SCHUR-CLASS MULTIPLIERS ON THE FOCK SPACE: DE BRANGES-ROVNYAK REPRODUCING KERNEL SPACES AND TRANSFER-FUNCTION REALIZATIONS

JOSEPH A. BALL, VLADIMIR BOLOTNIKOV, AND QUANLEI FANG

ABSTRACT. We introduce and study a Fock-space noncommutative analogue of reproducing kernel Hilbert spaces of de Branges-Rovnyak type. Results include: use of the de Branges-Rovnyak space $\mathcal{H}(K_S)$ as the state space for the unique (up to unitary equivalence) observable, coisometric transfer-function realization of the Schur-class multiplier S, realization-theoretic characterization of inner Schur-class multipliers, and a calculus for obtaining a realization for an inner multiplier with prescribed left zero-structure. In contrast with the parallel theory for the Arveson space on the unit ball $\mathbb{B}^d \subset \mathbb{C}^d$ (which can be viewed as the symmetrized version of the Fock space used here), the results here are much more in line with the classical univariate case, with the extra ingredient of the existence of all results having both a "left" and a "right" version.

Dedicated to the memory of Tiberiu Constantinescu

1. Introduction

Recently there has been much interest and an evolving theory of noncommutative function theory and associated multivariable operator theory and multidimensional system theory with evolution along a free semigroup; we mention [2, 22, 6, 10, 11, 18, 20, 24, 25, 26, 27, 29, 30]. A central player in many of these developments is the noncommutative Schur class consisting of formal power series in a set of noncommuting indeterminates which define contractive multipliers between (unsymmetrized) vector-valued Fock spaces; such Schur-class functions play the role of the characteristic function for the Popescu analogue for a row contraction of the Sz.-Nagy-Foiaş model theory for a single contraction operator (see [27, 15]). For the classical (univariate) case, there is an approach to operator-model theory complementary to the Sz.-Nagy-Foiaş approach which emphasizes constructions with reproducing kernel Hilbert spaces over the unit disk rather than the geometry of the unitary dilation space of a contraction operator. Our purpose here is to flesh out the ingredients of this approach for the Fock space setting. The appropriate noncommutative multivariable version of a reproducing kernel Hilbert space has already been worked out in [14] and certain other relevant background material appears in [7]. Unlike the work in some of the papers mentioned above, specifically [2, 3, 6, 11, 18, 19, 20, 22, 24, 25, 29], we shall deal with formal power series with operator coefficients as parts of some formal structure (e.g., as inducing multiplication operators between two Hilbert spaces whose elements are formal power series with vector coefficients) rather than as themselves functions on some collection of

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noncommutative operator-tuples. Before discussing the precise noncommutative results which we present here, we review the corresponding classical versions of the results.

For $\mathcal{U}$ and $\mathcal{Y}$ two Hilbert spaces, let $\mathcal{L}(\mathcal{U},\mathcal{Y})$ denote the space of bounded linear operators between $\mathcal{U}$ and $\mathcal{Y}$. We also let $H^2_{\mathcal{U}}(\mathbb{D})$ be the standard Hardy space of the $\mathcal{U}$-valued holomorphic functions on the unit disk $\mathbb{D}$. By the classical Schur class $\mathcal{S}(\mathcal{U},\mathcal{Y})$ we mean the set of $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued functions holomorphic on the unit disk $\mathbb{D}$ with values $S(\lambda)$ having norm at most 1 for each $\lambda \in \mathbb{D}$. There are several equivalent characterizations of the class $\mathcal{S}(\mathcal{U},\mathcal{Y})$; for convenience, we list some in the following theorem.

Theorem 1.1. *Let S be an $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function defined on the unit disk $\mathbb{D}$. Then the following are equivalent:*

1. *$S \in \mathcal{S}(\mathcal{U},\mathcal{Y})$, i.e., S is analytic on $\mathbb{D}$ with contractive values in $\mathcal{L}(\mathcal{U},\mathcal{Y})$.*
2. *The multiplication operator $M_S\colon f(z) \mapsto S(z)\cdot f(z)$ is a contraction from $H^2_{\mathcal{U}}(\mathbb{D})$ into $H^2_{\mathcal{Y}}(\mathbb{D})$.*
3. *The kernel*
$$K_S(\lambda,\zeta) := \frac{I_{\mathcal{Y}} - S(\lambda)S(\zeta)^*}{1-\lambda\overline{\zeta}}$$
is positive on $\mathbb{D}\times\mathbb{D}$, i.e., there exists an auxiliary Hilbert space $\mathcal{X}$ and a function $H\colon \mathbb{D} \to \mathcal{L}(\mathcal{X},\mathcal{Y})$ such that
$$K_S(\lambda,\zeta) = H(\lambda)H(\zeta)^* \quad \text{for all} \quad \lambda,\zeta \in \mathbb{D}. \tag{1.1}$$
4. *There exists a Hilbert space $\mathcal{X}$ and a unitary connection operator (or colligation) $\mathbf{U}$ of the form*
$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \tag{1.2}$$
so that $S(\lambda)$ can be realized in the form
$$S(\lambda) = D + \lambda C(I_{\mathcal{X}} - \lambda A)^{-1}B. \tag{1.3}$$
5. *There exists a Hilbert space $\mathcal{X}$ and a contractive connecting operator $\mathbf{U}$ of the form (1.2) so that (1.3) holds.*

A pair (C,A) is called *an output pair* if $C \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ and $A \in \mathcal{L}(\mathcal{X},\mathcal{X})$. An output pair (C,A) is called *contractive* if $A^*A + C^*C \le I_{\mathcal{X}}$, *isometric* if $A^*A + C^*C = I_{\mathcal{X}}$ and *observable* if $\bigcap_{n=0}^{\infty} \operatorname{Ker} CA^n = \{0\}$. We shall say that the realization (1.3) of $S(\lambda)$ is *observable* if the output pair (C,A) occurring in (1.3) is observable. Furthermore, with an output contractive pair (C,A), one can associate the positive kernel
$$K_{C,A}(\lambda,\zeta) = C(I-\lambda A)^{-1}(I-\overline{\zeta}A^*)^{-1}C^* \tag{1.4}$$
which is (as it is readily seen) defined on $\mathbb{D}\times\mathbb{D}$.

As also remarked in [8], the coisometric version of (4) $\Longrightarrow$ (2) is particularly transparent, since in this case a simple computation shows that then (1.1) holds with $H(\lambda) = C(I-\lambda A)^{-1}$, i.e., $K_S(\lambda,\zeta) = K_{C,A}(\lambda,\zeta)$. We have the following sort of converse of these observations.

Theorem 1.2. (1) *Suppose that $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and that (C, A) is an observable, contractive output-pair of operators such that*

$$K_S(\lambda, \zeta) = K_{C,A}(\lambda, \zeta). \tag{1.5}$$

Then there is a unique choice of $B\colon \mathcal{U} \to \mathcal{X}$ so that $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & S(0) \end{smallmatrix}\right]$ is coisometric and $\mathbf{U}$ provides a realization for S: $S(\lambda) = S(0) + \lambda C(I - \lambda A)^{-1}B$.

(2) *Suppose that we are given only an observable, contractive output-pair of operators (C, A) as above. Then there is a choice of an input space $\mathcal{U}$ and a Schur multiplier $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ so that* (1.5) *holds.*

As we see from Theorem 1.1, for any Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, we can associate the positive kernel $K_S(\lambda, \zeta)$ and therefore also by Aronszajn's construction the reproducing kernel Hilbert space $\mathcal{H}(K_S)$; this space is called the de Branges-Rovnyak space associated with S. It turns out that any observable coisometric realization $\mathbf{U}$ for S is unitarily equivalent to a certain canonical functional-model realization.

Theorem 1.3. *Let $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$. Then the operator*

$$\mathbf{U}_{dBR} = \begin{bmatrix} A_{dBR} & B_{dBR} \\ C_{dBR} & D_{dBR} \end{bmatrix} : \begin{bmatrix} \mathcal{H}(K_S) \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(K_S) \\ \mathcal{Y} \end{bmatrix}$$

with the entries given by

$$A_{dBR}\colon f(\lambda) \to \frac{f(\lambda) - f(0)}{\lambda}, \qquad B_{dBR}\colon u \to \frac{S(\lambda) - S(0)}{\lambda} u,$$
$$C_{dBR}\colon f \to f(0), \qquad D_{dBR}\colon u \to S(0)u$$

provides an observable and coisometric realization

$$S(\lambda) = D_{dBR} + \lambda C_{dBR}(I_{\mathcal{H}(K_S)} - \lambda A_{dBR})^{-1} B_{dBR}. \tag{1.6}$$

Moreover, any other observable coisometric realization of S is unitarily equivalent to (1.6).

Let us say that a Schur function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ is *inner* if the associated multiplication operator $M_S\colon H^2_{\mathcal{U}}(\mathbb{D}) \to H^2_{\mathcal{Y}}(\mathbb{D})$ is a partial isometry. Equivalently, $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and the almost everywhere existing boundary value function $S(\zeta) = \lim_{r \uparrow 1} S(r\zeta)$ is a partial isometry for almost all $\zeta \in \mathbb{T}$. The following characterization of inner functions in terms of realizations is well known (see [16, 17]).

Theorem 1.4. *A Schur multiplier $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ is inner if and only if its essentially unique observable, coisometric realization of the form* (1.3) *is such that A is strongly stable, i.e.,*

$$\lim_{n \to \infty} \|A^n x\| = 0 \text{ for all } x \in \mathcal{X}. \tag{1.7}$$

Inner functions come up in the representation of shift-invariant subspaces of $H^2_{\mathcal{Y}}$ as in the Beurling-Lax theorem. The following version of the Beurling-Lax theorem first identifies any shift-invariant subspace as the set of solutions of a collection of homogeneous interpolation conditions and then obtains a realization for the Beurling-Lax representer in terms of the data set for the homogeneous interpolation problem. The finite-dimensional version of this result can be found in [9, Chapter 14] while the details of the general case appear in [12]. We let M_λ denote the shift operator

$$M_\lambda \colon f(\lambda) \to \lambda f(\lambda) \quad \text{for} \quad f \in H^2_{\mathcal{Y}}(\mathbb{D})$$

and given a contractive pair (C, A) we let

$$\mathcal{M}_{A^*,C^*} = \{f \in H^2_{\mathcal{Y}}(\mathbb{D})\colon \ (C^* f)^{\wedge L}(A^*) = 0\} \tag{1.8}$$

where we have set

$$(C^* f)^{\wedge L}(A^*) := \sum_{n=0}^{\infty} A^{*n} C^* f_n \quad \text{if} \quad f(\lambda) = \sum_{n=0}^{\infty} f_n \lambda^n \in H^2_{\mathcal{Y}}(\mathbb{D}).$$

Theorem 1.5. (1) *Suppose that $\mathcal{M}$ is a subspace of $H^2_{\mathcal{Y}}(\mathbb{D})$ which is M_λ-invariant. Then there is an isometric pair (C, A) such that A is strongly stable (i.e., (1.7) holds) and such that $\mathcal{M} = \mathcal{M}_{A^*,C^*}$.*

(2) *Suppose that the shift-invariant subspace $\mathcal{M} \subset H^2_{\mathcal{Y}}(\mathbb{D})$ has the representation $\mathcal{M} = \mathcal{M}_{A^*,C^*}$ as in (1.8) where (C, A) is an isometric pair with A strongly stable. Choose an input space $\mathcal{U}$ and operators $B\colon \mathcal{U} \to \mathcal{X}$ and $D\colon \mathcal{U} \to \mathcal{Y}$ so that*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \colon \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

is unitary. Then the function $S(\lambda) = D + \lambda C(I_{\mathcal{X}} - \lambda A)^{-1} B$ is inner (i.e., M_S is isometric) and is a Beurling-Lax representer for $\mathcal{M}$:

$$S \cdot H^2_{\mathcal{U}}(\mathbb{D}) = \mathcal{M}_{A^*,C^*}.$$

Our goal here is to obtain noncommutative analogues of these results, where the classical Schur class is replaced by the noncommutative Schur class of contractive multipliers between Fock spaces of formal power series in noncommuting indeterminates and where the classical reproducing kernel Hilbert spaces become the noncommutative formal reproducing kernel Hilbert spaces introduced in [14]. Let $z = (z_1, \dots, z_d)$ and $w = (w_1, \dots, w_d)$ be two sets of noncommuting indeterminates. We let $\mathcal{F}_d$ denote the free semigroup generated by the d letters $\{1, \dots, d\}$. A generic element of $\mathcal{F}_d$ is a word w equal to a string of letters

$$\alpha = i_N \cdots i_1 \quad \text{where} \quad i_k \in \{1, \dots, d\} \ \text{ for } \ k = 1, \dots, N. \tag{1.9}$$

