon;i,\ell}]_{i,\ell=1}^p$ and $\Gamma_{j,\varepsilon}$ $(j = 1, \dots, n)$ are positive kernels; this equivalence follows in the same way as the equivalence between (1.16) and (1.28) explained in Remark 1.6.

Lemma 3.3. *Assume that Ω is finite and that $\varphi : \Omega \times \Omega \mapsto \mathscr{L}(\mathscr{E}_*)$ is in the cone $\mathscr{C}_\varepsilon$ for all ε sufficiently small. Then $\varphi \in \mathscr{C}$, i.e., φ has a representation* (3.1).

Proof. The assumption is that there are functions

$$H_\varepsilon = [H_{\varepsilon,1} \quad \dots \quad H_{\varepsilon,p}] : \Omega \mapsto \mathscr{L}(\mathscr{H}' \otimes \mathbb{C}^p, \mathscr{E}_*) \quad \text{and} \quad \gamma_{j,\varepsilon} : \Omega \mapsto \mathscr{L}(\mathscr{H}', \mathscr{E}_*)$$

so that φ has the representation

$$\varphi(z,w) = H_\varepsilon(z)(I - (1+\varepsilon)^2 \mathbf{Q}(z)\mathbf{Q}(w)^*)H_\varepsilon(w)^* + \sum_{j=1}^n (1 - \varepsilon^2 z_j \overline{w_j})\gamma_{j,\varepsilon}(z)\gamma_{j,\varepsilon}(w)^*$$

for all $\varepsilon > 0$ sufficiently small. One can adapt the proof of Lemma 3.1 to see that, for each fixed $z \in \Omega$, $H_\varepsilon(z)$ and $\gamma_{j,\varepsilon}(z)$ are bounded uniformly with respect to ε for all $0<\varepsilon<\delta$, where $\delta>0$ is chosen so that $|z_j|<1/\delta$ for $j=1,\dots,n$ for all $z\in\Omega$. Hence also

$$\mathbb{K}_{\varepsilon;i,\ell}(z,w) := H_{\varepsilon,i}(z)H_{\varepsilon,\ell}(w)^* \quad \text{and} \quad \Gamma_{j,\varepsilon}(z,w) := \gamma_{j,\varepsilon}(z)\gamma_{j,\varepsilon}(w)^*$$

are uniformly bounded with respect to ε for all $0<\varepsilon<\delta$ and give representation (3.5) with $\mathbb{K}_{\varepsilon;i,\ell}$ and $\Gamma_{j,\varepsilon}$ defined as above. We assume that $\mathscr{E}_*$ is a separable Hilbert space, and hence also the space of trace-class operators $\mathscr{L}_1(\mathscr{E}_*)$ (the pre-dual of $\mathscr{L}(\mathscr{E}_*)$) is separable. Then (see [27, Theorem 1, p. 426]) the unit ball of $\mathscr{L}(\mathscr{E}_*)$ in the weak-$*$ topology is metrizable. As the unit ball of $\mathscr{L}(\mathscr{E}_*)$ in the weak-$*$ topology is also compact by Alaoglu's Theorem (see [27, Theorem 2, p. 424]), there is a subsequence $\varepsilon_N\to 0$ as $N\to\infty$ such that $\mathbb{K}_{\varepsilon_N;j,k}(z,w)\to K_{j,k}(z,w)$ and $\Gamma_{j,\varepsilon_N}(z,w)\to\Gamma_j(z,w)$ in the weak-$*$ topology for each $z,w\in\Omega$. Moreover, from characterization (1.5) of positive kernels, we see that $\mathbb{K}=[\mathbb{K}_{j,k}]_{j,k=1}^p$ and Γ_j ($j=1,\dots,n$) are again positive kernels. By taking limits as $N\to\infty$ in (3.5) (with ε_N in place of ε), we see that φ has representation (3.2). We conclude that $\varphi\in\mathscr{C}$ as asserted. □

Lemma 3.4. *Assume that Ω is finite and choose $\varepsilon>0$ sufficiently small so that $|z_j|<1/\varepsilon$ for $j=1,\dots,n$ and $z\in\Omega$. Then*:

(1) *Any positive kernel φ is in $\mathscr{C}_\varepsilon$.*
(2) *If φ is a positive kernel, then the kernel $(1-\varepsilon^2 z_j\overline{w_j})\varphi(z,w)$ is also in $\mathscr{C}_\varepsilon$ for each $j=1,\dots,n$.*

Proof. For the first assertion, a simple adjustment of the proof of Lemma 3.2 gives $H:\Omega\mapsto\mathscr{L}(\mathscr{H}'\otimes\mathbb{C}^p,\mathscr{E}_*)$ so that

$$\varphi(z,w) = H(z)(I-(1+\varepsilon)^2\mathbf{Q}(z)\mathbf{Q}(w)^*)H(w)^*.$$

But this is the special case of the form required for membership in $\mathscr{C}_\varepsilon$ with $\gamma_j(z)=0$ for each $j=1,\dots,n$.

For the second assertion, use the defining form for membership in $\mathscr{C}_\varepsilon$ with $H(z)=0$, $\gamma_j(z)$ chosen so that $\varphi(z,w)=\gamma_j(z)\gamma_j(w)^*$ and $\gamma_k(z)=0$ for $k\neq j$. □

Lemma 3.5. *Assume that Ω is a finite set and choose $\varepsilon>0$ as in Lemma* 3.4. *Consider the cone $\mathscr{C}_\varepsilon$ as a subset of the linear space $\mathscr{X}$ of $\mathscr{L}(\mathscr{E}_*)$-valued functions on Ω, endowed with the locally convex topology of pointwise weak-$*$ convergence. Then $\mathscr{C}_\varepsilon$ is closed in $\mathscr{X}$.*

Proof. By the Kreĭn–Šmulian Theorem (see [27, Theorem 7, p. 429]), it suffices to show that the intersection of $\mathscr{C}_\varepsilon$ with each bounded subset of $\mathscr{X}$ is closed in $\mathscr{X}$. As noted in the proof of Lemma 3.3, the weak-$*$ topology restricted to bounded sets is metrizable. Hence, to show that $\mathscr{C}_\varepsilon$ is closed in $\mathscr{X}$, it suffices to show that, whenever a bounded sequence $\{\varphi_N\}$ of elements of $\mathscr{C}_\varepsilon$ converges weak-$*$ to an element φ of $\mathscr{X}$, then in fact $\varphi\in\mathscr{C}_\varepsilon$. By assumption, each φ_N has a representation as in (3.5)

$$\varphi_N(z,w) = \sum_{k=1}^{p} \mathbb{K}_{N;k,k}(z,w) - \sum_{i,\ell=1}^{q} (1+\varepsilon)^2 \mathbf{q}_{ik}(z)\overline{\mathbf{q}_{\ell k}(w)}\mathbb{K}_{N;i,\ell}(z,w) + \sum_{j=1}^{n} (1-\varepsilon^2 z_j\overline{w_j})\Gamma_{j,N}(z,w). \tag{3.6}$$

Since $\varphi_N(z,w)$ is uniformly bounded in norm by assumption, by an argument as in the proof of Lemma 3.1 we see that $\mathbb{K}_{N;i,\ell}(z,w)$ and $\Gamma_{j,N}(z,w)$ are uniformly bounded in norm as $N\to\infty$ for each $z,w\in\Omega$. By the weak-$*$ compactness of the unit ball of $\mathscr{L}(\mathscr{E}_*)$, we may then drop down to a subsequence $\{\varphi_{N_K}\}_{K=1,2,\dots}$ such that $\mathbb{K}_{N_K;i,\ell}(z,w)\to\mathbb{K}_{i,\ell}(z,w)$ and $\Gamma_{j,N_K}(z,w)\to\Gamma_j(z,w)$ in the weak-$*$ topology as $K\to\infty$. By the criterion (1.5) for positive kernels, we see that $\mathbb{K}(z,w) = [\mathbb{K}_{i,\ell}(z,w)]_{i,\ell=1}^{p}$ and $\Gamma_j(z,w)$ ($j=1,\dots,n$) are again positive kernels. Taking limits in (3.6) (with N_K in place of N) as $K\to\infty$ yields a representation for $\varphi(z,w)$ of the form (3.5), and we conclude that $\varphi\in\mathscr{C}_\varepsilon$ as claimed. □

Proof of (1) ⇒ (2) in Theorem 1.5. The proof is based on a Hahn–Banach separation argument adapted from the proofs of various other versions of this result in [2,8,15].