Given two words α and β with α as in (1.9) and β of the form $\beta = j_{N'} \cdots j_1$, say, the product $\alpha\beta$ is defined by concatenation:

$$\alpha\beta = i_N \cdots i_1 j_{N'} \cdots j_1 \in \mathcal{F}_d.$$

The unit element of $\mathcal{F}_d$ is the *empty word* denoted by $\emptyset$. For α a word of the form (1.9), we let z^α denote the monomial in noncommuting indeterminates

$$z^\alpha = z_{i_N} \cdots z_{i_1}$$

and we let $z^\emptyset = 1$. We extend this noncommutative functional calculus to a d-tuple of operators $\mathbf{A} = (A_1, \dots, A_d)$ on a Hilbert space $\mathcal{X}$:

$$\mathbf{A}^v = A_{i_N} \cdots A_{i_1} \quad \text{if} \quad v = i_N \cdots i_1 \in \mathcal{F}_d \setminus \{\emptyset\}; \quad \mathbf{A}^\emptyset = I_{\mathcal{X}}. \tag{1.10}$$

We will also have need of the *transpose operation* on $\mathcal{F}_d$:

$$\alpha^\top = i_1 \cdots i_N \quad \text{if} \quad \alpha = i_N \cdots i_1. \tag{1.11}$$

A natural analogue of the Szegö kernel is the noncommutative Szegö kernel

$$k_{\rm Sz}(z,w) = \sum_{\alpha\in\mathcal{F}_d} z^\alpha w^{\alpha^\top}. \tag{1.12}$$

The associated reproducing kernel Hilbert space $\mathcal{H}(k_{\rm Sz})$ (in the sense of [14]) is a natural analogue of the classical Hardy space $H^2(\mathbb{D})$; we recall all the relevant definitions and main properties more precisely in Section 2. Our main purpose here is to obtain the analogues the Theorems 1.1–1.5 above with the classical Szegö kernel replaced by its noncommutative analogue (1.12).

In particular, the analogue of Theorem 1.5 involves the study of shift-invariant subspaces of the Fock space $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ generated by a collection of homogeneous interpolation conditions defined via a functional calculus with noncommutative operator argument. We mention that interpolation problems in the noncommutative Schur-multiplier class defined by nonhomogeneous interpolation conditions associated with such a functional calculus have been studied recently by a number of authors, including the late Tiberiu Constantinescu to whom this paper is dedicated (see [6, 18, 29, 30]). While the Fock-space version of the Beurling-Lax theorem already appears in the work of Popescu [26] (see also [7]), the proof here through inner solution of a homogeneous interpolation problem gives an alternative approach.

The present paper (with the exception of the final Section 5) parallels our companion paper [8] where corresponding results are worked out with the noncommutative Szegö kernel (1.12) replaced by the so-called Arveson kernel $k_d(\boldsymbol{\lambda},\boldsymbol{\zeta}) = 1/(1-\langle \boldsymbol{\lambda},\boldsymbol{\zeta}\rangle_{\mathbb{C}^d})$ which is positive on the unit ball $\mathbb{B}^d = \{\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_d)\colon \sum_{k=1}^d |\lambda_k|^2 < 1\}$ of $\mathbb{C}^d$. There the corresponding results are more delicate; in particular, the observable, coisometric realization for a contractive multiplier is unique only in very special circumstances, but the nonuniqueness can be explicitly characterized. In contrast, the results obtained here for the setting of the noncommutative Szegö kernel $k_{\rm Sz}(z,w)$ parallel more directly the situation for the classical univariate case.

The paper is organized as follows. After the present Introduction, Section 2 recalls the main facts from [14] which are needed in the sequel. Section 3 introduces the noncommutative Schur class of contractive Fock-space multipliers S and the associated noncommutative positive kernel $K_S(z,w)$, and develops the noncommutative analogues of Theorems 1.1 and 1.2. In fact, various pieces of the noncommutative version of Theorem 1.1 (see theorem 3.1 below) are already worked out in [14, 27, 15]. In connection with the noncommutative analogue of Theorem 1.2 (see Theorems 3.5 and 3.8 below), we rely on our paper [7] where the structure of noncommutative formal reproducing kernel spaces of the type $\mathcal{H}(K_{C,A})$ were worked out. Section 4 introduces the noncommutative functional-model coisometric colligation $\mathbf{U}_{dBR}$ and obtains the analogue of Theorem 1.3 for the Fock space setting (see Theorem 4.3 below). This functional model is the Brangesian model parallel to the noncommutative Sz.-Nagy-Foiaş model for a row contraction found in [27, 15]. The final Section 5 uses previous results concerning $\mathcal{H}(K_S)$ and $\mathcal{H}(K_{C,A})$ to arrive at the Fock-space version of Theorem 1.5 (see Theorems 5.1 and 5.2 below) in a simple way.

2. Noncommutative formal reproducing kernel Hilbert spaces

We now recall some of the basic ideas from [14] concerning noncommutative formal reproducing kernel Hilbert spaces. We let $z = (z_1,\dots,z_d)$, $w = (w_1,\dots,w_d)$

be two sets of noncommuting indeterminates and we let $\mathcal{F}_d$ be the free semigroup generated by the alphabet $\{1, \dots, d\}$ with unit element equal to the empty word $\emptyset$ as in the introduction. Given a coefficient Hilbert space $\mathcal{Y}$ we let $\mathcal{Y}\langle z\rangle$ denote the set of all polynomials in $z = (z_1, \dots, z_d)$ with coefficients in $\mathcal{Y}$:

$$\mathcal{Y}\langle z\rangle = \left\{ p(z) = \sum_{\alpha \in \mathcal{F}_d} p_\alpha z^\alpha \colon p_\alpha \in \mathcal{Y} \text{ and } p_\alpha = 0 \text{ for all but finitely many } \alpha \right\},$$

while $\mathcal{Y}\langle\langle z\rangle\rangle$ denotes the set of all formal power series in the indeterminates z with coefficients in $\mathcal{Y}$:

$$\mathcal{Y}\langle\langle z\rangle\rangle = \left\{ f(z) = \sum_{\alpha \in \mathcal{F}_d} f_\alpha z^\alpha \colon f_\alpha \in \mathcal{Y} \right\}.$$

Note that vectors in $\mathcal{Y}$ can be considered as Hilbert space operators between $\mathbb{C}$ and $\mathcal{Y}$. More generally, if $\mathcal{U}$ and $\mathcal{Y}$ are two Hilbert spaces, we let $\mathcal{L}(\mathcal{U}, \mathcal{Y})\langle z\rangle$ and $\mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z\rangle\rangle$ denote the space of polynomials (respectively, formal power series) in the noncommuting indeterminates $z = (z_1, \dots, z_d)$ with coefficients in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. Given $S = \sum_{\alpha \in \mathcal{F}_d} s_\alpha z^\alpha \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z\rangle\rangle$ and $f = \sum_{\beta \in \mathcal{F}_d} f_\beta z^\beta \in \mathcal{U}\langle\langle z\rangle\rangle$, the product $S(z) \cdot f(z) \in \mathcal{Y}\langle\langle z\rangle\rangle$ is defined as an element of $\mathcal{Y}\langle\langle z\rangle\rangle$ via the noncommutative convolution:

$$S(z) \cdot f(z) = \sum_{\alpha, \beta \in \mathcal{F}_d} s_\alpha f_\beta z^{\alpha\beta} = \sum_{v \in \mathcal{F}_d} \left(\sum_{\alpha, \beta \in \mathcal{F}_d \colon \alpha \cdot \beta = v} s_\alpha f_\beta \right) z^v. \tag{2.1}$$

Note that the coefficient of z^v in (2.1) is well defined since any given word $v \in \mathcal{F}_d$ can be decomposed as a product $v = \alpha \cdot \beta$ in only finitely many distinct ways.

In general, given a coefficient Hilbert space $\mathcal{C}$, we use the $\mathcal{C}$ inner product to generate a pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{C} \times \mathcal{C}\langle\langle w\rangle\rangle} \colon \mathcal{C} \times \mathcal{C}\langle\langle w\rangle\rangle \to \mathcal{C}\langle\langle w\rangle\rangle$$

via

$$\left\langle c, \sum_{\beta \in \mathcal{F}_d} f_\beta w^\beta \right\rangle_{\mathcal{C} \times \mathcal{C}\langle\langle w\rangle\rangle} = \sum_{\beta \in \mathcal{F}_d} \langle c, f_\beta \rangle_{\mathcal{C}} w^{\beta^\top} \in \mathcal{C}\langle\langle w\rangle\rangle.$$

We also may use the pairing in the reverse order

$$\left\langle \sum_{\alpha \in \mathcal{F}_d} f_\alpha w^\alpha, c \right\rangle_{\mathcal{C}\langle\langle w\rangle\rangle \times \mathcal{C}} = \sum_{\alpha \in \mathcal{F}_d} \langle f_\alpha, c \rangle_{\mathcal{C}} w^\alpha \in \mathcal{C}\langle\langle w\rangle\rangle.$$

These are both special cases of the more general pairing

$$\left\langle \sum_{\alpha \in \mathcal{F}_d} f_\alpha w'^\alpha, \sum_{\beta \in \mathcal{F}_d} g_\beta w^\beta \right\rangle_{\mathcal{C}\langle\langle w'\rangle\rangle \times \mathcal{C}\langle\langle w\rangle\rangle} = \sum_{\alpha, \beta \in \mathcal{F}_d} \langle f_\alpha, g_\beta \rangle_{\mathcal{C}} w^{\beta^\top} w'^\alpha.$$

Suppose that $\mathcal{H}$ is a Hilbert space whose elements are formal power series in $\mathcal{Y}\langle\langle z\rangle\rangle$ and that $K(z, w) = \sum_{\alpha, \beta \in \mathcal{F}_d} K_{\alpha,\beta} z^\alpha w^{\beta^\top}$ is a formal power series in the two sets of d noncommuting indeterminates $z = (z_1, \dots, z_d)$ and $w = (w_1, \dots, w_d)$. We say that $K(z, w)$ *is a reproducing kernel for* $\mathcal{H}$ if, for each $\beta \in \mathcal{F}_d$ the formal power series

$$K_\beta(z) := \sum_{\alpha \in \mathcal{F}_d} K_{\alpha,\beta} z^\alpha \quad \text{belongs to } \mathcal{H}$$

and we have the reproducing property

$$\langle f, K(\cdot,w)y\rangle_{\mathcal{H}\times\mathcal{H}\langle\langle w\rangle\rangle} = \langle f(w), y\rangle_{\mathcal{Y}\langle\langle w\rangle\rangle\times\mathcal{Y}} \quad \text{for every } f \in \mathcal{H}.$$

As a consequence we then also have

$$\langle K(\cdot,w')y', K(\cdot,w)y\rangle_{\mathcal{H}\langle\langle w'\rangle\rangle\times\mathcal{H}\langle\langle w\rangle\rangle} = \langle K(w,w')y', y\rangle_{\mathcal{Y}\langle\langle w,w'\rangle\rangle\times\mathcal{Y}}.$$

It is not difficult to see that a reproducing kernel for a given $\mathcal{H}$ is necessarily unique.

Let us now suppose that $\mathcal{H}$ is a Hilbert space whose elements are formal power series $f(z) = \sum_{\alpha\in\mathcal{F}_d} f_v z^v \in \mathcal{Y}\langle\langle z\rangle\rangle$ for a coefficient Hilbert space $\mathcal{Y}$. We say that $\mathcal{H}$ is a NFRKHS (*noncommutative formal reproducing kernel Hilbert space*) if, for each $\alpha \in \mathcal{F}_d$, the linear operator $\Phi_\alpha \colon \mathcal{H} \to \mathcal{Y}$ defined by $f(z) = \sum_{v\in\mathcal{F}_d} f_v z^v \mapsto f_\alpha$ is continuous. In this case, define $K(z,w) \in \mathcal{L}(\mathcal{Y})\langle\langle z,w\rangle\rangle$ by

$$K(z,w) = \sum_{\beta\in\mathcal{F}_d} \Phi_\beta^* w^{\beta^\top} =: \sum_{\alpha,\beta\in\mathcal{F}_d} K_{\alpha,\beta} z^\alpha w^{\beta^\top}.$$

Then one can check that $K(z,w)$ is a reproducing kernel for $\mathcal{H}$ in the sense defined above. Conversely (see [14, Theorem 3.1]), a given formal kernel $K(z,w) = \sum_{\alpha,\beta\in\mathcal{F}_d} K_{\alpha,\beta} z^\alpha w^{\beta^\top} \in \mathcal{L}(\mathcal{Y})\langle\langle z,w\rangle\rangle$ is the reproducing kernel for some NFRKHS $\mathcal{H}$ if and only if K is positive definite in either one of the equivalent senses:

(1) $K(z,w)$ has a factorization

$$K(z,w) = H(z)H(w)^* \tag{2.2}$$

for some $H \in \mathcal{L}(\mathcal{X},\mathcal{Y})\langle\langle z\rangle\rangle$ for some auxiliary Hilbert space $\mathcal{X}$. Here

$$H(w)^* = \sum_{\beta\in\mathcal{F}_d} H_\beta^* w^{\beta^\top} = \sum_{\beta\in\mathcal{F}_d} H_{\beta^\top}^* w^\beta \quad \text{if} \quad H(z) = \sum_{\alpha\in\mathcal{F}_d} H_\alpha z^\alpha.$$

(2) For all finitely supported $\mathcal{Y}$-valued functions $\alpha \mapsto y_\alpha$ it holds that

$$\sum_{\alpha,\alpha'\in\mathcal{F}_d} \langle K_{\alpha,\alpha'} y_{\alpha'}, y_\alpha\rangle \geq 0. \tag{2.3}$$

If K is such a positive kernel, we denote by $\mathcal{H}(K)$ the associated NFRKHS consisting of elements of $\mathcal{Y}\langle\langle z\rangle\rangle$.