Suppose that the function $F:\Omega\mapsto\mathscr{L}(\mathscr{E},\mathscr{E}_*)$ extends to a function $S\in\mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E},\mathscr{E}_*)$. Our goal is to show that $\varphi_F(z,w) = I_{\mathscr{E}_*} - F(z)F(w)^*$ is in $\mathscr{C}$. Consider first the case where Ω is finite. By Lemma 3.3 it suffices to show that φ_F is in $\mathscr{C}_\varepsilon$ for any $\varepsilon>0$. By the Hahn–Banach separation principle (see part (b) of Theorem 3.4 in [47]), it suffices to show: *given* $\varepsilon>0$ *and a continuous linear functional* $L:\mathscr{X}\mapsto\mathbb{C}$ *such that* $\Re L(\varphi)\geqslant 0$ *for all* $\varphi\in\mathscr{C}_\varepsilon$, *it follows that* $\Re L(\varphi_F)\geqslant 0$ where $\Re$ indicates "real part".

Fix $\varepsilon>0$ and let L be any weak-$*$ continuous linear functional $L:\mathscr{X}\mapsto\mathbb{C}$ such that $\Re L|_{\mathscr{C}_\varepsilon}\geqslant 0$. Define $L_1:\mathscr{X}\mapsto\mathbb{C}$ by

$$L_1(\varphi) = \tfrac{1}{2}(L(\varphi) + \overline{L(\breve{\varphi})}),$$

where we have set

$$\breve{\varphi}(z,w) = \varphi(w,z)^*.$$

Note that $L_1(\varphi) = \Re L(\varphi)$ in case $\breve{\varphi} = \varphi$.

Define a sesquilinear form $\langle\cdot,\cdot\rangle_{\mathrm{L}}$ on the linear space $\mathscr{H}_0$ of $\mathscr{L}(\mathscr{E}_*,\mathbb{C})$-valued functions on Ω by

$$\langle f,g\rangle_{\mathrm{L}} = L_1(g(z)^* f(w)).$$

Note that any function φ of the form $\varphi(z,w) = f(z)^* f(w)$ has the property that $\breve{\varphi} = \varphi$ and by part (1) of Lemma 3.4 any such φ is in $\mathscr{C}_\varepsilon$. We conclude that

$$\langle f,f\rangle_{\mathrm{L}} = \Re L(f(z)^* f(w))\geqslant 0 \quad \text{for all } f\in\mathscr{H}_0.$$

We may thus identify elements of 0-norm and then take a completion in the L-norm to get a Hilbert space $\mathscr{H}_{\mathrm{L}}$.

We next attempt to define operators $T_1, \dots, T_n$ on $\mathscr{H}_{\mathrm{L}}$ with adjoints given by

$$T_j^* : f(w) \mapsto \overline{w_j} f(w) \quad \text{for } f \in \mathscr{H}_0.$$

By part (2) of Lemma 3.4 we know that the kernel $(1-\varepsilon^2 z_j \overline{w_j}) f(z)^* f(w)$ belongs to $\mathscr{C}_\varepsilon$, and hence

$$\|f\|^2_{\mathscr{H}_{\mathrm{L}}} - \varepsilon^2 \|T_j^* f\|^2_{\mathscr{H}_{\mathrm{L}}} = \Re L((1-\varepsilon^2 z_j \overline{w_j}) f(z)^* f(w)) \geqslant 0.$$

Thus T_j extends to a bounded operator defined on all of $\mathscr{H}_{\mathrm{L}}$ with $\|T_j\| = \|T_j^*\| \leqslant 1/\varepsilon$.

Then the action of $\mathbf{Q}(T)^* : \mathbb{C}^p \otimes \mathscr{H}_{\mathrm{L}} \mapsto \mathbb{C}^q \otimes \mathscr{H}_{\mathrm{L}}$ is given simply as

$$\mathbf{Q}(T)^* : f(w) \mapsto \mathbf{Q}(w)^* f(w) \quad \text{for } f(w) = \begin{bmatrix} f_1(w) \\ \vdots \\ f_p(w) \end{bmatrix} \in \mathscr{H}_0 \otimes \mathbb{C}^p \subset \mathscr{H}_{\mathrm{L}} \otimes \mathbb{C}^p.$$

For f a block-column vector of the form $f = \mathrm{col}_{i=1,\dots,p} f_i \in \mathscr{H}_0 \otimes \mathbb{C}^p$, note that for a fixed $w \in \Omega$ the value $f(w)$ can be viewed as an operator from $\mathscr{E}_*$ into $\mathbb{C}^p$ (i.e., $f(w) \in \mathscr{L}(\mathscr{E}_*, \mathbb{C}^p)$), and

$$\|f\|^2_{\mathscr{H}_{\mathrm{L}} \otimes \mathbb{C}^p} = \sum_{i=1}^p \|f_i\|^2_{\mathscr{H}_0} = \sum_{i=1}^p \Re L(f_i(z)^* f_i(w)) = \Re L(f(z)^* f(w)).$$

Similarly

$$\|\mathbf{Q}(T)^* f\|^2_{\mathscr{H}_{\mathrm{L}} \otimes \mathbb{C}^p} = \Re L(f(z)^* \mathbf{Q}(z) \mathbf{Q}(w)^* f(w)).$$

Hence

$$\begin{aligned} &\|f\|^2_{\mathbb{C}^p \otimes \mathscr{H}_{\mathrm{L}}} - (1+\varepsilon)^2 \|\mathbf{Q}(T)^* f\|^2_{\mathbb{C}^q \otimes \mathscr{H}_{\mathrm{L}}} \\ &\quad = \Re L(f(z)^* (I_{\mathbb{C}^p} - (1+\varepsilon)^2 \mathbf{Q}(z) \mathbf{Q}(w)^*) f(w)). \end{aligned} \tag{3.7}$$

Clearly, any function $\varphi(z,w)$ of the form

$$\varphi(z,w) = f(z)^* (I_{\mathbb{C}^p} - (1+\varepsilon)^2 \mathbf{Q}(z) \mathbf{Q}(w)^*) f(w)$$

is in $\mathscr{C}_\varepsilon$ (simply take $\gamma_j(z) = 0$ in the defining representation for $\mathscr{C}_\varepsilon$). Hence by construction

$$\Re L(f(z)^* (I_{\mathbb{C}^p} - (1+\varepsilon)^2 \mathbf{Q}(z) \mathbf{Q}(w)^*) f(w)) \geqslant 0$$

and we see from (3.7) that

$$||\mathbf{Q}(T_1, \dots, T_n)|| \leqslant \frac{1}{1+\varepsilon} < 1.$$

Since by assumption $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$, we therefore have that $||S(T_1, \dots, T_n)|| \leqslant 1$ where $S(T_1, \dots, T_d) \in \mathcal{L}(\mathcal{H}_{\mathrm{L}} \otimes \mathcal{E}, \mathcal{H}_{\mathrm{L}} \otimes \mathcal{E}_*)$.

If we are in the scalar case ($\mathcal{E} = \mathcal{E}_* = \mathbb{C}$), we can now finish the proof quite simply. From the fact that T_j^* is given by multiplication by $\bar{w}_j$ on $\mathcal{X} \subset \mathcal{H}_{\mathrm{L}}$, we see that necessarily $S(T_1, \dots, T_d)^*$ is given by

$$S(T_1, \dots, T_n)^* : f(w) \mapsto S(w)^* f(w) \quad \text{for } f \in \mathcal{X} \subset \mathcal{H}_{\mathrm{L}}.$$

For the particular case where $f \in \mathcal{X}$ is the constant function $f(w) = 1$, we compute

$$\begin{aligned} 0 &\leqslant ||f||^2_{\mathcal{H}_{\mathrm{L}}} - ||S(T_1, \dots, T_d) f||^2_{\mathcal{H}_{\mathrm{L}}} \\ &= \Re L(f(z)f(w)^* - f(z)S(z)S(w)^* f(w)) \\ &= \Re L(1 - S(z)S(w)^*) \\ &= \Re L(\varphi_F(z,w)) \quad (\text{since } S \text{ extends } F). \end{aligned}$$

We have thus shown that $\Re L(\varphi_F(z,w)) \geqslant 0$ for any $L : \mathcal{X} \mapsto \mathbb{C}$ with $\Re L|_{\mathscr{C}_\varepsilon} \geqslant 0$ as desired.