3. The noncommutative Schur class: associated positive kernels and transfer-function realization

A natural analogue of the vector-valued Hardy space over the unit disk (see e.g. [26]) is the Fock space with coefficients in $\mathcal{Y}$ which we denote here by $H^2_{\mathcal{Y}}(\mathcal{F}_d)$:

$$H^2_{\mathcal{Y}}(\mathcal{F}_d) = \left\{ f(z) = \sum_{\alpha\in\mathcal{F}_d} f_\alpha z^v \colon \sum_{\alpha\in\mathcal{F}_d} \|f_\alpha\|^2 < \infty \right\}.$$

When $\mathcal{Y} = \mathbb{C}$ we write simply $H^2(\mathcal{F}_d)$. As explained in [14], $H^2(\mathcal{F}_d)$ is a NFRKHS with reproducing kernel equal to the following noncommutative analogue of the classical Szegö kernel:

$$k_{\mathrm{Sz}}(z,w) = \sum_{\alpha\in\mathcal{F}_d} z^\alpha w^{\alpha^\top}. \tag{3.1}$$

Thus we have in general $H^2_{\mathcal{Y}}(\mathcal{F}_d) = \mathcal{H}(k_{\mathrm{Sz}} \otimes I_{\mathcal{Y}})$. We let S_j denote the shift operator

$$S_j \colon f(z) = \sum_{v\in\mathcal{F}_d} f_v z^v \mapsto f(z)\cdot z_j = \sum_{v\in\mathcal{F}_d} f_v z^{v\cdot j} \text{ for } j = 1,\ldots,d \tag{3.2}$$

on $H^2_{\mathcal{Y}}(\mathcal{F}_d)$; when we wish to specify the coefficient space $\mathcal{Y}$ explicitly, we write $S_j \otimes I_{\mathcal{Y}}$ rather than only S_j. The adjoint of $S_j\colon H^2_{\mathcal{Y}}(\mathcal{F}_d) \to H^2_{\mathcal{Y}}(\mathcal{F}_d)$ is then given by

$$S_j^*\colon \sum_{v\in\mathcal{F}_d} f_v z^v \mapsto \sum_{v\in\mathcal{F}_d} f_{v\cdot j} z^v \quad \text{for} \quad j=1,\dots,d. \tag{3.3}$$

We let $\mathcal{M}_{nc,d}(\mathcal{U},\mathcal{Y})$ denote the set of formal power series $S(z)=\sum_{\alpha\in\mathcal{F}_d} s_\alpha z^\alpha$ with coefficients $s_\alpha \in \mathcal{L}(\mathcal{U},\mathcal{Y})$ such that the associated multiplication operator $M_S\colon f(z)\mapsto S(z)\cdot f(z)$ (see (2.1)) defines a bounded operator from $H^2_{\mathcal{U}}(\mathcal{F}_d)$ to $H^2_{\mathcal{Y}}(\mathcal{F}_d)$. It is not difficult to show that $\mathcal{M}_{nc,d}(\mathcal{U},\mathcal{Y})$ is the intertwining space for the two tuples $\mathbf{S}\otimes I_{\mathcal{U}} = (S_1\otimes I_{\mathcal{U}},\dots,,S_d\otimes I_{\mathcal{U}})$ and $\mathbf{S}\otimes I_{\mathcal{Y}} = (S_1\otimes I_{\mathcal{Y}},\dots,S_d\otimes I_{\mathcal{Y}})$: *an operator $X\in\mathcal{L}(\mathcal{U},\mathcal{Y})$ equals $X=M_S$ for some $S\in\mathcal{M}_{nc,d}(\mathcal{U},\mathcal{Y})$ whenever $S_j\otimes I_{\mathcal{Y}})X = X(S_j\otimes I_{\mathcal{U}})$ for $j=1,\dots,d$* (see e.g. [27] where, however, the conventions are somewhat different). We define the noncommutative Schur class $\mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y})$ to consist of such multipliers S for which M_S has operator norm at most 1:

$$\mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y}) = \{S\in\mathcal{L}(\mathcal{U},\mathcal{Y})\colon M_S\colon H^2_{\mathcal{Y}}(\mathcal{F}_d)\to H^2_{\mathcal{Y}}(\mathcal{F}_d) \text{ with } \|M_S\|_{op}\le 1\}. \tag{3.4}$$

The following is the noncommutative analogue of Theorem 1.1 for this setting.

Theorem 3.1. *Let $S(z)\in\mathcal{L}(\mathcal{U},\mathcal{Y})\langle\langle z\rangle\rangle$ be a formal power series in $z=(z_1,\dots,z_d)$ with coefficients in $\mathcal{L}(\mathcal{U},\mathcal{Y})$. Then the following are equivalent:*

(1) *$S\in\mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y})$, i.e., $M_S\colon \mathcal{U}\langle z\rangle \to \mathcal{Y}\langle\langle z\rangle\rangle$ given by $M_S\colon p(z)\to S(z)p(z)$ extends to define a contraction operator from $H^2_{\mathcal{U}}(\mathcal{F}_d)$ into $H^2_{\mathcal{Y}}(\mathcal{F}_d)$.*

(2) *The kernel*

$$K_S(z,w) := k_{Sz}(z,w) - S(z)k_{Sz}(z,w)S(w)^* \tag{3.5}$$

is a noncommutative positive kernel (see (2.2) and (2.3)).

(3) *There exists a Hilbert space $\mathcal{X}$ and a unitary connection operator $\mathbf{U}$ of the form*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D\end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D\end{bmatrix} \colon \begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix} \to \begin{bmatrix}\mathcal{X}\\ \vdots \\ \mathcal{X} \\ \mathcal{Y}\end{bmatrix} \tag{3.6}$$

so that $S(z)$ can be realized as a formal power series in the form

$$S(z) = D + \sum_{j=1}^d \sum_{v\in\mathcal{F}_d} CA^vB_jz^v\cdot z_j = D + C(I-Z(z)A)^{-1}Z(z)B \tag{3.7}$$

where we have set

$$Z(z) = \begin{bmatrix} z_1 I_{\mathcal{X}} & \dots & z_d I_{\mathcal{X}}\end{bmatrix}, \quad A = \begin{bmatrix}A_1\\ \vdots \\ A_d\end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_d\end{bmatrix}. \tag{3.8}$$

(4) *There exists a Hilbert space $\mathcal{X}$ and a contractive block operator matrix $\mathbf{U}$ as in (3.6) such that $S(z)$ is given as in (3.7)*

Proof. (1) $\Longrightarrow$ (2) is Theorem 3.15 in [14]. A proof of (2) $\Longrightarrow$ (3) is done in [15, Theorem 5.4.1] as an application of the Sz.-Nagy-Foiaş model theory for row contractions worked out there following ideas of Popescu [26, 27]; an alternative proof via the "lurking isometry argument" can be found in [14, Theorem 3.16].

The implication (3) $\Longrightarrow$ (4) is trivial. The content of (4) $\Longrightarrow$ (1) amounts to Proposition 4.1.3 in [15]. $\square$

We note that formula (3.7) has the interpretation that $S(z)$ is the *transfer function* of the multidimensional linear system with evolution along $\mathcal{F}_d$ given by the input-state-output equations

$$\Sigma\colon \left\{ \begin{array}{ccc} x(1\cdot\alpha) & = & A_1x(\alpha)+B_1u(\alpha) \\ \vdots & & \vdots \\ x(d\cdot\alpha) & = & A_dx(\alpha)+B_du(\alpha) \\ y(\alpha) & = & Cx(\alpha)+Du(\alpha) \end{array} \right. \tag{3.9}$$

initialized with $x(\emptyset)=0$. Here $u(\alpha)$ takes values in the input space $\mathcal{U}$, $x(\alpha)$ takes values in the state space $\mathcal{X}$, and $y(\alpha)$ takes values in the output space $\mathcal{Y}$ for each $\alpha\in\mathcal{F}_d$. If we introduce the noncommutative Z-transform

$$\{x(\alpha)\}_{\alpha\in\mathcal{F}_d}\mapsto \widehat{x}(z):=\sum_{\alpha\in\mathcal{F}_d}x(\alpha)z^\alpha$$

and apply this transform to each of the system equations in (3.9), one can solve for $\widehat{y}(z)$ in terms of $\widehat{u}(z)$ to arrive at

$$\widehat{y}(z)=T_\Sigma(z)\cdot\widehat{u}(z)$$

where the *transfer function* $T_\Sigma(z)$ of the system (3.9) is the formal power series with coefficients in $\mathcal{L}(\mathcal{U},\mathcal{Y})$ given by

$$T_\Sigma(z)=D+\sum_{j=1}^d\sum_{\alpha\in\mathcal{F}_d}C\mathbf{A}^vB_jz^vz_j=D+C(I-Z(z)A)^{-1}Z(z)B. \tag{3.10}$$

For complete details, we refer to [15, 10, 11].

The implication (4) $\Longrightarrow$ (2) can be seen directly via the explicit identity (3.11) given in the next proposition; for the commutative case we refer to [1, Lemma 2.2].

Proposition 3.2. *Suppose that* $\mathbf{U}=\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]:\mathcal{X}\oplus\mathcal{U}\to\mathcal{X}^d\oplus\mathcal{Y}$ *is contractive with associated transfer function* $S\in\mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y})$ *given by* (3.7)*. Then the kernel* $K_S(z,w)$ *given by* (3.5) *is can also be represented as*

$$K_S(z,w)=C(I_\mathcal{X}-Z(z)A)^{-1}(I_\mathcal{X}-A^*Z(w)^*)^{-1}C^*+D_S(z,w) \tag{3.11}$$

where

$$\begin{aligned} D_S(z,w)=&\begin{bmatrix}C(I_\mathcal{X}-Z(z)A)^{-1}Z(z) & I_\mathcal{Y}\end{bmatrix}k_{Sz}(z,w)\\ &\cdot(I-\mathbf{U}\mathbf{U}^*)\begin{bmatrix}Z(w)^*(I-A^*Z(w)^*)^{-1}C^*\\ I_\mathcal{Y}\end{bmatrix}. \end{aligned} \tag{3.12}$$

Proof. For a fixed $\alpha\in\mathcal{F}_d$, let us set

$$X_\alpha=z^\alpha w^{\alpha^\top}I_\mathcal{Y}-S(z)z^\alpha w^{\alpha^\top}S(w)^*, \tag{3.13}$$

$$Y_\alpha=\begin{bmatrix}C(I-Z(z)A)^{-1}Z(z) & I_\mathcal{Y}\end{bmatrix}z^\alpha w^{\alpha^\top}(I-\mathbf{U}\mathbf{U}^*)\begin{bmatrix}Z(w)^*(I-A^*Z(w)^*)^{-1}C^*\\ I_\mathcal{Y}\end{bmatrix}.$$

Note that by (3.5) and (3.1),

$$\sum_{\alpha\in\mathcal{F}_d}X_\alpha=K_S(z,w)\quad\text{and}\quad\sum_{\alpha\in\mathcal{F}_d}Y_\alpha=D_S(z,w).$$

Therefore (3.11) is verified once we show that

$$\sum_{\alpha\in\mathcal{F}_d} X_\alpha - \sum_{\alpha\in\mathcal{F}_d} Y_\alpha = C(I - Z(z)A)^{-1}(I - A^*Z(w)^*)^{-1}C^*. \tag{3.14}$$