For the general case we use a somewhat more indirect argument. For $\Phi \in \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$ a collection of nonnegative integers, the tensor product operator $T^{*k} \otimes \Phi^*$ acts on an element $f(w) \otimes e_*$ of $\mathcal{H}_{\mathrm{L}} \otimes \mathcal{E}_*$. We assume that the function f has a constant value $f(w) = \ell$ for an element ℓ of $\mathcal{L}(\mathcal{E}_*, \mathbb{C})$. We compute the $(\mathcal{H}_{\mathrm{L}} \otimes \mathcal{E})$-inner product of $(T^{*k} \otimes \Phi^*)(\ell \otimes e_*)$ against another such object $(T^{*k'} \otimes \Phi'^*)(\ell' \otimes e'_*)$ as follows:

$$\begin{aligned} &\langle (T^{*k} \otimes \Phi^*)(\ell \otimes e_*), (T^{*k'} \otimes \Phi'^*)(\ell' \otimes e'_*) \rangle_{\mathcal{H}_{\mathrm{L}} \otimes \mathcal{E}} \\ &= \langle \bar{w}^k \ell \otimes \Phi^* e_*, \bar{w}^{k'} \ell' \otimes \Phi'^* e'_* \rangle_{\mathcal{H}_{\mathrm{L}} \otimes \mathcal{E}} \\ &= \langle \bar{w}^k \ell, \bar{w}^{k'} \ell' \rangle_{\mathcal{H}_{\mathrm{L}}} \cdot \langle \Phi^* e_*, \Phi'^* e'_* \rangle_{\mathcal{E}} \\ &= L_1(z^{k'} \bar{w}^k \ell'^* \ell) \cdot \langle \Phi' \Phi^* e_*, e'_* \rangle_{\mathcal{E}_*} \\ &= L_1(z^{k'} \bar{w}^k \ell'^* \cdot \langle \Phi' \Phi^* e_*, e'_* \rangle_{\mathcal{E}_*} \cdot \ell) \\ &= L_1(\ell'^* (e'_*)^* (\Phi' z^k)(\Phi^* \bar{w}^k) e_* \ell). \end{aligned} \tag{3.8}$$

Here we view the vector $e'_* \in \mathcal{E}_*$ as the operator $e'_* : \alpha \mapsto \alpha e'_*$ from $\mathbb{C}$ to $\mathcal{E}_*$ with adjoint operator $(e'_*)^* : \mathcal{E}_* \mapsto \mathbb{C}$ given by $(e'_*)^* : e''_* \mapsto \langle e''_*, e'_* \rangle_{\mathcal{E}_*} \in \mathbb{C}$. In this way, the inner product $\langle \Phi' \Phi^* e_*, e'_* \rangle_{\mathscr{E}_*}$, when viewed as an operator on $\mathbb{C}$, can be written as the operator composition

$$\langle \Phi' \Phi^* e_*, e'_* \rangle_{\mathscr{E}_*} = (e'_*)^* \Phi' \Phi^* e_* : \mathbb{C} \mapsto \mathbb{C}.$$

By linearity we see that we can generalize (3.8) to

$$\langle \mathbb{G}_1(T)^*(\ell \otimes e_*), \mathbb{G}_2(T)^*(\ell' \otimes e'_*) \rangle_{\mathscr{H}_{\mathrm{L}} \otimes \mathscr{U}} = L_1(\ell'^*(e'_*)^* \mathbb{G}_2(z) \mathbb{G}_1(w)^* e_* \ell) \tag{3.9}$$

for any polynomials $\mathbb{G}_1(z)$ and $\mathbb{G}_2(z)$ with coefficients in $\mathscr{L}(\mathscr{E}, \mathscr{E}_*)$. It is easily seen that (3.9) continues to hold if $\mathbb{G}_j(z) = \frac{1}{q_j(z)} \widetilde{\mathbb{G}}_j(z)$ for $j = 1, 2$, where q_j is a rational scalar function in $z = (z_1, \dots, z_n)$ with no zeros in $\mathscr{D}_{\mathbf{Q}}$ and $\widetilde{\mathbb{G}}_j(z)$ is a polynomial in z with coefficients in $\mathscr{L}(\mathscr{E}, \mathscr{E}_*)$. By the continuity properties of the Taylor functional calculus (see e.g. Theorem 5.20 in [24]) and the weak-$*$ continuity of L, it follows that (3.9) continues to hold for $\mathbb{G}_1$ and $\mathbb{G}_2$ equal to any $\mathscr{L}(\mathscr{E}, \mathscr{E}_*)$-valued functions holomorphic on $\mathscr{D}_{\mathbf{Q}}$.

We now apply (3.9) to the case where $\mathbb{G}_1 = \mathbb{G}_2 = S \in \mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*)$ and $\ell^* = e_* = e_{*j}$, $(\ell')^* = e_*{}' = e_{*i}$ where $e_{*1}, e_{*2}, \dots, e_{*N}, \dots$ is an orthonormal basis for $\mathscr{E}_*$, to get

$$\langle S(T)^*(e^*_{*j} \otimes e_{*j}), S(T)^*(e^*_{*i} \otimes e_{*i}) \rangle_{\mathscr{H}_{\mathrm{L}} \otimes \mathscr{E}} = L_1(e_{*i} e^*_{*i} S(z) S(w)^* e_{*j} e^*_{*j}).$$

Summing over $i, j = 1, \dots, N$ then gives

$$\begin{aligned} \left\| S(T)^* \left(\sum_{j=1}^N e^*_{*j} \otimes e_{*j} \right) \right\|^2_{\mathscr{H}_{\mathrm{L}} \otimes \mathscr{E}} &= \sum_{i,j=1}^N L_1(e_{*i} e^*_{*i} S(z) S(w)^* e_{*j} e^*_{*j}) \\ &= \Re L(P_N S(z) S(w)^* P_N), \end{aligned} \tag{3.10}$$

where $P_N \in \mathscr{L}(\mathscr{E}_*)$ is the orthogonal projection onto the span of $\{e_{*1}, \dots, e_{*N}\}$.

Moreover, we compute

$$\langle e^*_{*j} \otimes e_{*j}, e^*_{*i} \otimes e_{*i} \rangle_{\mathscr{H}_{\mathrm{L}} \otimes \mathscr{E}_*} = \langle e^*_{*j}, e^*_{*i} \rangle_{\mathscr{H}_{\mathrm{L}}} \cdot \langle e_{*j}, e_{*i} \rangle_{\mathscr{E}_*} = \delta_{i,j} L_1(e_{*i} e^*_{*j}).$$

Summing this over $i, j = 1, \dots, N$ gives

$$\left\| \sum_{j=1}^N e^*_{*j} \otimes e_{*j} \right\|^2_{\mathscr{H}_{\mathrm{L}} \otimes \mathscr{E}_*} = \sum_{j=1}^N L_1(e_{*j} e^*_{*j}) = \Re L(P_N). \tag{3.11}$$

Using that $S \in \mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*)$ and combining (3.10) and (3.11) now gives

$$\begin{aligned} 0 &\leqslant \left\| \sum_{j=1}^N e^*_{*j} \otimes e_{*j} \right\|^2_{\mathscr{H}_{\mathrm{L}} \otimes \mathscr{E}_*} - \left\| S(T)^* \left(\sum_{j=1}^N e^*_{*j} \otimes e_{*j} \right) \right\|^2_{\mathscr{H}_{\mathrm{L}} \otimes \mathscr{E}} \\ &= \Re L(P_N (I - S(z) S(w)^*) P_N). \end{aligned} \tag{3.12}$$

By the pointwise weak-$*$ continuity of L, upon letting $N \to \infty$ in (3.12) we see that

$$\Re L(\varphi_F(z,w)) = \Re L(I - S(z)S(w)^*) \geqslant 0.$$

as desired.

It remains only to remove the assumption that Ω is finite. This is done by considering the net of all finite subsets ω of Ω. For each of these finite subsets ω we have a representation of type (3.1) holding on ω with associated coefficients H_ω depending on ω. Without loss of generality we may assume that the auxiliary Hilbert space $\mathscr{H}'$ is independent of ω: $H_\omega(z) \in \mathscr{L}(\mathscr{H}', \mathscr{E}_*)$. From Lemma 3.1 we see that, for each fixed z, $||H_\omega(z)||$ is bounded independently of the finite set ω for which $z \in \omega$. Then it follows that the associated positive kernel $\mathbb{K}_\omega(z,w) = H_\omega(z)H_\omega(w)^*$ is bounded independently of the choice of finite subset ω containing z and w, where the positive kernel $\mathbb{K}_\omega(z,w)$ gives a representation of φ of the form (3.2). A compactness argument can then be used to arrive at a pointwise weak-$*$ limit point $\mathbb{K}(z,w)$ for all $\mathbb{K}_\omega(z,w)$. Since property (1.5) is preserved under such pointwise limits, we see that $\mathbb{K}(z,w)$ is again a positive kernel. Moreover, we see that the limiting process leads to a representation for $\varphi(z,w)$ of the form (3.2) on all of Ω, and hence $\varphi \in \mathscr{C}$ as wanted. This completes the proof of (1) $\Rightarrow$ (2) in Theorem 1.5. $\square$

Proof of (2) $\Rightarrow$ (5) **in Theorem 1.5.** Assume that $F : \Omega \mapsto \mathscr{L}(\mathscr{E}, \mathscr{E}_*)$ is given such that representation (1.16) holds for all $z, w \in \Omega$ for some $H : \Omega \mapsto \mathscr{L}(\mathscr{H}' \otimes \mathbb{C}^p, \mathscr{E})$. We rewrite (1.16) in the form

$$H(z)\mathbf{Q}(z)\mathbf{Q}(w)^*H(w)^* + I_{\mathscr{E}_*} = H(z)H(w)^* + F(z)F(w)^*.$$