Substituting (3.7) into (3.13) gives

$$\begin{aligned} X_\alpha &= z^\alpha w^{\alpha^\top} I_{\mathcal{Y}} - [D + C(I - Z(z)A)^{-1}Z(z)B] \cdot z^\alpha w^{\alpha^\top} \cdot \\ &\qquad \cdot [D^* + B^*Z(w)^*(I - A^*Z(w)^*)^{-1}C^*] \\ &= z^\alpha w^{\alpha^\top}(I_{\mathcal{Y}} - DD^*) - C(I - Z(z)A)Z(z)BD^*z^\alpha w^{\alpha^\top} \\ &\qquad - z^\alpha w^{\alpha^\top} DB^*Z(w)^*(I - A^*Z(w)^*)^{-1}C^* \\ &\qquad - C(I - Z(z)A)^{-1}Z(z)B \cdot z^\alpha w^{\alpha^\top} \cdot B^*Z(w)^*(I - A^*Z(w)^*)^{-1}C^*. \end{aligned}$$

On the other hand, careful bookkeeping and use of the identity

$$I - \mathbf{U}\mathbf{U}^* = \begin{bmatrix} I - AA^* - BB^* & -AC^* - BD^* \\ -CA^* - DB^* & I - CC^* - DD^* \end{bmatrix}$$

gives that

$$\begin{aligned} Y_\alpha = C(I - Z(z)A)^{-1}Z(z) \cdot z^\alpha w^{\alpha^\top} \cdot (I - AA^* - BB^*)Z(w)^*(I - A^*Z(w)^*)^{-1}C^* \\ - C(I - Z(z)A)^{-1}Z(z)(AC^* + BD^*)z^\alpha w^{\alpha^\top} \\ - z^\alpha w^{\alpha^\top}(CA^* + DB^*)Z(w)^*(I - A^*Z(w)^*)^{-1}C^* \\ + z^\alpha w^{\alpha^\top}(I - CC^* - DD^*). \end{aligned}$$

Further careful bookkeeping then shows that

$$\begin{aligned} X_\alpha - Y_\alpha &= z^\alpha w^{\alpha^\top} CC^* + C(I - Z(z)A)^{-1}Z(z)AC^*z^\alpha w^{\alpha^\top} \\ &\quad + z^\alpha w^{\alpha^\top} CA^*Z(w)^*(I - A^*Z(w)^*)^{-1}C^* \\ &\quad - C(I - Z(z)A)^{-1}Z(z) \cdot z^\alpha w^{\alpha^\top} \cdot (I - AA^*)Z(w)^*(I - A^*Z(w)^*)^{-1}C^* \\ &= C(I - Z(z)A)^{-1}(z^\alpha w^{\alpha^\top} I_{\mathcal{X}} - Z(z)z^\alpha w^{\alpha^\top} Z(w)^*)(I - A^*Z(w)^*)^{-1}C^*. \end{aligned} \tag{3.15}$$

Note that

$$Z(z) \cdot z^\alpha w^{\alpha^\top} \cdot Z(w)^* = \sum_{k=1}^d z_k z^\alpha w^{\alpha^\top} w_k$$

and hence

$$\sum_{\alpha\in\mathcal{F}_d\colon |\alpha|=N} Z(z)z^\alpha w^{\alpha^\top} Z(w)^* = \sum_{\alpha\in\mathcal{F}_d\colon |\alpha|=N+1} z^\alpha w^{\alpha^\top} I_{\mathcal{X}}.$$

Therefore,

$$\begin{aligned} &\sum_{\alpha\in\mathcal{F}_d} z^\alpha w^{\alpha^\top} I_{\mathcal{X}} - \sum_{\alpha\in\mathcal{F}_d} Z(z)z^\alpha w^{\alpha^\top} Z(w)^* \\ &\qquad = \sum_{N=0}^\infty \sum_{\alpha\in\mathcal{F}_d\colon |w|=N} z^\alpha w^{\alpha^\top} I_{\mathcal{X}} - \sum_{N=1}^\infty \sum_{\alpha\in\mathcal{F}_d\colon |w|=N} z^\alpha w^{\alpha^\top} I_{\mathcal{X}} = I_{\mathcal{X}}. \end{aligned} \tag{3.16}$$

Summing (3.15) and combining with (3.16) gives the result (3.14) as wanted. $\square$

Given a d-tuple of operators $A_1, \dots, A_d$ on the Hilbert space $\mathcal{X}$, we let $\mathbf{A} = (A_1, \dots, A_d)$ denote the operator d-tuple while A denotes the associated column matrix as in (3.8) considered as an operator from $\mathcal{X}$ into $\mathcal{X}^d$. If C is an operator from $\mathcal{X}$ into an output space $\mathcal{Y}$, we say that $(C, \mathbf{A})$ is an output pair. The paper [7] studied output pairs and connections with the associated state-output noncommutative linear system (3.9). We are particularly interested in the case where in addition $(C, \mathbf{A})$ is *contractive*, i.e.,

$$A_1^* A_1 + \cdots + A_d^* A_d + C^* C \leq I_{\mathcal{X}}. \tag{3.17}$$

In this case we have the following result.

Proposition 3.3. *Suppose that $(C, \mathbf{A})$ is a contractive output pair. Then:*

(1) *The observability operator*

$$\mathcal{O}_{C,\mathbf{A}} \colon x \mapsto \sum_{\alpha \in \mathcal{F}_d} (C\mathbf{A}^v x) z^\alpha = C(I - Z(z)A)^{-1} x \tag{3.18}$$

maps $\mathcal{X}$ contractively into $H^2_{\mathcal{Y}}(\mathcal{F}_d)$.

(2) *The space* Ran $\mathcal{O}_{C,\mathbf{A}}$ *is a NFRKHS with norm given by*

$$\|\mathcal{O}_{C,\mathbf{A}} x\|_{\mathcal{H}(K_{C,\mathbf{A}})} = \|Qx\|_{\mathcal{X}}$$

where Q is the orthogonal projection onto (Ker $\mathcal{O}_{C,\mathbf{A}})^\perp$ *and with formal reproducing kernel $K_{C,A}$ given by*

$$K_{C,\mathbf{A}}(z, w) = C(I - Z(z)A)^{-1}(I - Z(w)^* A^*)^{-1} C^*. \tag{3.19}$$

(3) *$\mathcal{H}(K_{C,\mathbf{A}})$ is invariant under the backward shift operators S_j^* given by* (3.3) *for $j = 1, \dots, d$ and moreover the difference-quotient inequality*

$$\sum_{j=1}^d \|S_j^* f\|^2_{\mathcal{H}(K_{C,\mathbf{A}})} \leq \|f\|^2_{\mathcal{H}(K_{C,\mathbf{A}})} - \|f_\emptyset\|^2_{\mathcal{Y}} \quad \textit{for all} \quad f \in \mathcal{H}(K_{C,\mathbf{A}}) \tag{3.20}$$

is satisfied.

(4) *$\mathcal{H}(K_{C,\mathbf{A}})$ is isometrically included in $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ if and only if in addition $\mathbf{A}$ is* strongly stable*, i.e.,*

$$\lim_{N \to \infty} \sum_{\alpha \in \mathcal{F}_d \colon |\alpha| = N} \|\mathbf{A}^v x\|^2 = 0 \quad \textit{for all} \quad x \in \mathcal{X}. \tag{3.21}$$

Proof. We refer the reader to [7, Theorem 2.10] for complete details of the proof. Here we only note that the backward-shift-invariance property in part (3) is a consequence of the intertwining relation

$$S_j^* \mathcal{O}_{C,\mathbf{A}} = \mathcal{O}_{C,\mathbf{A}} A_j \quad \text{for} \quad j = 1, \dots, d \tag{3.22}$$

and that, in the observable case, (3.20) is equivalent to the contractivity property (3.17) of $(C, \mathbf{A})$. □

The paper [7] studies the NFRKHSs $\mathcal{H}(K)$ where the kernel K has the special form $K_{C,\mathbf{A}}$ for a contractive output pair as in (3.19). Here we wish to study the noncommutative analogues of de Branges-Rovnyak spaces $\mathcal{H}(K_S)$ with K_S given by (3.5).

The following corollary to Proposition 3.2 gives a connection between kernels of the form $K_{C,\mathbf{A}}$ for a contractive output pair $(C, \mathbf{A})$ and kernels of the form K_S for a noncommutative Schur-class multiplier $S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$.

Corollary 3.4. *Suppose that the operator* $\mathbf{U}$ *of the form* (3.6) *is contractive with associated noncommutative Schur multiplier* $S(z)$ *given by* (3.7). *Suppose that the associated output-pair* $(C, \mathbf{A})$ *with* $\mathbf{A} = (A_1, \dots, A_d)$ *is observable (i.e., the observability operator* $\mathcal{O}_{C,\mathbf{A}}$ *given by* (3.18) *is injective). Then the associated kernels* $K_S(z,w)$ *and* $K_{C,\mathbf{A}}(z,w)$ *given by* (3.5) *and* (3.19) *are the same*

$$K_S(z,w) = K_{C,\mathbf{A}}(z,w) \tag{3.23}$$

if and only if $\mathbf{U}$ *is coisometric.*

Proof. By Proposition 3.2 the identity of kernels (3.23) holds if and only if the defect kernel $D_S(z,w)$ defined in (3.12) is zero. Let us partition $I - \mathbf{U}\mathbf{U}^*$ as a $(d+1) \times (d+1)$ block matrix with respect to the $(d+1)$-fold decomposition $\mathcal{X}^d \oplus \mathcal{Y}$ of its domain and range spaces

$$I - \mathbf{U}\mathbf{U}^* = [M_{i,j}]_{1 \le i,j \le d+1}$$

and let us write $D_S(z,w)$ as a formal power series

$$D_S(z,w) = \sum_{v,v' \in \mathcal{F}_d} D_{v,v'} z^v w^{v'}.$$

It follows from (3.12) that $D_{v,v'}$ is given by

$$\begin{aligned} D_{v,v'} = & \sum_{\beta,\alpha,\gamma \in \mathcal{F}_d, i,j \in \{1,\dots,d\}\colon \beta i \alpha = v, \alpha^\top j \gamma^\top = v'} CA^\beta M_{i,j} A^{*\gamma^\top} C^* \\ & + \sum_{\beta \in \mathcal{F}_d, i \in \{1,\dots,d\}\colon \beta i = v(v'^\top)^{-1}} M_{i,d+1} \\ & + \sum_{j \in \{1,\dots,d\}, \beta \in \mathcal{F}_d\colon j\gamma^\top = v'(v^\top)^{-1}} M_{d+1,j} + M_{d+1,d+1}, \end{aligned}$$

where in general we write

$$vw^{-1} = \begin{cases} v' & \text{if } v = v'w \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Considering the case $v = v' = \emptyset$ leads to $M_{d+1,d+1} = 0$. Considering next the case $v = i_0$, $v' = \emptyset$ leads to $M_{i_0,d+1} = 0$ for $i_0 = 1, \dots, d$. Similarly, the case $v = \emptyset$, $v' = j_0$ leads to $M_{d+1,j_0} = 0$ for $j_0 = 1, \dots, d$. Considering next the case $v = i_0$, $v' = j_0$ leads to $CM_{i_0,j_0}C^* = 0$ for all $i_0, j_0 = 1, \dots, d$, and hence $C(I - \mathbf{U}\mathbf{U}^*)C^* = 0$. The general case together with an induction argument on the length of words leads to the general collapsing

$$CA^\beta (I - \mathbf{U}\mathbf{U}^*) A^{*\gamma^\top} C^* = 0.$$

The observability assumption then forces $I - \mathbf{U}\mathbf{U}^* = 0$, i.e., that $\mathbf{U}$ is coisometric as wanted. □

Alternatively, we can suppose that we know only the contractive output pair $(C, \mathbf{A})$ and we seek to find a noncommutative Schur multiplier $S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ so that (3.23) holds. We start with a preliminary result.

Theorem 3.5. *Let* $(C, \mathbf{A})$ *with* $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *be a contractive output-pair. Then there exists an input space* $\mathcal{U}$ *and an* $S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ *so that*

$$K_S(z,w) = K_{C,\mathbf{A}}(z,w). \tag{3.24}$$

Proof. By the result of Corollary 3.4, it suffices to find an input space $\mathcal{U}$ and an operator $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] : \mathcal{U} \to \mathcal{X}^d \oplus \mathcal{Y}$ so that $\mathbf{U} := \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathcal{X} \oplus \mathcal{U} \to \mathcal{X}^d \oplus \mathcal{Y}$ is a coisometry. The details for such a coisometry-completion problem are carried out in the proof of Theorem 2.1 in [8]. $\square$

We now consider the situation where we are given a contractive output-pair $(C, \mathbf{A})$ and a noncommutative Schur multiplier $S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ so that (3.24) holds.

Lemma 3.6. *Let*

$$F(z) = \sum_{v \in \mathcal{F}_d} F_v z^v \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \quad \textit{and} \quad G(z) = \sum_{v \in \mathcal{F}_d} G_v z^v \in \mathcal{L}(\mathcal{U}', \mathcal{Y})$$

be two formal power series. Then the formal power series identity

$$F(z)F(w)^* = G(z)G(w)^* \tag{3.25}$$

is equivalent to the existence of a (necessarily unique) isometry V from

$$\mathcal{D}_V := \overline{\text{span}}_{v \in \mathcal{F}_d} \, \text{Ran } F_v^* \subset \mathcal{U} \quad \textit{onto} \quad \mathcal{R}_V := \overline{\text{span}}_{v \in \mathcal{F}_d} \, \text{Ran } G_v^* \subset \mathcal{U}'$$

so that the identity of formal power series

$$V F(w)^* = G(w)^* \tag{3.26}$$

holds.

Proof. If there is an isometry V satisfying (3.26), equating coefficients of $v^\top$ gives

$$V F_v^* = G_v^*.$$

The isometric property of V then leads to

$$F_{v'} F_v^* = G_{v'} G_v^* \quad \text{for all} \quad v, v' \in \mathcal{F}_d \tag{3.27}$$

from which we get

$$\sum_{v', v \in \mathcal{F}_d} F_{v'} F_v^* z^{v'} w^{v^\top} = \sum_{v', v \in \mathcal{F}_d} G_{v'} G_v^* z^{v'} w^{v^\top}$$

which is the same as (3.25) written out in coefficient form.

Conversely, the assumption (3.25) leads to (3.27). Then the formula

$$V \colon F_v^* y \mapsto G_v^* y \quad \text{for} \quad v \in \mathcal{F}_d \text{ and } y \in \mathcal{Y} \tag{3.28}$$

extends by linearity and continuity to a well-defined isometry (still denoted by V) from $\mathcal{D}_V$ onto $\mathcal{R}_V$. Since identification of coefficients of z^v on both sides of (3.26) reduces to (3.28), we see that (3.26) follows as wanted. $\square$

Lemma 3.7. *Let $(C, \mathbf{A})$ be a contractive output pair and $S \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ a formal power series. Then the following are equivalent:*

(1) (3.24) *holds, i.e.,*

$$C(I - Z(z)A)^{-1}(I - A^* Z(w)^*)^{-1} C^* = k_{Sz}(z, w) I_{\mathcal{Y}} - S(z) k_{Sz}(z, w) S(w)^*. \tag{3.29}$$

(2) *The alternative version of* (3.29) *holds:*

$$C(I - Z(z)A)^{-1}(I_{\mathcal{X}} - Z(z)Z(w)^*)(I - A^* Z(w)^*)^{-1} C^* = I - S(z)S(w)^*. \tag{3.30}$$

(3) *There is an isometry*

$$V = \begin{bmatrix} A_V & B_V \\ C_V & D_V \end{bmatrix} : [\overline{\mathrm{Ran}}(\mathcal{O}_{C,\mathbf{A}})^*]^d \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{U}$$

so that we have the identity of formal power series:

$$\begin{bmatrix} A_V & B_V \\ C_V & D_V \end{bmatrix} \begin{bmatrix} Z(w)^*(I - A^*Z(w)^*C^* \\ I_{\mathcal{Y}} \end{bmatrix} = \begin{bmatrix} (I - A^*Z(w)^*)^{-1}C^* \\ S(w)^* \end{bmatrix}. \tag{3.31}$$

Proof. **(1)** $\Longleftrightarrow$ **(2):** Suppose that (3.29) holds. Then

$$\begin{aligned} & C(I - Z(z)A)^{-1}Z(z)Z(w)^*(I - A^*Z(w)^*)^{-1}C^* \\ & \quad = \sum_{k=1}^d w_k C(I - Z(z)A)^{-1}(I - A^*Z(w)^*)^{-1}C^* z_k \\ & \quad = \sum_{k=1}^d w_k k_{\mathrm{Sz}}(z,w) z_k - S(z)\left(\sum_{k=1}^d w_k k_{\mathrm{Sz}}(z,w) z_k\right) S(w)^* \\ & \quad = (k_{\mathrm{Sz}}(z,w) - 1)I_{\mathcal{Y}} - S(z)\left(k_{\mathrm{Sz}}(z,w) - 1\right) S(w)^* \end{aligned}$$

and consequently,

$$\begin{aligned} & C(I - Z(z)A)^{-1}(I - Z(z)Z(w)^*)(I - A^*Z(w)^*)^{-1}C^* \\ & \quad = k_{\mathrm{Sz}}(z,w)I_{\mathcal{Y}} - S(z)k_{\mathrm{Sz}}(z,w)S(w)^* \\ & \qquad - [(k_{\mathrm{Sz}}(z,w) - 1)I_{\mathcal{Y}} - S(z)\left(k_{\mathrm{Sz}}(z,w) - 1\right) S(w)^*] \\ & \quad = I_{\mathcal{Y}} - S(z)S(w)^* \end{aligned}$$

and we recover (3.30) as desired.

Conversely, assume that (3.30) holds. Multiplication of (3.30) on the left by $w^{\gamma^\top}$ and on the right by z^γ gives

$$\begin{aligned} & C(I - Z(z)A)^{-1}\left(z^\gamma w^{\gamma^\top} I_{\mathcal{X}} - Z(z)z^\gamma w^{\gamma^\top} Z(w)^*\right)(I - A^*Z(w)^*)^{-1}C^* \\ & \quad = z^\gamma w^{\gamma^\top} I_{\mathcal{Y}} - S(z)z^\gamma w^{\gamma^\top} S(w)^*. \end{aligned} \tag{3.32}$$

Summing up (3.32) over all $\gamma \in \mathcal{F}_d$ leaves us with (3.29). This completes the proof of (1) $\Longleftrightarrow$ (2).

(2) $\Longleftrightarrow$ **(3):** Observe that (3.30) can be written in equivalent block matrix form as

$$\begin{aligned} & \begin{bmatrix} C(I - Z(z)A)^{-1}Z(z) & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} Z(w)^*(I - A^*Z(w)^*)^{-1}C^* \\ I_{\mathcal{Y}} \end{bmatrix} \\ & \quad = \begin{bmatrix} C(I - Z(z)A)^{-1} & S(z) \end{bmatrix} \begin{bmatrix} (I - A^*Z(w)^*)^{-1}C^* \\ S(w)^* \end{bmatrix}. \end{aligned}$$

Now we apply Lemma 3.6 to the particular case

$$F(w)^* = \begin{bmatrix} Z(w)^*(I - A^*Z(w)^*)^{-1}C^* \\ I_{\mathcal{Y}} \end{bmatrix}, \qquad G(w)^* = \begin{bmatrix} (I - A^*Z(w)^*)^{-1}C^* \\ S(w)^* \end{bmatrix}$$

to see the equivalence of (2) and (3). It is easily checked that $\mathcal{D}_V$ for our case here is the d-fold inflation of the observability subspace inside $\mathcal{X}^d$:

$$\mathcal{D}_V = [\overline{\mathrm{span}}_{v \in \mathcal{F}_d} \, \mathrm{Ran}\, \mathbf{A}^{*v} C^*]^d \oplus \mathcal{Y} = [\overline{\mathrm{Ran}}\, \mathcal{O}_{C,\mathbf{A}})^*]^d \oplus \mathcal{Y}.$$

□

Theorem 3.8. *Suppose that $S(z) \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ and that $(C, \mathbf{A})$ is an observable, contractive output-pair such that (3.24) holds. Then there exists a unique operator $B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix} : \mathcal{U} \to \mathcal{X}^d$ so that $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & s_\emptyset \end{smallmatrix}\right]$ is a coisometry and $\mathbf{U}$ provides a realization for S: $S(z) = s_\emptyset + C(I - Z(z)A)^{-1}Z(z)B$.*

Proof. We are given the operators $A\colon \mathcal{X} \to \mathcal{X}^d$, $C\colon \mathcal{X} \to \mathcal{Y}$ and $D = s_\emptyset\colon \mathcal{U} \to \mathcal{Y}$ and seek an operator $B\colon \mathcal{U} \to \mathcal{X}^d$ so that $S(z) = D + C(I - Z(z)A)^{-1}Z(z)B$, or, what is the same, so that

$$S(w)^* = D^* + B^*Z(w)^*(I - A^*Z(w)^*)^{-1}C^*.$$

This last identity can be rewritten as

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} Z(w)^*(I - A^*Z(w)^*)^{-1}C^* \\ I_\mathcal{Y} \end{bmatrix} = \begin{bmatrix} (I - A^*Z(w)^*)^{-1}C^* \\ S(w)^* \end{bmatrix} \tag{3.33}$$

since the identity

$$A^*Z(w)^*(I - A^*Z(w)^*)^{-1}C^* + C^* = (I - A^*Z(w)^*)^{-1}C^*$$

expressing equality of the top components holds true automatically. Lemma 3.7 tells us that there is an isometry $V = \left[\begin{smallmatrix} A_V & B_V \\ C_V & D_V \end{smallmatrix}\right] : \mathcal{X}^d \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{U}$ which has the same action as desired by $\left[\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right]$ in (3.33). It suffices to set $B^* = C_V$. □

We say that two colligations $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathcal{X} \oplus \mathcal{U} \to \mathcal{X}^d \oplus \mathcal{Y}$ and $\mathbf{U}' = \left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]$ are *unitarily equivalent* if there is a unitary operator $U\colon \mathcal{X} \to \mathcal{X}'$ such that

$$\begin{bmatrix} \oplus_{k=1}^d U & 0 \\ 0 & I_\mathcal{Y} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_\mathcal{U} \end{bmatrix}.$$

Corollary 3.9. *Any two observable, coisometric realizations $\mathbf{U}$ and $\mathbf{U}'$ for the same $S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ are unitarily equivalent.*

Proof. Suppose that $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and $\mathbf{U}' = \left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]$ are two such realizations. From Proposition 3.2 we see that

$$K_{C,\mathbf{A}}(z, w) = K_{C',\mathbf{A}'}(z, w).$$

Then Theorem 2.13 of [7] implies that $(C, \mathbf{A})$ is unitarily equivalent to $(C', \mathbf{A}')$, so there is a unitary operator $U\colon \mathcal{X} \to \mathcal{X}'$ such that

$$C' = CU^* \quad \text{and} \quad A'_j = UA_jU^* \quad \text{for} \quad j = 1, \ldots, d.$$

Then $\widetilde{\mathbf{U}} = \left[\begin{smallmatrix} A' & (\oplus_{k=1}^d U)B \\ C' & D \end{smallmatrix}\right]$ and $\mathbf{U}' = \left[\begin{smallmatrix} A' & B' \\ C' & D \end{smallmatrix}\right]$ both give coisometric realizations of S with the same observable output pair $(C', \mathbf{A}')$. By the uniqueness assertion of Theorem 3.5, it follows that $B' = (\oplus_{k=1}^d U)B$ as well, and hence $\mathbf{U}$ and $\mathbf{U}'$ are unitarily equivalent. □

4. de Branges-Rovnyak model colligations

In this section we show that any $S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ has a canonical observable, coisometric realization which uses $\mathcal{H}(K_S)$ as the state space. We first need some preliminaries concerning the finer structure of the noncommutative de Branges-Rovnyak functional-model spaces $\mathcal{H}(K_S)$. Let us denote the Taylor coefficients of $S(z)$ as s_v, so

$$S(z) = \sum_{v \in \mathcal{F}_d} s_v z^v,$$

to avoid confusion with the (right) shift operators $S_j \colon f(z) \mapsto f(z) \cdot z_j$.

Just as in the classical case, the de Branges-Rovnyak space $\mathcal{H}(K_S)$ has several equivalent characterizations.

Proposition 4.1. *Let $S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ and let $\mathcal{H}$ be a Hilbert space of formal power series in $\mathcal{Y}\langle\langle z \rangle\rangle$. Then the following are equivalent.*

(1) *$\mathcal{H}$ is equal to the NFRKHS $\mathcal{H}(K_S)$ isometrically, where $K_S(z,w)$ is the noncommutative positive kernel given by* (3.5).

(2) *$\mathcal{H} = \operatorname{Ran}(I - M_S M_S^*)^{1/2}$ with lifted norm*

$$\|(I - M_S M_S)^{1/2} g\|_{\mathcal{H}} = \|Qg\|_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} \tag{4.1}$$

where Q is the orthogonal projection of $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ onto $(\operatorname{Ker}(I - M_S M_S^*)^{1/2})^{\perp}$.