If we set

$$\mathscr{D} = \bigvee\left\{ \begin{bmatrix} \mathbf{Q}(w)^*H(w)^* \\ I_{\mathscr{E}_*} \end{bmatrix} e_* : w \in \Omega,\ e_* \in \mathscr{E}_* \right\},$$
$$\mathscr{R} = \bigvee\left\{ \begin{bmatrix} H(w)^* \\ F(w)^* \end{bmatrix} e_* : w \in \Omega,\ e_* \in \mathscr{E}_* \right\}$$

(where $\bigvee$ denotes “closed linear span”), we see that the formula

$$\mathbf{V} : \begin{bmatrix} \mathbf{Q}(w)^*H(w)^* \\ I_{\mathscr{E}_*} \end{bmatrix} e_* \mapsto \begin{bmatrix} H(w)^* \\ F(w)^* \end{bmatrix} e_*$$

extends by linearity to define an isometry from $\mathscr{D}$ onto $\mathscr{R}$. Extend $\mathbf{V}$ to a unitary operator

$$\mathbf{U}^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} : \begin{bmatrix} \mathbb{C}^p \otimes \mathscr{H} \\ \mathscr{E}_* \end{bmatrix} \mapsto \begin{bmatrix} \mathbb{C}^q \otimes \mathscr{H} \\ \mathscr{E} \end{bmatrix},$$

where $\mathscr{H}$ is a Hilbert space containing $\mathscr{H}'$. Since $\mathbf{U}^*$ extends $\mathbf{V}$ we have the operator equation

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} \mathbf{Q}(w)^* H(w)^* \\ I_{\mathscr{E}_*} \end{bmatrix} = \begin{bmatrix} H(w)^* \\ F(w)^* \end{bmatrix}. \tag{3.13}$$

Since $\|\mathbf{Q}(w)\| < 1$ for $w \in \Omega$ and $\|A^*\| \leqslant 1$ as $\mathbf{U}^*$ is unitary, we can use the equation from the first block row of (3.13) to solve for $H(w)^*$:

$$H(w)^* = (I - A^*\mathbf{Q}(w)^*)^{-1} C^*.$$

From the second block row of (3.13) we then get

$$B^*\mathbf{Q}(w)^*(I - A^*\mathbf{Q}(w)^*)^{-1} C^* + D^* = F(w)^*.$$

Take adjoints and replace w by z to arrive at the representation (1.21) for F.
Given that $F(z)$ has the form (1.21) for a unitary $\mathbf{U}$, one computes

$$\begin{aligned} I - F(z)F(w)^* &\\ &= I - [D + C(I - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)B][D^* + B^*\mathbf{Q}(w)^*(I - A^*\mathbf{Q}(w)^*)^{-1}C^*] \\ &= I - DD^* - DB^*\mathbf{Q}(w)^*(I - A^*\mathbf{Q}(w)^*)^{-1}C^* - C(I - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)BD^* \\ &\quad - C(I - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)BB^*\mathbf{Q}(w)^*(I - A^*\mathbf{Q}(w)^*)^{-1}C^*. \end{aligned} \tag{3.14}$$

From the fact that $\mathbf{U}$ is unitary we have the relations

$$I - DD^* = CC^*, \quad -DB^* = CA^*, \quad -BD^* = AC^*, \quad -BB^* = -I + AA^*.$$

Plugging this into (3.14) leaves us with

$$\begin{aligned} I - F(z)F(w)^* &\\ &= CC^* + CA^*\mathbf{Q}(w)^*(I - A^*\mathbf{Q}(w)^*)^{-1}C^* + C(I - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)AC^* \\ &\quad + C(I - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)(-I + AA^*)\mathbf{Q}(w)^*(I - A^*\mathbf{Q}(w)^*)^{-1}C^* \\ &= C(I - \mathbf{Q}(z)A)^{-1}[(I - \mathbf{Q}(z)A)(I - A^*\mathbf{Q}(w)^*) + (I - \mathbf{Q}(z)A)A^*\mathbf{Q}(w)^* \\ &\quad + \mathbf{Q}(z)A(I - A^*\mathbf{Q}(w)^*) - \mathbf{Q}(z)\mathbf{Q}(w)^* + \mathbf{Q}(z)AA^*\mathbf{Q}(w)^*] \\ &\quad \cdot (I - A^*\mathbf{Q}(w)^*)^{-1}C^* \\ &= C(I - \mathbf{Q}(z)A)^{-1}(I - \mathbf{Q}(z)\mathbf{Q}(w)^*)(I - A^*\mathbf{Q}(w)^*)^{-1}C^* \end{aligned} \tag{3.15}$$

and representation (1.16) holds with $H(z) = C(I - \mathbf{Q}(z)A)^{-1}$. A similar argument can be used to show that the representation (1.18) holds with $G(z) = (I - A\mathbf{Q}(z))^{-1}B$ and that (1.19) holds with this choice of $H(z)$ and $G(z)$. Thus we have also shown that (5) $\Rightarrow$ (2), (3) and (4) in Theorem 1.5. $\square$

Proof of (5) $\Rightarrow$ (1) **in Theorem 1.5.** Assume that $F : \Omega \mapsto \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ has realization (1.21) for all $z \in \Omega$ for some unitary U. As $\|\mathbf{Q}(z)\| < 1$ for $z \in \Omega$ and $\|A\| \leqslant 1$, the inverse $(I - \mathbf{Q}(z)A)^{-1}$ is well defined as the sum of the geometric series

$$(I - \mathbf{Q}(z)A)^{-1} = \sum_{k=0}^{\infty} (\mathbf{Q}(z)A)^k$$

and hence F has a natural extension to all of $\mathcal{D}_{\mathbf{Q}}$ given by

$$F(z) = D + C(I - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)B \quad (z \in \mathcal{D}_{\mathbf{Q}}).$$

Similarly, if $T = (T_1, \dots, T_n)$ is an n-tuple of commuting operators on a Hilbert space $\mathcal{K}$ for which $\mathbf{Q}(T) \in \mathcal{L}(\mathbb{C}^q \otimes \mathcal{K}, \mathbb{C}^p \otimes \mathcal{K})$ has $\|\mathbf{Q}(T)\| < 1$, by the same argument, the operator $S(T) \in \mathcal{L}(\mathcal{K} \otimes \mathcal{E}, \mathcal{K} \otimes \mathcal{E}_*)$ given by

$$\begin{aligned} S(T) = (I_{\mathcal{K}} \otimes D) + (I_{\mathcal{K}} \otimes C)(I_{\mathbb{C}^p \otimes \mathcal{K} \otimes \mathcal{H}} - (\mathbf{Q}(T) \otimes I_{\mathcal{H}})(I_{\mathcal{K}} \otimes A))^{-1} \\ \times (\mathbf{Q}(T) \otimes I_{\mathcal{H}})(I_{\mathcal{K}} \otimes B) \end{aligned}$$

is well defined. Moreover we see that $S(T)$ has the form

$$S(T) = D' + C'(I - XA')^{-1}XB',$$

where $X = \mathbf{Q}(T) \otimes I_{\mathcal{H}}$ is a strict contraction and where

$$U' = \begin{bmatrix} I_{\mathcal{K}} \otimes A & I_{\mathcal{K}} \otimes B \\ I_{\mathcal{K}} \otimes C & I_{\mathcal{K}} \otimes D \end{bmatrix} : \begin{bmatrix} \mathbb{C}^q \otimes \mathcal{K} \otimes \mathcal{H} \\ \mathcal{K} \otimes \mathcal{E} \end{bmatrix} \mapsto \begin{bmatrix} \mathbb{C}^p \otimes \mathcal{K} \otimes \mathcal{H} \\ \mathcal{K} \otimes \mathcal{E}_* \end{bmatrix}$$

is unitary. As has been pointed out and used in [8] as well as many other places, computation (3.14)–(3.15) is more general than noted there; what is actually shown is: *If* $U = \left[\begin{smallmatrix} A & B \\ C & C \end{smallmatrix}\right] : \mathcal{H} \oplus \mathcal{E} \mapsto \mathcal{H}' \oplus \mathcal{E}_*$ *is unitary and* $X \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ *is a strict contraction, then the operator*

$$Y := D + C(I_{\mathcal{H}} - XA)^{-1}XB \in \mathcal{L}(\mathcal{E}, \mathcal{E}_*) \tag{3.16}$$

is well defined and satisfies

$$I - YY^* = C(I - XA)^{-1}(I - XX^*)(I - A^*X^*)^{-1}C^* \tag{3.17}$$

and hence, in particular, $\|Y\| \leqslant 1$.

Applying this general principal with $S(T)$ in place of X and U' in place of U, we arrive at the desired result that $\|S(T)\| \leqslant 1$.