(3) *$\mathcal{H}$ is the space of all formal power series $f(z) \in \mathcal{Y}\langle\langle z \rangle\rangle$ with finite $\mathcal{H}$-norm, where the $\mathcal{H}$-norm is given by*

$$\|f\|^2_{\mathcal{H}} = \sup_{g \in H^2_{\mathcal{U}}(\mathcal{F}_d)} \left\{ \|f + M_S g\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} - \|g\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)} \right\}. \tag{4.2}$$

Proof. **(1) $\Longleftrightarrow$ (2):** It is straightforward to verify the identity

$$(I - M_S M_S^*)(k_{\mathrm{Sz}}(\cdot, w) y) = K_S(\cdot, w) y \text{ for each } y \in \mathcal{Y}.$$

(The interpretation for this is that, for each word γ, the coefficient of w^γ of the left hand side agrees with the coefficient of w^γ on the right hand side as elements of $\mathcal{H}(k_{\mathrm{Sz}} I_{\mathcal{Y}}) = H^2_{\mathcal{Y}}(\mathcal{F}_d)$—see [14]). We then see that

$$\begin{aligned}
&\langle (I - M_S M_S^*) k_{\mathrm{Sz}}(\cdot, w') y', (I - M_S M_S^*) k_{\mathrm{Sz}}(\cdot, w) y \rangle_{\mathcal{H}(K_S)\langle\langle w' \rangle\rangle \times \mathcal{H}(K_S)\langle\langle w \rangle\rangle)} \\
&\quad = \langle K_S(\cdot, w') y', K_S(\cdot, w) y \rangle_{\mathcal{H}(K_S)\langle\langle w' \rangle\rangle \times \mathcal{H}(K_S)\langle\langle w \rangle\rangle} \\
&\quad = \langle K_S(w, w') y', y \rangle_{\mathcal{Y}\langle\langle w', w \rangle\rangle \times \mathcal{Y}} \\
&\quad = \langle K_S(\cdot, w') y', k_{\mathrm{Sz}}(\cdot, w) y \rangle_{H^2_{\mathcal{Y}}(\mathcal{F}_d)\langle\langle w' \rangle\rangle \times H^2_{\mathcal{Y}}(\mathcal{F}_d)\langle\langle w \rangle\rangle} \\
&\quad = \langle (I - M_S M_S^*) k_{\mathrm{Sz}}(\cdot, w') y', k_{\mathrm{Sz}}(\cdot, w) y \rangle_{(H^2_{\mathcal{Y}}(\mathcal{F}_d)\langle\langle w' \rangle\rangle \times H^2_{\mathcal{Y}}(\mathcal{F}_d)\langle\langle w \rangle\rangle)}.
\end{aligned}$$

It follows that $\operatorname{Ran}(I - M_S M_S^*) \subset \mathcal{H}(K_S)$ with

$$\langle (I - M_S M_S^*) g, (I - M_S M_S^*) g' \rangle_{\mathcal{H}(K_S)} = \langle (I - M_S M_S^*) g, g' \rangle_{H^2_{\mathcal{Y}}(\mathcal{F}_d)}$$

for $g, g' \in H^2_{\mathcal{Y}}(\mathcal{F}_d)$. The precise characterization $\mathcal{H}(K_S) = \operatorname{Ran}(I - M_S M_S^*)^{1/2}$ with the lifted norm (4.1) now follows via a completion argument.

(2) $\Longleftrightarrow$ (3): This follows from the argument in [31, NI-6].

$\square$

Proposition 4.2. *Suppose that $S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ and let $\mathcal{H}(K_S)$ be the associated NFRKHS where K_S is given by* (3.5). *Then the following conditions hold:*

(1) *The NFRKHS $\mathcal{H}(K_S)$ is contained contractively in $H^2_{\mathcal{Y}}(\mathcal{F}_d)$:*

$$\|f\|^2_{\mathcal{H}_{\mathcal{Y}}(\mathcal{F}_d)} \le \|f\|^2_{\mathcal{H}(K_S)} \quad \textit{for all } f \in \mathcal{H}(K_S).$$

(2) *$\mathcal{H}(K_S)$ is invariant under each of the backward-shift operators S_j^* given by (3.3) for $j = 1, \dots d$, and moreover, the difference-quotient inequality* (3.20) *holds for $\mathcal{H}(K_S)$:*

$$\sum_{j=1}^{d} \|S_j^* f\|^2_{\mathcal{H}(K_S)} \le \|f\|^2_{\mathcal{H}(K_S)} - \|f_\emptyset\|^2. \tag{4.3}$$

(3) *For each $u \in \mathcal{U}$ and $j = 1, \dots, d$, the vector $S_j^*(M_S u)$ belongs to $\mathcal{H}(K_S)$ with the estimate*

$$\sum_{j=1}^{d} \|S_j^*(M_S u)\|^2_{\mathcal{H}(K_S)} \le \|u\|^2_{\mathcal{U}} - \|s_\emptyset u\|^2_{\mathcal{Y}}. \tag{4.4}$$

Proof. We know from Theorem 3.1 that $S(z)$ can be realized as in (3.6) and (3.7) with $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ a coisometry (or even unitary). From Proposition 3.2 it follows that $K_S(z,w) = K_{C,\mathbf{A}}(z,w)$ and hence $\mathcal{H}(K_S) = \mathcal{H}(K_{C,\mathbf{A}})$ isometrically. Conditions (1) and (2) now follow from the properties of $\mathcal{H}(K_{C,\mathbf{A}})$ listed in Proposition 3.3 and the discussion immediately following.

One can also prove points (1) and (2) directly from the characterization of $\mathcal{H}(K_S)$ in part (3) of Proposition 4.1 (and thereby bypass realization theory) as follows; these proofs follow the proofs for the classical case in [16, 17]. For the contractive inclusion property (part (1)), note that

$$\begin{aligned} \|f\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} &= \left[\|f + M_S g\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} - \|g\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)}\right]\Big|_{g=0} \\ &\le \sup_{g \in H^2_{\mathcal{U}}(\mathcal{F}_d)} \left\{\|f + M_S g\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} - \|g\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)}\right\} = \|f\|^2_{\mathcal{H}(K_S)}. \end{aligned}$$

To verify part (2), we compute

$$\begin{aligned} \sum_{j=1}^{d} \|S_j^* f\|^2_{\mathcal{H}(K_S)} &= \sup_{g_j} \left\{\sum_{j=1}^{d} \left[\|S_j^* f + M_S g_j\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} - \|g_j\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)}\right]\right\} \\ &= \sup_{g_j} \left\{\sum_{j=1}^{d} \left[\|S_j S_j^* f + M_S(g_j z_j)\|^2 - \|g_j z_j\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)}\right]\right\} \\ &= \sup_{g_j} \left\{\sum_{j=1}^{d} \|S_j S_j^* f + M_S(g_j z_j)\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} + \|f_\emptyset\|^2_{\mathcal{Y}} - \sum_{j=1}^{d} \|g_j z_j\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)}\right\} - \|f_\emptyset\|^2_{\mathcal{Y}} \\ &= \sup_{g \in H^2_{\mathcal{Y}}(\mathcal{F}_d) \colon g_\emptyset = 0} \left\{\|f + M_S g\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} - \|g\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)}\right\} - \|f_\emptyset\|^2_{\mathcal{Y}} \\ &\le \sup_{g \in H^2_{\mathcal{Y}}(\mathcal{F}_d)} \left\{\|f + M_S g\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} - \|g\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)}\right\} - \|f_\emptyset\|^2_{\mathcal{Y}} = \|f\|^2_{\mathcal{H}(K_S)} - \|f_\emptyset\|^2_{\mathcal{Y}} \end{aligned}$$

and part (2) of Proposition 4.2 follows.

To verify part (3), we again use the third characterization of $\mathcal{H}(K_S)$ in Proposition 4.1. Pick $g_1, \dots, g_d \in H^2_{\mathcal{U}}(\mathcal{F}_d)$ and let

$$\widetilde{g} = \sum_{j=1}^{d} g_j z_j = \sum_{j=1}^{d} S_j g_j.$$

Since $S_j^* S_i = \delta_{ij} I$ for $i, j = 1, \dots, d$ where δ_{ij} is the Kronecker's symbol, we have

$$\|\widetilde{g}\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)} = \sum_{j=1}^{d} \|S_j g_j\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)} = \sum_{j=1}^{d} \|g_j\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)} \tag{4.5}$$

and, since the multiplication operator M_S commutes with S_j for $j = 1, \dots, d$, we have also

$$\|M_S \widetilde{g}\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)} = \sum_{j=1}^{d} \|M_S g_j\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)}. \tag{4.6}$$

Next we note that

$$\begin{aligned}\|S_j^*(M_S u) + M_S g_j\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} &= \|S_j^*(M_S u)\|^2 + 2\Re\langle S_j^*(M_S u),\, M_S g_j\rangle + \|M_S g_j\|^2 \\ &= \langle S_j S_j^*(M_S u),\, M_S u\rangle + 2\Re\langle M_S u,\, M_S g_j z_j\rangle + \|M_S g_j\|^2.\end{aligned}$$

Summing up the latter equalities for $j = 1, \dots, d$, making use of (4.6) and applying the identity

$$f - f_\emptyset = \sum_{j=1}^{d} S_j S_j^* f \qquad (f \in H^2_{\mathcal{Y}}(\mathcal{F}_d)$$

to $f = M_S u$, we get

$$\begin{aligned}\sum_{j=1}^{d} \|S_j^*(M_S u) + M_S g_j\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} &= \langle M_S u - s_\emptyset u,\, M_S u\rangle + 2\Re\langle M_S u,\, M_S \widetilde{g}\rangle + \|M_S \widetilde{g}\|^2 \\ &= \|M_S u\|^2 - \|s_\emptyset u\|^2 + 2\Re\langle M_S u,\, M_S \widetilde{g}\rangle + \|M_S \widetilde{g}\|^2 \\ &= \|M_S u + M_S \widetilde{g}\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} - \|s_\emptyset u\|^2_{\mathcal{Y}}.\end{aligned} \tag{4.7}$$

Since $\|M_S\|_{op} \le 1$ and since $\widetilde{g}_\emptyset = 0$, we have

$$\begin{aligned}\|M_S u + M_S \widetilde{g}\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} &= \|M_S(u + \widetilde{g})\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} \\ &\le \|u + \widetilde{g}\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)} = \|u\|^2_{\mathcal{U}} + \|\widetilde{g}\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)}.\end{aligned} \tag{4.8}$$

Adding (4.5), (4.7) and (4.8) gives

$$\sum_{j=1}^{d} \Big[\|S_j^*(M_S u) + M_S g_j\|^2_{H^2_{\mathcal{Y}}(\mathcal{F}_d)} - \|g_j\|^2_{H^2_{\mathcal{U}}(\mathcal{F}_d)} \Big] \le \|u\|^2_{\mathcal{U}} - \|s_\emptyset u\|^2_{\mathcal{Y}}.$$

The latter estimate is uniform with respect to g_j's and then taking suprema we conclude (by the third characterization of $\mathcal{H}(K_S)$ in Proposition 4.1) that $S_j^*(M_S u) \in \mathcal{H}(K_S)$ for each $j = 1, \dots, d$ with the estimate

$$\sum_{j=1}^{d} \|S_j^*(M_S u)\|^2_{\mathcal{H}(K_S)} \le \|u\|^2_{\mathcal{U}} - \|s_\emptyset u\|^2_{\mathcal{Y}},$$

This concludes the proof of Proposition 4.2. □

Let us define an operator $E\colon H^2_{\mathcal{Y}}(\mathcal{F}_d) \to \mathcal{Y}$ by

$$E\colon \sum_{v \in \mathcal{F}_d} f_v z^v \mapsto f_\emptyset. \tag{4.9}$$

As is observed in [7, Proposition 2.9] and can be observed directly,

$$E\mathbf{S}^{*v}f = E\left(\sum_{\alpha\in\mathcal{F}_d} f_{\alpha v^\top} z^\alpha\right) = f_{v^\top} \text{ for all } f(z) = \sum_{\alpha\in\mathcal{F}_d} f_\alpha z^\alpha \in \mathcal{H}^2_{\mathcal{Y}}(\mathcal{F}_d) \text{ and } v \in \mathcal{F}_d. \tag{4.10}$$

Hence the observability operator $\mathcal{O}_{E,\mathbf{S}}\colon H^2_{\mathcal{Y}}(\mathcal{F}_d) \to H^2_{\mathcal{Y}}(\mathcal{F}_d)$ defined as in (3.18) works out to be

$$\mathcal{O}_{E,\mathbf{S}^*} = \tau$$

where τ is the involution on $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ given by

$$\tau\colon \sum_{v\in\mathcal{F}_d} f_v z^v \mapsto \sum_{v\in\mathcal{F}_d} f_{v^\top} z^v. \tag{4.11}$$

For this reason we use the "reflected" de Branges-Rovnyak space

$$\mathcal{H}^\tau(K_S) = \tau \circ \mathcal{H}(K_S) := \{\tau(f)\colon f \in \mathcal{H}(K_S)\} \tag{4.12}$$

as the state space for our de Branges-Rovnyak-model realization of S rather than simply $\mathcal{H}(K_S)$ as in the classical case. We define

$$\|\tau(f)\|_{\mathcal{H}^\tau(K_S)} = \|f\|_{\mathcal{H}(K_S)}.$$

Recall that the operator of multiplication on the right by the variable z_j on $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ was denoted in (3.2) by S_j rather than by S_j^R for simplicity. We shall now need its left counterpart, denoted by S_j^L and given by

$$S_j^L\colon f(z) = \sum_{v\in\mathcal{F}_d} f_v z^v \mapsto z_j \cdot f(z) = \sum_{v\in\mathcal{F}_d} f_v z^{j\cdot v} \tag{4.13}$$

with adjoint (as an operator on $H^2_{\mathcal{Y}}(\mathcal{F}_d)$) given by

$$(S_j^L)^*\colon \sum_{v\in\mathcal{F}_d} f_v z^v \mapsto \sum_{v\in\mathcal{F}_d} f_{j\cdot v} z^v. \tag{4.14}$$

For emphasis we now write S_j^R rather than simply S_j. We then have the following result.