This completes the proof of (5) $\Rightarrow$ (1) in Theorem 1.5. □

Completion of the proof of Theorem 1.5. We have now verified (1) $\Rightarrow$ (2) $\Rightarrow$ (5) $\Rightarrow$ (1) and along the way we have observed that (5) $\Rightarrow$ (2)–(4). Trivially (4) $\Rightarrow$ (2), (3) (simply focus on the diagonal block entries in (1.19)). It remains only to show (3) $\Rightarrow$ (5). This can be done by a parallel version of the "lurking isometry" argument used in the proof of (2) $\Rightarrow$ (5) given above.

Finally, assume that $S(z)$ has the form (1.22) for some unitary coupling matrix $\mathbf{U}$ as in (1.20). Then, from the general principle (3.17) with Y of the form (3.16) with X taken to be $X = \mathbf{Q}(z)$, relation (1.23) follows. Relation (1.25) follows similarly (with $\mathbf{U}^*$ in place of $\mathbf{U}$ and with $X = \mathbf{Q}(z)^*$). Relation (1.24) is straightforward algebra:

$$\begin{aligned} S(z) - S(w) &= C(I - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)B - C(I - \mathbf{Q}(w)A)^{-1}\mathbf{Q}(w)B \\ &= C(I - \mathbf{Q}(z)A)^{-1}[\mathbf{Q}(z)(I - A\mathbf{Q}(w)) \\ &\quad - (I - \mathbf{Q}(z)A)\mathbf{Q}(w)](I - A\mathbf{Q}(w))^{-1}B \\ &= C(I - \mathbf{Q}(z)A)^{-1}[\mathbf{Q}(z) - \mathbf{Q}(w)](I - A\mathbf{Q}(w))^{-1}B. \end{aligned}$$

This completes the proof of Theorem 1.5. □

4. The solvability criterion in Theorems 1.10 and 1.16

Solvability criteria for interpolation Problems 1.7–1.9 were given in Theorem 1.10 in terms of positive definite kernels satisfying certain Stein identities. However, these solvability criteria can be formulated in terms similar to those in Theorem 1.5. Moreover, in certain situations these alternative formulations are more convenient to apply.

Theorem 4.1. *Suppose that we are given subsets Ω_{L} and Ω_{R} of $\mathscr{D}_{\mathbf{Q}}$ and data functions $\mathbf{a}, \mathbf{c}, \mathbf{b}, \mathbf{d}$ as in* (1.38) *and* (1.40). *Then*:

(1′) *Problem* 1.7 *has a solution if and only if there exist a Hilbert space $\mathscr{H}$ and an $\mathscr{L}(\mathbb{C}^p \otimes \mathscr{H}, \mathscr{E}_{\mathrm{L}})$-valued function*

$$R(z) = [R_1(z) \quad \cdots \quad R_p(z)] \tag{4.1}$$

defined on Ω_{L} so that

$$R(z)(I_{\mathbb{C}^p \otimes \mathscr{H}} - \mathbf{Q}(z)\mathbf{Q}(w)^*)R(w)^* = \mathbf{a}(z)^*\mathbf{a}(w) - \mathbf{c}(z)^*\mathbf{c}(w) \tag{4.2}$$

for $z, w \in \Omega_{\mathrm{L}}$.

($2'$) *Problem* 1.8 *has a solution if and only if there exist a Hilbert space* $\mathcal{H}$ *and an* $\mathcal{L}(\mathbb{C}^q \otimes \mathcal{H}, \mathcal{E}_{\mathrm{R}})$*-valued function*

$$D(z) = \begin{bmatrix} D_1(z) \\ \vdots \\ D_q(z) \end{bmatrix} \tag{4.3}$$

defined on Ω *so that*

$$D(z)^*(I_{\mathbb{C}^q \otimes \mathcal{H}} - \mathbf{Q}(z)^*\mathbf{Q}(w))D(w) = \mathbf{b}(z)^*\mathbf{b}(w) - \mathbf{d}(z)^*\mathbf{d}(w) \tag{4.4}$$

for $z, w \in \Omega_{\mathrm{R}}$.

($3'$) *Problem* 1.9 *has a solution if and only if there exist a Hilbert space* $\mathcal{H}$ *and functions* $R(z)$ *and* $D(z)$ *as in* (4.1) *and* (4.3), *so that*

$$\begin{aligned} &\begin{bmatrix} R(\xi_{\mathrm{L}}) \\ D(\xi_{\mathrm{R}})^*\mathbf{Q}(\xi_{\mathrm{R}})^* \end{bmatrix} \begin{bmatrix} R(\mu_{\mathrm{L}})^* & \mathbf{Q}(\mu_{\mathrm{R}})D(\mu_{\mathrm{R}}) \end{bmatrix} \\ &\quad - \begin{bmatrix} R(\xi_{\mathrm{L}})\mathbf{Q}(\xi_{\mathrm{L}}) \\ D(\xi_{\mathrm{R}})^* \end{bmatrix} \begin{bmatrix} \mathbf{Q}(\mu_{\mathrm{L}})^*R(\mu_{\mathrm{L}})^* & D(\mu_{\mathrm{R}}) \end{bmatrix} \\ &\quad = \begin{bmatrix} \mathbf{a}(\xi_{\mathrm{L}})^* \\ \mathbf{d}(\xi_{\mathrm{R}})^* \end{bmatrix} \begin{bmatrix} \mathbf{a}(\mu_{\mathrm{L}}) & \mathbf{d}(\mu_{\mathrm{R}}) \end{bmatrix} - \begin{bmatrix} \mathbf{c}(\xi_{\mathrm{L}})^* \\ \mathbf{b}(\xi_{\mathrm{R}})^* \end{bmatrix} \begin{bmatrix} \mathbf{c}(\mu_{\mathrm{L}}) & \mathbf{b}(\mu_{\mathrm{R}}) \end{bmatrix} \end{aligned} \tag{4.5}$$

for all $\xi_{\mathrm{L}}, \mu_{\mathrm{L}} \in \Omega_{\mathrm{L}}$ *and* $\xi_{\mathrm{R}}, \mu_{\mathrm{R}} \in \Omega_{\mathrm{R}}$.

Proof. The equivalence of the various statements in Theorem 4.1 to the respective corresponding statements in Theorem 1.10 is based on a simple observation that positive kernels K_{L}, K_{R} and K of the special form (1.53) with factorizations

$$\begin{aligned} K_{\mathrm{L}}(\xi, \mu) &= \begin{bmatrix} R_1(\xi) \\ \vdots \\ R_p(\xi) \end{bmatrix} \begin{bmatrix} R_1(\mu)^* & \cdots & R_p(\mu)^* \end{bmatrix}, \\ K_{\mathrm{R}}(\xi, \mu) &= \begin{bmatrix} D_1(\xi)^* \\ \vdots \\ D_q(\xi)^* \end{bmatrix} \begin{bmatrix} D_1(\mu) & \cdots & D_q(\mu) \end{bmatrix} \end{aligned} \tag{4.6}$$

and

$$K(\xi_{\mathrm{L}}, \xi_{\mathrm{R}}, \mu_{\mathrm{L}}, \mu_{\mathrm{R}}) = T(\xi_{\mathrm{L}}, \xi_{\mathrm{R}})^* T(\mu_{\mathrm{L}}, \mu_{\mathrm{R}}),$$

where

$$T(\mu_{\mathrm{L}}, \mu_{\mathrm{R}}) = \begin{bmatrix} R_1(\mu_{\mathrm{L}})^* & \cdots & R_p(\mu_{\mathrm{L}})^* & D_1(\mu_{\mathrm{R}}) & \cdots & D_q(\mu_{\mathrm{R}}) \end{bmatrix}$$

satisfy Stein equations (1.50) (respectively, (1.51) and (1.52)) if and only if the $R(z)$ and $D(z)$ constructed from these representation via formulas (4.1) and (4.3) are subject to equalities (4.2) (respectively, (4.4) and (4.5)). $\square$

Now we make explicit how the various parts of Theorems 1.10 and 1.16 can be reduced to part (3) of Theorem 1.16 and then we will establish the solution criterion for Problem 1.15 given in part (3) of Theorem 1.16.

As was observed in the Introduction, the left interpolation problem (Problem 1.7) is the special case of the bitangential interpolation problem (Problem 1.9) where $\Omega_{\mathrm{R}} = \emptyset$ and the right interpolation problem (Problem 1.8) is the special case of the bitangential interpolation problem (Problem 1.9) where $\Omega_{\mathrm{L}} = \emptyset$. From these observations we see immediately that parts (1) and (2) of Theorem 1.10 are specializations of part (3) of Theorem 1.10. Similarly, parts (1) and (2) of Theorem 1.16 are specializations of part (3) of Theorem 1.16 corresponding to the respective cases $\Omega_{\mathrm{R}} = \emptyset$ and $\Omega_{\mathrm{L}} = \emptyset$. Moreover, it is not difficult to see that part (3) of Theorem 1.10 is an immediate consequence of part (3) of Theorem 1.16. Indeed, given an interpolation data set **a**, **c**, **b** and **d** for Problem 1.9 as in (1.38) and (1.40) and a function $S \in \mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E}, \mathscr{E}_*)$, use (1.59)–(1.61) to define additional data functions $\Psi_{j\ell}(\xi,\mu)$, $\Lambda_{j\ell}(\xi,\mu)$ and $\Phi_{j\ell}(\xi,\mu)$ and thereby generate a data set for Problem 1.15. Then trivially, S solves Problem 1.9 if and only if S solves Problem 1.15. The solution criterion for Problem 1.15 in part (3) of Theorem 1.16, with the extraneous interpolation data $\Psi_{j\ell}(\xi,\mu)$, $\Lambda_{j\ell}(\xi,\mu)$ and $\Phi_{j\ell}(\xi,\mu)$ ignored, then gives the solution criterion for solvability of Problem 1.9 in part (3) of Theorem 1.10. We conclude: *to prove all parts of Theorems* 1.10 *and* 1.16, *we need only to prove part* (3) *of Theorem* 1.16.

Proof of Theorem 1.16. (3) We start with the necessity part. Let S be a solution of Problem 1.15, that is let relations (1.38), (1.40) and (1.59)–(1.61) be in force, where $H_1(z), \dots, H_p(z)$, $G_1(z), \dots, G_q(z)$ are functions arising in representations (1.16) and (1.18) (with S in place of F) associated with S. By Theorem 1.5 and Remark 1.6, we know that (1.49) holds (with S in place of F) with the kernel $\mathbb{K}$ factored as in (1.36), i.e., with

$$\mathbb{K}(\xi_{\mathrm{L}},\xi_{\mathrm{R}},\mu_{\mathrm{L}},\mu_{\mathrm{R}}) = \mathbb{T}(\xi_{\mathrm{L}},\xi_{\mathrm{R}})^*\mathbb{T}(\mu_{\mathrm{L}},\mu_{\mathrm{R}}), \tag{4.7}$$

where, according to (1.37),

$$\mathbb{T}(\mu_{\mathrm{L}},\mu_{\mathrm{R}}) = [H_1(\mu_{\mathrm{L}})^* \quad \cdots \quad H_p(\mu_{\mathrm{L}})^* \quad G_1(\mu_{\mathrm{R}}) \quad \cdots \quad G_q(\mu_{\mathrm{R}})].$$

Let us now define a kernel

$$K(\xi_{\mathrm{L}},\xi_{\mathrm{R}},\mu_{\mathrm{L}},\mu_{\mathrm{R}}) = \begin{bmatrix} K_{\mathrm{L}}(\xi_{\mathrm{L}},\mu_{\mathrm{L}}) & K_{\mathrm{LR}}(\xi_{\mathrm{L}},\mu_{\mathrm{R}}) \\ K_{\mathrm{LR}}(\mu_{\mathrm{L}},\xi_{\mathrm{R}})^* & K_{\mathrm{R}}(\xi_{\mathrm{R}},\mu_{\mathrm{R}}) \end{bmatrix}$$

on $(\Omega_{\rm L}\times\Omega_{\rm R})\times(\Omega_{\rm L}\times\Omega_{\rm R})$ by

$$K(\xi_{\rm L},\xi_{\rm R},\mu_{\rm L},\mu_{\rm R})=\begin{bmatrix}\mathbf{a}(\xi_{\rm L})^* & 0\\ 0 & \mathbf{b}(\xi_{\rm R})^*\end{bmatrix}\mathbb{K}(\xi_{\rm L},\xi_{\rm R},\mu_{\rm L},\mu_{\rm R})\begin{bmatrix}\mathbf{a}(\mu_{\rm L}) & 0\\ 0 & \mathbf{b}(\mu_{\rm R})\end{bmatrix},\tag{4.8}$$

or equivalently,

$$K(\xi_{\rm L},\xi_{\rm R},\mu_{\rm L},\mu_{\rm R})=T(\xi_{\rm L},\xi_{\rm R})^*T(\mu_{\rm L},\mu_{\rm R}),\tag{4.9}$$

where we have set

$$\begin{aligned}&T(\mu_{\rm L},\mu_{\rm R})\\ &\quad=[H_1(\mu_{\rm L})^*\mathbf{a}(\mu_{\rm L})\quad\cdots\quad H_p(\mu_{\rm L})^*\mathbf{a}(\mu_{\rm L})\quad G_1(\mu_{\rm R})\mathbf{b}(\mu_{\rm R})\quad\cdots\quad G_q(\mu_{\rm R})\mathbf{b}(\mu_{\rm R})].\end{aligned}\tag{4.10}$$

From the factored forms (4.8) and (4.7) of K and $\mathbb{K}$ we see that K is a positive. We also read off from (4.8) together with (4.10) that $K(\xi_{\rm L},\xi_{\rm R},\mu_{\rm L},\mu_{\rm R})$ is alternatively given in terms of the interpolation data $\Psi_{j\ell}$, $\Lambda_{j\ell}$ and $\Phi_{j\ell}$ as in (1.55)–(1.57). Finally, by restricting the Stein equation (1.49) (with S in place of F) to $\xi_{\rm L},\mu_{\rm L}\in\Omega_{\rm L}$ and $\xi_{\rm R},\mu_{\rm R}\in\Omega_{\rm R}$ and multiplying the result on the left by $\begin{bmatrix}\mathbf{a}(\xi_{\rm L})^* & 0\\ 0 & \mathbf{b}(\xi_{\rm R})^*\end{bmatrix}$ and on the right by $\begin{bmatrix}\mathbf{a}(\mu_{\rm L}) & 0\\ 0 & \mathbf{b}(\mu_{\rm R})\end{bmatrix}$, we see that $K(\xi_{\rm L},\xi_{\rm R},\mu_{\rm L},\mu_{\rm R})$ satisfies the Stein equation (1.52). In this way we see the necessity of the solvability criterion in Theorem 1.10.

To prove the sufficiency part, we assume that the kernel K of the form (1.53) with the block entries expressed in terms of interpolation data as in (1.55)–(1.57), is positive on $(\Omega_{\rm L}\times\Omega_{\rm R})\times(\Omega_{\rm L}\times\Omega_{\rm R})$ and satisfies the Stein equation (1.52). We fix a factorization

$$K(\xi_{\rm L},\xi_{\rm R},\mu_{\rm L},\mu_{\rm R})=T(\xi_{\rm L},\xi_{\rm R})^*T(\mu_{\rm L},\mu_{\rm R})\tag{4.11}$$

of K with an operator valued function T decomposed conformally with (4.10)

$$T(\mu_{\rm L},\mu_{\rm R})=[R_1(\mu_{\rm L})^*\quad\cdots\quad R_p(\mu_{\rm L})^*\quad D_1(\mu_{\rm R})\quad\cdots\quad D_q(\mu_{\rm R})],\tag{4.12}$$

where

$$R_1,\ldots,R_p:\Omega_{\rm L}\mapsto\mathcal{L}(\mathcal{H}',\mathcal{E}_{\rm L}),\quad D_1,\ldots,D_q:\Omega_{\rm R}\mapsto\mathcal{L}(\mathcal{E}_{\rm R},\mathcal{H}')$$

and $\mathcal{H}'$ is an auxiliary Hilbert space, and define functions R and D via formulas (4.1) and (4.3). As it was explained in the proof of Theorem 4.1, these functions satisfy identity (4.5), which can be written equivalently as

$$\begin{bmatrix} R(\xi_{\rm L}) & \mathbf{c}(\xi_{\rm L})^* \\ D(\xi_{\rm R})^*\mathbf{Q}(\xi_{\rm R})^* & \mathbf{b}(\xi_{\rm R})^* \end{bmatrix}\begin{bmatrix} R(\mu_{\rm L})^* & \mathbf{Q}(\mu_{\rm R})D(\mu_{\rm R}) \\ \mathbf{c}(\mu_{\rm L}) & \mathbf{b}(\mu_{\rm R}) \end{bmatrix}$$

$$= \begin{bmatrix} R(\xi_{\rm L})\mathbf{Q}(\xi_{\rm L}) & \mathbf{a}(\xi_{\rm L})^* \\ D(\xi_{\rm R})^* & \mathbf{d}(\xi_{\rm R})^* \end{bmatrix}\begin{bmatrix} \mathbf{Q}(\mu_{\rm L})^*R(\mu_{\rm L})^* & D(\mu_{\rm R}) \\ \mathbf{a}(\mu_{\rm L}) & \mathbf{d}(\mu_{\rm R}) \end{bmatrix}. \tag{4.13}$$