Theorem 4.3. *Let $S(z) \in \mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y})$ and let $\mathcal{H}^\tau(K_S)$ be the associated de Branges-Rovnyak space given by* (4.12)*. Define operators*

$$A_{dBR,j}\colon \mathcal{H}^\tau(K_S) \to \mathcal{H}^\tau(K_S), \qquad B_{dBR,j}\colon \mathcal{U} \to \mathcal{H}^\tau(K_S) \quad (j = 1,\dots,d),$$
$$C_{dBR}\colon \mathcal{H}^\tau(K_S) \to \mathcal{Y}, \qquad D_{dBR}\colon \mathcal{U} \to \mathcal{Y}$$

by

$$\begin{aligned} A_{dBR,j} &= (S_j^L)^*|_{\mathcal{H}^\tau(K_S)}, & B_{dBR,j} &= \tau(S_j^R)^* M_S|_{\mathcal{U}} = (S_j^L)^*\tau M_S|_{\mathcal{U}}, \\ C_{dBR} &= E|_{\mathcal{H}^\tau(K_S)}, & D_{dBR} &= s_\emptyset \end{aligned} \tag{4.15}$$

where E is given by (4.9)*, and set*

$$A_{dBR} = \begin{bmatrix} A_{dBR,1} \\ \vdots \\ A_{dBR,d} \end{bmatrix} : \mathcal{H}^\tau(K_S) \to \mathcal{H}^\tau(K_S)^d, \quad B_{dBR} = \begin{bmatrix} B_{dBR,1} \\ \vdots \\ B_{dBR,d} \end{bmatrix} \mathcal{U} \to \mathcal{H}^\tau(K_S)^d.$$

Then

$$\mathbf{U}_{dBR} = \begin{bmatrix} A_{dBR} & B_{dBR} \\ C_{dBR} & D_{dBR} \end{bmatrix} : \begin{bmatrix} \mathcal{H}^\tau(K_S) \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}^\tau(K_S)^d \\ \mathcal{Y} \end{bmatrix}$$

is an observable coisometric colligation with transfer function equal to $S(z)$:

$$S(z) = D_{dBR} + C_{dBR}(I_{\mathcal{H}^{\tau}(K_S)} - Z(z)A_{dBR})^{-1}Z(z)B_{dBR}. \tag{4.16}$$

Any other observable, coisometric realization of S is unitarily equivalent to this functional-model realization of S.

Proof. As observed in Proposition 4.2, $\mathcal{H}(K_S)$ is invariant under S_j^* for each $j = 1, \dots, d$. From the easily checked intertwining relations

$$(S_j^L)^*\tau = \tau(S_j^R)^* \text{ for } j = 1, \dots, d, \tag{4.17}$$

the fact that $\mathcal{H}(K_S)$ is invariant under each $(S_j^R)^*$ implies that $\mathcal{H}^{\tau}(K_S)$ is invariant under each $(S_j^L)^*$ for $j = 1, \dots, d$. Hence the formula for $A_{\mathrm{dBR},j}$ in (4.15) defines an operator on $\mathcal{H}^{\tau}(K_S)$. The first formula for $B_{\mathrm{dBR},j}$ in (4.15) defines an operator from $\mathcal{U}$ into $\mathcal{H}^{\tau}(K_S)$ by part (3) of Proposition 4.2; this is consistent with the second formula as a consequence of (4.17). From (4.10) it follows that the pair $(E, \mathbf{S}^*)$ is observable and therefore, since C and $\mathbf{A}$ are restrictions of E and $\mathbf{S}$ respectively, the pair $(C, \mathbf{A})$ is also observable. Hence, for $u \in \mathcal{U}$, making use of (4.10) gives

$$C_{\mathrm{dBR}}\mathbf{A}_{\mathrm{dBR}}^{*v}B_{\mathrm{dBR},j}u = E(\mathbf{S}^L)^{*v}\tau S_j^*(M_S \cdot u) = s_{v\cdot j}u$$

and it follows that

$$\begin{aligned} D_{\mathrm{dBR}} + C_{\mathrm{dBR}}(I - Z(z)\mathbf{A}_{\mathrm{dBR}})^{-1}Z(z)B_{\mathrm{dBR}} &= s_{\emptyset} + \sum_{j=1}^{\infty}\sum_{v\in\mathcal{F}_d} C_{\mathrm{dBR}}\mathbf{A}_{\mathrm{dBR}}^{v}B_{\mathrm{dBR},j}z^v z_j \\ &= s_{\emptyset} + \sum_{j=1}^{d}\sum_{v\in\mathcal{F}_d} s_{v\cdot j}z^v z_j = S(z) \end{aligned}$$

and (4.16) follows.

By Proposition 4.2 we know that $\mathcal{H}(K_S)$ is contractively included in $H^2_{\mathcal{Y}}(\mathcal{F}_d)$, is invariant under the backward-shift operators $(S_j^R)^*$ given by (3.3) for $j = 1, \dots, d$ with the difference-quotient inequality (4.3) satisfied. Hence, by part (4) of Theorem 2.8 in [7], it follows that the kernels K_S and $K_{C_{\mathrm{dBR}},\mathbf{A}_{\mathrm{dBR}}}$ match:

$$K_S(z, w) = K_{C_{\mathrm{dBR}},\mathbf{A}_{\mathrm{dBR}}}(z, w). \tag{4.18}$$

The fact that $\mathbf{U}_{\mathrm{dBR}}$ is coisometric now follows from Corollary 3.4. Finally, the uniqueness statement in Theorem 4.3 follows from Corollary 3.9. □

Remark 4.4. The proof of Theorem 4.3 assumed knowledge of the candidate operators (4.15) for a realization of S and then amounted to a check that these operators work. We remark here that, once A_{dBR} and B_{dBR} are chosen so that (4.18) holds, one can then solve for $B_{\mathrm{dBR},1} \dots B_{\mathrm{dBR},d}$ according to the prescription (3.33) in the proof of Theorem 3.8:

$$B_{\mathrm{dBR}}^* Z(w)^*(I - A_{\mathrm{dBR}}^* Z(w)^*)^{-1}C^* = S(w)^* - s_{\emptyset}^*$$

to arrive at the formula for $B_{\mathrm{dBR},j}$ $(j = 1, \dots, d)$ in formula (4.15).

Remark 4.5. It is possible to make all the ideas and results of this paper symmetric with respect to "left versus right". Then the multiplication operator M_S given by (2.1) is really the *left* multiplication operator

$$M_S^L = \sum_{v\in\mathcal{F}_d} s_{\alpha}(\mathbf{S}^L)^v \colon f(z) \mapsto S(z) \cdot f(z).$$

It is natural to define the corresponding *right* multiplication operator M_S^R by

$$M_S^R = \sum_{v \in \mathcal{F}_d} s_\alpha (\mathbf{S}^R)^v.$$

In the scalar case $\mathcal{U} = \mathcal{Y} = \mathbb{C}$ where $f(z) \cdot S(z)$ makes sense, we have

$$M_S^R \colon f(z) \mapsto f(z) \cdot (\tau \circ S)(z)$$

while in general we have

$$M_S^R \colon \sum_{v \in \mathcal{F}_d} f_v z^v \mapsto \sum_{v \in \mathcal{F}_d} \left[\sum_{\alpha, \beta \in \mathcal{F}_d \colon \alpha\beta = v} s_{\beta^\top} f_\alpha \right] z^v.$$

The Schur-class $\mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ is really the *left* Schur class $\mathcal{S}^L_{nc,d}(\mathcal{U}, \mathcal{Y})$. The *right* Schur class $\mathcal{S}^R_{nc,d}(\mathcal{U}, \mathcal{Y})$ consists of all formal power series $S(z) = \sum_{v \in \mathcal{F}_d} s_v z^v$ for which the associated *right* multiplication operator $M_S^R = \sum_{v \in \mathcal{F}_d} s_v (\mathbf{S}^R)^v$ has operator norm at most 1. The kernel $K_S(z,w)$ given by (3.5) is really the *left* kernel $K_S^L(z,w)$ given by

$$K_S(z,w) = K_S^L(z,w) = \{[I_\mathcal{Y} - M_S^L (M_S^L)^*](k_{\rm Sz}(\cdot, w))\}(z).$$

It is then natural to define the corresponding *right* kernel

$$K_S^R(z,w) = \{[I_\mathcal{Y} - M_S^R (M_S^R)^*](k_{\rm Sz}(\cdot, w))\}(z).$$

Given an output pair $(C, \mathbf{A})$, the observability operator $\mathcal{O}_{C,\mathbf{A}}$ given by (3.18) is really the *left* observability operator $\mathcal{O}^L_{C,\mathbf{A}}$ with range space invariant under the *right* backward-shift operators $(S_j^R)^*$; the corresponding *right* observability operator $\mathcal{O}^R_{C,\mathbf{A}}$ is given by

$$\mathcal{O}^R_{C,\mathbf{A}} \colon x \mapsto \sum_{\alpha \in \mathcal{F}_d} (C \mathbf{A}^{v^\top} x) z^\alpha = C(I - Z(\mathbf{S}^R)A)^{-1} x$$

and has range space invariant under the *left* backward shifts $(S_j^L)^*$. The system (3.9) is really a *left* noncommutative multidimensional linear system with *left* transfer function (3.10)

$$T_{\Sigma^L}(z) = D + C(I - Z(\mathbf{S}^L)A)^{-1} Z(\mathbf{S}^L) B.$$

For a given colligation $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, there is an associated right transfer function

$$T_{\Sigma^R}(z) = D + C(I - Z(\mathbf{S}^R)A)^{-1} Z(\mathbf{S}^R) B$$

associated with the *right* noncommutative multidimensional linear system

$$\Sigma^R \colon \left\{ \begin{array}{ccc} x(\alpha \cdot 1) & = & A_1 x(\alpha) + B_1 u(\alpha) \\ \vdots & & \vdots \\ x(\alpha \cdot d) & = & A_d x(\alpha) + B_d u(\alpha) \\ y(\alpha) & = & C x(\alpha) + D u(\alpha) \end{array} \right. \tag{4.19}$$

initialized with $x(\emptyset) = 0$. With these definitions in place, it is straightforward to formulate and prove mirror-reflected versions of Theorem 3.1, Proposition 3.3, Theorem 3.5, Theorem 3.8 (as well as Theorems 5.1 and 5.2 to come below); we leave the details to the reader. With all this in hand, it is then possible to identify the state-space $\mathcal{H}^\tau(K_S) = \tau \circ \mathcal{H}(K_S^L)$ appearing in Theorem 4.3 as nothing other than $\mathcal{H}(K_S^R)$. Thus, the functional-model realization for a given S as an element of the *left* Schur class $\mathcal{S}^L_{nc,d}(\mathcal{U}, \mathcal{Y})$ uses as state space the functional-model space $\mathcal{H}(K_S^R)$

based on the *right* kernel K_S^R while the realization of S as a member of the *right* Schur-class $\mathcal{S}_{nc,d}^R(\mathcal{U},\mathcal{Y})$ uses as the state space the functional-model $\mathcal{H}(K_S^L)$ based on the *left* kernel K_S^L. Presumably it is possible to have an S in the left Schur-class $\mathcal{S}_{nc,d}^L(\mathcal{U},\mathcal{Y})$ but not in the right Schur-class $\mathcal{S}_{nc,d}^R(\mathcal{U},\mathcal{Y})$ and vice-versa, although we have not worked out an example. With this interpretation, the functional-model realization in Theorem 4.3 becomes a more canonical extension of the classical univariate case.