If we set

$$\mathscr{D} = \bigvee\left\{\begin{bmatrix} R(\mu_{\rm L})^* & \mathbf{Q}(\mu_{\rm R})D(\mu_{\rm R}) \\ \mathbf{c}(\mu_{\rm L}) & \mathbf{b}(\mu_{\rm R}) \end{bmatrix}\begin{bmatrix} e_{\rm L} \\ e_{\rm R} \end{bmatrix},\ \mu_{\rm L}\in\Omega_{\rm L},\ \mu_{\rm R}\in\Omega_{\rm R},\ e_{\rm L}\in\mathscr{E}_{\rm L},\ e_{\rm R}\in\mathscr{E}_{\rm R}\right\},$$

$$\mathscr{R} = \bigvee\left\{\begin{bmatrix} \mathbf{Q}(\mu_{\rm L})^*R(\mu_{\rm L})^* & D(\mu_{\rm R}) \\ \mathbf{a}(\mu_{\rm L}) & \mathbf{d}(\mu_{\rm R}) \end{bmatrix}\begin{bmatrix} e_{\rm L} \\ e_{\rm R} \end{bmatrix},\ \mu_{\rm L}\in\Omega_{\rm L}, \mu_{\rm R}\in\Omega_{\rm R},\ e_{\rm L}\in\mathscr{E}_{\rm L},\ e_{\rm R}\in\mathscr{E}_{\rm R}\right\},$$

we conclude from (4.13) that the formula

$$\mathbf{V}: \begin{bmatrix} R(\mu_{\rm L})^* & \mathbf{Q}(\mu_{\rm R})D(\mu_{\rm R}) \\ \mathbf{c}(\mu_{\rm L}) & \mathbf{b}(\mu_{\rm R}) \end{bmatrix}\begin{bmatrix} e_{\rm L} \\ e_{\rm R} \end{bmatrix} \mapsto \begin{bmatrix} \mathbf{Q}(\mu_{\rm L})^*R(\mu_{\rm L})^* & D(\mu_{\rm R}) \\ \mathbf{a}(\mu_{\rm L}) & \mathbf{d}(\mu_{\rm R}) \end{bmatrix}\begin{bmatrix} e_{\rm L} \\ e_{\rm R} \end{bmatrix}$$

extends by linearity to define an isometry from $\mathscr{D}$ onto $\mathscr{R}$. Extend $\mathbf{V}$ to a unitary operator

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C}^p \otimes \mathscr{H} \\ \mathscr{E} \end{bmatrix} \mapsto \begin{bmatrix} \mathbb{C}^q \otimes \mathscr{H} \\ \mathscr{E}_* \end{bmatrix},$$

where $\mathscr{H}$ is a Hilbert space containing $\mathscr{H}'$. We will show that the characteristic function of the unitary $\mathbf{Q}$-colligation $\mathbf{U}$

$$S(z) = D + C(I_{\mathbb{C}^p\otimes\mathscr{H}} - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)B \tag{4.14}$$

is a solution of Problem 1.15. By Theorem 1.5, S belongs to the class $\mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathscr{E},\mathscr{E}_*)$ and thus, it remains to show that S satisfies interpolation conditions (1.39), (1.41) and (1.59)–(1.61).

To this end, we note that since $\mathbf{U}$ extends $\mathbf{V}$, we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} R(\mu_{\rm L})^* & \mathbf{Q}(\mu_{\rm R})D(\mu_{\rm R}) \\ \mathbf{c}(\mu_{\rm L}) & \mathbf{b}(\mu_{\rm R}) \end{bmatrix} = \begin{bmatrix} \mathbf{Q}(\mu_{\rm L})^*R(\mu_{\rm L})^* & D(\mu_{\rm R}) \\ \mathbf{a}(\mu_{\rm L}) & \mathbf{d}(\mu_{\rm R}) \end{bmatrix}$$

and since $\mathbf{U}$ is unitary, we have also

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\begin{bmatrix} \mathbf{Q}(\mu_{\rm L})^*R(\mu_{\rm L})^* & D(\mu_{\rm R}) \\ \mathbf{a}(\mu_{\rm L}) & \mathbf{d}(\mu_{\rm R}) \end{bmatrix} = \begin{bmatrix} R(\mu_{\rm L})^* & \mathbf{Q}(\mu_{\rm R})D(\mu_{\rm R}) \\ \mathbf{c}(\mu_{\rm L}) & \mathbf{b}(\mu_{\rm R}) \end{bmatrix}.$$

Thus,

$$A^*\mathbf{Q}(\mu_{\mathrm{L}})^*R(\mu_{\mathrm{L}})^* + C^*\mathbf{a}(\mu_{\mathrm{L}}) = R(\mu_{\mathrm{L}})^*, \tag{4.15}$$

$$B^*\mathbf{Q}(\mu_{\mathrm{L}})^*R(\mu_{\mathrm{L}})^* + D^*\mathbf{a}(\mu_{\mathrm{L}}) = \mathbf{c}(\mu_{\mathrm{L}}), \tag{4.16}$$

$$A\mathbf{Q}(\mu_{\mathrm{R}})D(\mu_{\mathrm{R}}) + B\mathbf{b}(\mu_{\mathrm{L}}) = D(\mu_{\mathrm{R}}), \tag{4.17}$$

$$C\mathbf{Q}(\mu_{\mathrm{R}})D(\mu_{\mathrm{R}}) + D\mathbf{b}(\mu_{\mathrm{R}}) = \mathbf{d}(\mu_{\mathrm{R}}). \tag{4.18}$$

It follows from (4.15) that

$$R(\mu_{\mathrm{L}})^* = (I_{\mathbb{C}^p\otimes\mathscr{H}} - A^*\mathbf{Q}(\mu_{\mathrm{L}})^*)^{-1}C^*\mathbf{a}(\mu_{\mathrm{L}}) \tag{4.19}$$

which, when substituted into (4.16), gives

$$D^*\mathbf{a}(\mu_{\mathrm{L}}) + B^*\mathbf{Q}(\mu_{\mathrm{L}})^*(I_{\mathbb{C}^p\otimes\mathscr{H}} - A^*\mathbf{Q}(\mu_{\mathrm{L}})^*)^{-1}C^*\mathbf{a}(\mu_{\mathrm{L}}) = \mathbf{c}(\mu_{\mathrm{L}}).$$

Since μ_{L} is an arbitrary point in Ω_{L}, the latter equality coincides with (1.39), due to (4.14). Furthermore, it follows from (4.17) that

$$D(\mu_{\mathrm{R}}) = (I_{\mathbb{C}^q\otimes\mathscr{H}} - A\mathbf{Q}(\mu_{\mathrm{R}}))^{-1}B\mathbf{b}(\mu_{\mathrm{R}}), \tag{4.20}$$

which being substituted into (4.18), leads to

$$D\mathbf{b}(\mu_{\mathrm{R}}) + C\mathbf{Q}(\mu_{\mathrm{R}})(I_{\mathbb{C}^q\otimes\mathscr{H}} - A\mathbf{Q}(\xi_{\mathrm{R}}))^{-1}B\mathbf{b}(\mu_{\mathrm{R}}) = \mathbf{d}(\mu_{\mathrm{R}}).$$

This equality coincides with (1.41), due to (4.14). It remains to show that it satisfies also conditions (1.59)–(1.61). But it follows from (1.26), (4.19) and (4.20) that

$$R_j(\xi_{\mathrm{L}}) = H_j(\xi_{\mathrm{L}})^*\mathbf{a}(\xi_{\mathrm{L}}) \quad (j = 1, \dots, p),$$

$$D_k(\xi_{\mathrm{R}}) = G_k(\xi_{\mathrm{R}})\mathbf{b}(\xi_{\mathrm{R}}) \quad (k = 1, \dots, q)$$

and these last relations together with factorization (4.9) imply (1.59)–(1.61). □

As an illustration of the solvability criterion, we now show how Corollary 1.11 follows from Theorem 1.10.