Let us say that $S \in \mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y})$ is *inner* if the multiplication operator

$$M_S\colon H^2_{\mathcal{U}}(\mathcal{F}_d) \to H^2_{\mathcal{Y}}(\mathcal{F}_d)$$

is isometric; such multipliers are the representers for shift-invariant subspaces in Popescu's Fock-space analogue of the Beurling-Lax theorem [26] (see also [7]). It is now an easy matter to characterize which functional-model realizations as in Theorem 4.3 go with inner multipliers.

Theorem 4.6. *The Schur-class multiplier $S \in \mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y})$ is inner if and only if S has an observable, coisometric realization* (3.7) *such that $\mathbf{A} = (A_1,\dots,A_d)$ is strongly stable (see* (3.21)*).*

Proof. By Corollary 3.9, any observable, coisometric realization is unitarily equivalent to the functional-model realization given in Proposition 4.2. Note that S is inner if and only if $I - M_S M_S^*$ is an orthogonal projection. From the characterization of $\mathcal{H}(K_S)$ in part (2) of Proposition 4.1, we see that this last condition occurs if and only if $\mathcal{H}(K_S)$ is contained isometrically in $H^2_{\mathcal{Y}}(\mathcal{F}_d)$. By part (3) of Proposition 3.3, this in turn is equivalent to strong stability of $\mathbf{A}$, and Theorem 4.6 follows. □

5. Shift-invariant subspaces and Beurling-Lax representation theorems

Suppose that $(\mathbf{Z},X)$ is an isometric input pair, i.e., $\mathbf{Z} = (Z_1,\dots,Z_d)$ where each $Z_j\colon \mathcal{X}\to\mathcal{X}$ and $X\colon \mathcal{Y}\to\mathcal{X}$. We say that the input pair $(\mathbf{Z},X)$ is *input-stable* if the associated controllability operator

$$\mathcal{C}_{\mathbf{Z},X}\colon \sum_{v\in\mathcal{F}_d} f_v z^v \mapsto \sum_{v\in\mathcal{F}_d} \mathbf{Z}^{v^\top} X f_v$$

maps $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ into $\mathcal{X}$. We say that the pair $(\mathbf{Z},X)$ is *exactly controllable* if in addition $\mathcal{C}_{\mathbf{Z},X}$ maps $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ onto $\mathcal{X}$. In this case the associated controllability gramian

$$\mathcal{G}_{\mathbf{Z},X} := \mathcal{C}_{\mathbf{Z},X}(\mathcal{C}_{\mathbf{Z},X})^*$$

is strictly positive-definite on $\mathcal{X}$. and is the unique solution $H = \mathcal{G}_{\mathbf{Z},X}$ of the Stein equation

$$H - Z_1 H Z_1^* - \cdots - Z_d H Z_d^* = XX^*. \tag{5.1}$$

By considering the similar pair

$$(\mathbf{Z}',X') \text{ with } \mathbf{Z}' = (Z_1',\dots,Z_d') \text{ where } Z_j' = H^{-1/2} Z_j H^{1/2} \text{ and } X' = H^{-1/2}X,$$

without loss of generality we may assume that the input pair $(\mathbf{Z},X)$ is *isometric*, i.e., (5.1) is satisfied with $H = I_{\mathcal{X}}$. We are interested in the case when in addition $\mathbf{Z}^*$ is *strongly stable* in the sense of (3.21); in this case $\mathcal{G}_{\mathbf{Z},X}$ is the unique solution of the Stein equation (5.1). We remark that all these statements are dual to the analogous statements made for observability operators $\mathcal{O}_{C,\mathbf{A}}$ since the adjoint $(C, \mathbf{A}) := (X^*, \mathbf{Z}^*)$ of any input pair $(\mathbf{Z}, X)$ is an output pair.

Given any isometric input pair $(\mathbf{Z}, X)$ with $\mathbf{Z}^*$ strongly stable, we define a *left functional calculus with operator argument* as follows. Given $f \in H^2_{\mathcal{Y}}(\mathcal{F}_d)$ of the form $f(z) = \sum_{v \in \mathcal{F}_d} f_v z^v$, define

$$(Xf)^{\wedge L}(\mathbf{Z}) = \sum_{v \in \mathcal{F}_d} \mathbf{Z}^{v^\top} X f_v =: \mathcal{C}_{\mathbf{Z},X} f.$$

We define a subspace $\mathcal{M}_{\mathbf{Z},X}$ to be the set of all solutions of the associated homogeneous interpolation condition:

$$\mathcal{M}_{\mathbf{Z},X} := \{f \in H^2_{\mathcal{Y}}(\mathcal{F}_d) \colon (Xf)^{\wedge L}(\mathbf{Z}) = 0\}.$$

That $\mathcal{M}_{\mathbf{Z},X}$ is invariant under the (right) shift operator S_j follows from the intertwining property $\mathcal{C}_{\mathbf{Z},X} S_j = Z_j \mathcal{C}_{\mathbf{Z},X}$ verified by the following computation:

$$\begin{aligned}\mathcal{C}_{\mathbf{Z},X} S_j f = (X S_j f)^{\wedge L}(\mathbf{Z}) &= \sum_{v \in \mathcal{F}_d} \mathbf{Z}^{(vj)^\top} X f_v = Z_j \cdot \sum_{v \in \mathcal{F}_d} \mathbf{Z}^{v^\top} X f_v \\ &= Z_j \cdot (Xf)^{\wedge L}(\mathbf{Z}) = Z_j \mathcal{C}_{\mathbf{Z},X} f.\end{aligned}$$

It is easily checked that $\mathcal{M}_{\mathbf{Z},X}$ is closed in the metric of $H^2_{\mathcal{Y}}(\mathcal{F}_d)$. Hence, by Popescu's Beurling-lax theorem for the Fock space (see [26]) it is guaranteed that $\mathcal{M}_{\mathbf{Z},X}$ has a representation of the form

$$\mathcal{M}_{\mathbf{Z},X} = \theta \cdot H^2_{\mathcal{U}}(\mathcal{F}_d) = \text{Ran } M_\theta$$

for an inner multiplier $\theta \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$. Our goal is to understand how to compute a transfer-function realization for θ directly from the homogeneous interpolation data $(\mathbf{Z}, X)$. First, however, we show that shift-invariant subspaces $\mathcal{M} \subset H^2_{\mathcal{Y}}(\mathcal{F}_d)$ of the form $\mathcal{M} = \mathcal{M}_{\mathbf{Z},X}$ for an admissible input pair $(\mathbf{Z}, X)$ as above are not as special as may at first appear.

Theorem 5.1. *Suppose that $\mathcal{M}$ is a closed, shift-invariant subspace of $H^2_{\mathcal{Y}}(\mathcal{F}_d)$. Then there is an isometric input-pair $(\mathbf{Z}, X)$ with $\mathbf{Z}^*$ strongly stable so that $\mathcal{M} = \mathcal{M}_{\mathbf{Z},X}$.*

Proof. If $\mathcal{M}$ is invariant for the operators S_j, then $\mathcal{M}^\perp$ is invariant for the operators S_j^* for each $j = 1, \dots, d$. Hence by Theorem 2.8 from [7] there is an observable, contractive output pair $(C, \mathbf{A})$ so that $\mathcal{M}^\perp = \mathcal{H}(K_{C,\mathbf{A}}) = \text{Ran } \mathcal{O}_{C,\mathbf{A}}$ isometrically. As $\mathcal{M}^\perp \subset H^2_{\mathcal{Y}}(\mathcal{F}_d)$ isometrically, Proposition 3.3 tells us that we may take $(C, \mathbf{A})$ isometric and that $\mathbf{A}$ is strongly stable. Let $(\mathbf{Z}, X)$ be the input pair $(\mathbf{Z}, X) = (\mathbf{A}^*, C^*)$. As $\mathcal{M}^\perp = \text{Ran } \mathcal{O}_{C,\mathbf{A}}$, we may compute $\mathcal{M}$ as

$$\mathcal{M} = (\text{Ran } \mathcal{O}_{C,\mathbf{A}})^\perp = \text{Ker} \, (\mathcal{O}_{C,\mathbf{A}})^* = \text{Ker } \mathcal{C}_{\mathbf{A}^*, C^*} = \text{Ker } \mathcal{C}_{\mathbf{Z},X}$$

and Theorem 5.1 follows. □

We now suppose that a shift-invariant subspace is given in the form $\mathcal{M} = \mathcal{M}_{\mathbf{Z},X}$ for an admissible homogeneous interpolation data set and we construct a realization for the associated Beurling-Lax representer.

Theorem 5.2. *Suppose that $(\mathbf{Z}, X)$ is an admissible homogeneous interpolation data set and $\mathcal{M}_{\mathbf{Z},X} = \text{Ker } \mathcal{C}_{\mathbf{Z},X}$ is the associated shift-invariant subspace. Let $(C, \mathbf{A})$ be the output pair defined by*

$$(C, \mathbf{A}) = (X^*, \mathbf{Z}^*)$$

and choose an input space $\mathcal{U}$ *with* $\dim \mathcal{U} = \operatorname{rank}(I_{\mathcal{X}^d \oplus \mathcal{Y}} - \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right] \left[\begin{smallmatrix} A^* & C^* \end{smallmatrix}\right])$ *and define an operator* $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] : \mathcal{U} \to \mathcal{X}^d \oplus \mathcal{Y}$ *as a solution of the Cholesky factorization problem*

$$\begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B^* & D^* \end{bmatrix} = I_{\mathcal{X}^d \oplus \mathcal{Y}} - \begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} A^* & C^* \end{bmatrix}.$$

Set $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *and let* $\theta \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ *be the transfer function of* $\mathbf{U}$*:*

$$\theta(z) = D + C(I - Z(z)A)^{-1} Z(z) B.$$

Then θ *is inner and* $\mathcal{M}_{\mathbf{Z},X} = \theta \cdot H^2_{\mathcal{U}}(\mathcal{F}_d)$*.*

Proof. If $(\mathbf{Z}, X)$ is an admissible homogeneous interpolation data set, then $(\mathbf{Z}, X)$ is controllable and $\mathbf{Z}^*$ is strongly stable. Since $(C, \mathbf{A}) = (X^*, \mathbf{Z}^*)$, we have $(C, \mathbf{A})$ is observable and $\mathbf{A}$ is strongly stable. ¿From the construction of $\mathbf{U}$, we know $\mathbf{U}$ is coisometric. Then by Theorem 4.6, θ is inner and hence $I - M_\theta M_\theta^*$ is the orthogonal projection of $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ onto $(\text{Ran } M_\theta)^\perp$. From part (2) of Proposition (4.1) it then follows that

$$\mathcal{H}(K_\theta) = H^2_{\mathcal{Y}} \ominus \theta \cdot H^2_{\mathcal{U}}(\mathcal{F}_d) \text{ isometrically.} \tag{5.2}$$

On the other hand, again since $\mathbf{U}$ is coisometric, from Corollary 3.4 we see that $K_\theta = K_{C,\mathbf{A}}$ and hence $\mathcal{H}(K_\theta) = \mathcal{H}(K_{C,\mathbf{A}})$. Since $\mathbf{A}$ is strongly stable, Proposition 3.3 tells us that $\mathcal{H}(K_{C,\mathbf{A}})$ is isometrically included in $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ and is characterized by

$$\mathcal{H}(K_\theta) = \mathcal{H}(K_{C,\mathbf{A}}) = \text{Ran } \mathcal{O}_{C,\mathbf{A}} = \text{Ran}\,(\mathcal{C}_{\mathbf{Z},X)})^*. \tag{5.3}$$

Comparing (5.2) with (5.3) and taking orthogonal complements finally leaves us with

$$\theta \cdot H^2_{\mathcal{U}}(\mathcal{F}_d) = (\text{Ran}\,(\mathcal{C}_{\mathbf{Z},X})^*)^\perp = \text{Ker } \mathcal{C}_{\mathbf{Z},X} = \mathcal{M}_{\mathbf{Z},X}$$

and Theorem 5.2 follows. □

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Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123, USA
E-mail address: `ball@math.vt.edu`

Department of Mathematics, The College of William and Mary, Williamsburg VA 23187-8795, USA
E-mail address: `vladi@math.wm.edu`

Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123, USA
E-mail address: `qlfang@math.vt.edu`