Proof of Corollary 1.11. We formulate the scalar problem as a left tangential problem as follows. We take $\mathscr{E} = \mathscr{E}_* = \mathscr{E}_{\mathrm{L}} = \mathbb{C}$. We take Ω_{L} to be the finite set

$$\Omega_{\mathrm{L}} = \{z^1, \dots, z^N\} \subset \mathscr{D}_{\mathbf{Q}}.$$

For $i = 1, \dots, N$, set $\mathbf{a}(z^i) = 1$ and $\mathbf{c}(z^i) = \bar{w}_i$. Then the interpolation conditions (1.39) reduce to

$$S(z^i) = w_i \quad \text{for } i = 1, \dots, N.$$

The solution criterion in Theorem 1.10 calls for a positive kernel $K_{\mathrm{L}} : \Omega_{\mathrm{L}} \times \Omega_{\mathrm{L}} \mapsto \mathscr{L}(\mathbb{C}^p \otimes \mathbb{C}) \cong \mathbb{C}^{p \times p}$. Let us define numbers $\gamma_{k,\ell;k',\ell'}$ by

$$\gamma_{k,\ell;k',\ell'} = K_{\mathrm{L};k,k'}(z^{\ell}, z^{\ell'}).$$

Now it is straightforward to verify that the Stein equation (1.50) collapses to the system of equations (1.54), and Corollary 1.11 follows immediately from Theorem 1.10. □

5. Toeplitz Corona theorem

Suppose that $a_1, \dots, a_k$ are given functions in $H^\infty(\mathbb{D})$. The corona problem asks for conditions on $\{a_1, \dots, a_k\}$ so that there exist functions $f_1, \dots, f_k$ analytic and uniformly bounded on $\mathbb{D}$ so that

$$a_1 f_1 + \cdots + a_k f_k = 1. \tag{5.1}$$

The *Toeplitz corona theorem* (see [46] and [31]) asserts that *there exist such* $f_1, \dots, f_k \in H^\infty(\mathbb{D})$ *satisfying* (5.1) *with* $\sup_{z \in \mathbb{D}} \{|f_1(z)|^2 + \cdots + |f_k(z)|^2\} \leqslant \frac{1}{\delta^2}$ (i.e., with $F = [f_1 \quad \cdots \quad f_k]^\top \in \frac{1}{\delta} \mathscr{S}_{\mathbb{D}}(\mathbb{C}, \mathbb{C}^p)$) *if and only if*

$$T_{a_1} T_{a_1}^* + \cdots + T_{a_k} T_{a_k}^* \geqslant \delta^2 I > 0, \tag{5.2}$$

where $T_{a_i} : h(z) \to a_i(z) h(z)$ is the analytic Toeplitz operator on the Hardy space $H^2(\mathbb{D})$ with symbol a_i for $i = 1, \dots, k$. Equivalently, by looking at the gramian of the left-hand side of (5.2) with respect to an arbitrary finite collection of reproducing kernel functions $k_{z_i}(z) = \frac{1}{1 - z\bar{z}_i}$ in $H^2(\mathbb{D})$, we see that condition (5.2) alternatively can be expressed as

$$\sum_{i,j=1}^{N} \frac{a_1(z_i)\overline{a_1(z_j)} + \cdots + a_k(z_i)\overline{a_k(z_j)} - \delta^2}{1 - z_i \bar{z}_j} \bar{c}_i c_j \geqslant 0$$

for all complex scalars $c_1, \dots, c_N$ and all points $z_1, \dots, z_N \in \mathbb{D}$ for $N = 1, 2, 3, \dots$, i.e., the function

$$k(z, w) = \frac{a_1(z)\overline{a_1(w)} + \cdots + a_n(z)\overline{a_n(w)} - \delta^2}{1 - z\bar{w}}$$

is a positive kernel on $\mathbb{D} \times \mathbb{D}$. The *Carleson corona theorem* (see [23]), on the other hand, asserts: *there exists* $f_1, \dots, f_k \in H^\infty(\mathbb{D})$ *with*

$$\max_{1 \leqslant i \leqslant k} \sup_{|z|<1} |f_i(z)| \leqslant M(\delta) < \infty$$

if and only if

$$\inf_{|z|<1} \{|a_1(z)| + \cdots + |a_k(z)|\} \geqslant \delta > 0.$$

Unlike as in the formulation of the Toeplitz corona theorem, the relation between δ and $M(\delta)$ is rather complicated in the Carleson corona theorem.

As explained in the Introduction, for the case of the unit disk ($\mathbb{D} = \mathscr{D}_{\mathbf{Q}}$ with $n = 1$ and $\mathbf{Q}(z_1) = z_1$), the Schur class $\mathscr{S}_{\mathbb{D}}(\mathscr{E}, \mathscr{E}_*)$ and the Schur–Agler class $\mathscr{S}\mathscr{A}_{\mathbb{D}}(\mathscr{E}, \mathscr{E}_*)$ coincide. Thus the conclusion of the Toeplitz corona theorem can equivalently be expressed as $F = [f_1 \quad \cdots \quad f_k]^\top \in \frac{1}{\delta}\mathscr{S}\mathscr{A}_{\mathbb{D}}(\mathbb{C}, \mathbb{C}^p)$.

In this section we present an extension of the Toeplitz corona theorem to the case where the unit disk $\mathbb{D}$ is replaced by a domain of the general type $\mathscr{D}_{\mathbf{Q}}$ for a matrix polynomial $\mathbf{Q}$. The result is as follows.

Theorem 5.1. *Suppose that we are given analytic functions* $a_1, \dots, a_k$ *uniformly bounded on a domain* $\mathscr{D}_{\mathbf{Q}}$, *and a positive number* $\delta > 0$. *Then there exist bounded, analytic functions* $f_1, \dots, f_k$ *on* $\mathscr{D}_{\mathbf{Q}}$ *such that*

$$a_1(z)f_1(z) + \cdots + a_k(z)f_k(z) = 1 \quad \textit{for all } z \in \mathscr{D}_{\mathbf{Q}}$$

and

$$F := [f_1 \quad \cdots \quad f_k]^\top \in \tfrac{1}{\delta}\mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathbb{C}, \mathbb{C}^k)$$

if and only if there is an auxiliary Hilbert space $\mathscr{H}$ *and an analytic* $\mathscr{L}(\mathbb{C}^p \otimes \mathscr{H}, \mathbb{C})$-*valued function* $z \mapsto H(z)$ *on* $\mathscr{D}_{\mathbf{Q}}$ *so that*

$$a_1(z)\overline{a_1(w)} + \cdots + a_k(z)\overline{a_k(w)} - \delta^2 = H(z)(I_{\mathbb{C}^p \otimes \mathscr{H}} - (\mathbf{Q}(z)\mathbf{Q}(w)^*) \otimes I_{\mathscr{H}})H(w)^*,$$

or, equivalently, there exists a positive kernel

$$K = \begin{bmatrix} K_{11} & \cdots & K_{1p} \\ \vdots & & \vdots \\ K_{p1} & \cdots & K_{pp} \end{bmatrix} : \mathscr{D}_{\mathbf{Q}} \times \mathscr{D}_{\mathbf{Q}} \mapsto \mathscr{L}(\mathbb{C}^p)$$

so that

$$a_1(z)\overline{a_1(w)} + \cdots + a_k(z)\overline{a_k(w)} - \delta^2 = \sum_{i=1}^{k} K_{i,i}(z,w) - \sum_{\ell=1}^{q} \sum_{i,j=1}^{p} [\mathbf{q}_{i\ell}(z)\overline{\mathbf{q}_{j\ell}(w)}K_{i,j}(z,w)].$$

Proof. Take $\Omega_{\rm L} = \mathscr{D}_{\mathbf{Q}}$, $\mathscr{E}_* = \mathbb{C}^k$, $\mathscr{E} = \mathscr{E}_{\rm L} = \mathscr{E}_{\rm R} = \mathbb{C}$, $\mathbf{a}(z) = [a_1(z) \ \cdots \ a_k(z)]^*$, $\mathbf{c}(z) = \delta$ in Theorem 1.10 part (1). Note that $S(z) = [s_1(z) \ \cdots \ s_k(z)]^\top$ is in the Schur–Agler class $\mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathbb{C}, \mathbb{C}^k)$ and satisfies the left interpolation condition (1.39) if and only if $F(z) = [f_1(z) \ \cdots \ f_k(z)]^\top$ is in the scaled Schur–Agler class $\frac{1}{\delta}\mathscr{S}\mathscr{A}_{\mathbf{Q}}(\mathbb{C}, \mathbb{C}^k)$ and satisfies the corona condition

$$a_1(z)f_1(z) + \cdots + a_k(z)f_k(z) = 1,$$

where we have set $f_i(z) := \frac{1}{\delta}s_i(z)$ for $i = 1, \ldots, k$. Thus Theorem 5.1 amounts to a straightforward specialization of Theorem 1.10 part (1). □

Remark 5.2. For the case where $\mathscr{D}_{\mathbf{Q}} = \mathbb{D}^n$ as in Example 2.4, Theorem 5.1 appears in [18]; the result also can be seen as an application of the commutant lifting theorem for the polydisk obtained in [15] in a standard way (see [46]). For the case where $\mathscr{D}_{\mathbf{Q}} = \mathbb{B}^n$ as in Example 2.1, the result does not appear explicitly in [19] but can be derived from the commutant lifting theorem given there for multipliers for the reproducing kernel Hilbert space associated with the positive kernel

$$k(z, w) = \frac{1}{1 - \langle z, w \rangle_{\mathbb{C}^n}}$$

on $\mathbb{B}^n$. A parallel application of the commutant lifting theorem in [7] leads to an alternative derivation of Theorem 5.1.

Remark 5.3. The Toeplitz Corona Theorem (also called the Operator Corona Theorem) can be used as a stepping stone toward proving the Carleson corona theorem; we refer to [49] and the references found there.

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