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Schur-class multipliers on the Arveson space: De Branges–Rovnyak reproducing kernel spaces and commutative transfer-function realizations

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Abstract

An interesting and recently much studied generalization of the classical Schur class is the class of contractive operator-valued multipliers $S(\boldsymbol{\lambda})$ for the reproducing kernel Hilbert space $\mathcal{H}(k_d)$ on the unit ball $\mathbb{B}^d \subset \mathbb{C}^d$, where k_d is the positive kernel $k_d(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = 1/(1 - \langle \boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle)$ on $\mathbb{B}^d$. The reproducing kernel space $\mathcal{H}(K_S)$ associated with the positive kernel $K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = (I - S(\boldsymbol{\lambda})S(\boldsymbol{\zeta})^*) \cdot k_d(\boldsymbol{\lambda}, \boldsymbol{\zeta})$ is a natural multivariable generalization of the classical de Branges–Rovnyak canonical model space. A special feature appearing in the multivariable case is that the space $\mathcal{H}(K_S)$ in general may not be invariant under the adjoints $M^*_{\lambda_j}$ of the multiplication operators $M_{\lambda_j} : f(\boldsymbol{\lambda}) \mapsto \lambda_j f(\boldsymbol{\lambda})$ on $\mathcal{H}(k_d)$. We show that invariance of $\mathcal{H}(K_S)$ under $M^*_{\lambda_j}$ for each $j = 1, \dots, d$ is equivalent to the existence of a realization for $S(\boldsymbol{\lambda})$ of the form $S(\boldsymbol{\lambda}) = D + C(I - \lambda_1 A_1 - \cdots - \lambda_d A_d)^{-1}(\lambda_1 B_1 + \cdots + \lambda_d B_d)$ such that connecting operator $\mathbf{U} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix}$ has adjoint $\mathbf{U}^*$ which is isometric on a certain natural subspace ($\mathbf{U}$ is "weakly coisometric") and has the additional property that the state operators $A_1, \dots, A_d$ pairwise commute; in this case one can take the state space to be the functional-model space $\mathcal{H}(K_S)$ and the state operators $A_1, \dots, A_d$ to be given by $A_j = M^*_{\lambda_j}|_{\mathcal{H}(K_S)}$ (a de Branges–Rovnyak functional-model realization). We show that this special situation always occurs for the case of inner functions S (where the associated multiplication operator M_S is a partial isometry), and that inner multipliers are characterized by the existence of such a realization such that the state operators $A_1, \dots, A_d$ satisfy an additional stability property.

Keywords: Operator-valued functions; Schur-class multipliers; Inner multipliers

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1. Introduction

A multivariable generalization of the Szegő kernel $k(\lambda,\zeta)=(1-\lambda\bar{\zeta})^{-1}$ much studied of late is the positive kernel

$$k_d(\boldsymbol{\lambda},\boldsymbol{\zeta})=\frac{1}{1-\langle\boldsymbol{\lambda},\boldsymbol{\zeta}\rangle}$$

on $\mathbb{B}^d\times\mathbb{B}^d$ where $\mathbb{B}^d=\{\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_d)\in\mathbb{C}^d\colon \langle\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle<1\}$ is the unit ball of the d-dimensional Euclidean space $\mathbb{C}^d$. By $\langle\boldsymbol{\lambda},\boldsymbol{\zeta}\rangle=\sum_{j=1}^d\lambda_j\bar{\zeta}_j$ we mean the standard inner product in $\mathbb{C}^d$. The reproducing kernel Hilbert space (RKHS) $\mathcal{H}(k_d)$ associated with k_d via Aronszajn's construction [3] is a natural multivariable analogue of the Hardy space H^2 of the unit disk and coincides with H^2 if $d=1$.

For $\mathcal{Y}$ an auxiliary Hilbert space, we consider the tensor product Hilbert space $\mathcal{H}_{\mathcal{Y}}(k_d):=\mathcal{H}(k_d)\otimes\mathcal{Y}$ whose elements can be viewed as $\mathcal{Y}$-valued functions in $\mathcal{H}(k_d)$. Then $\mathcal{H}_{\mathcal{Y}}(k_d)$ can be characterized as follows:

$$\mathcal{H}_{\mathcal{Y}}(k_d)=\left\{f(\boldsymbol{\lambda})=\sum_{\mathbf{n}\in\mathbb{Z}_+^d}f_{\mathbf{n}}\boldsymbol{\lambda}^{\mathbf{n}}\colon \|f\|^2=\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{\mathbf{n}!}{|\mathbf{n}|!}\cdot\|f_{\mathbf{n}}\|_{\mathcal{Y}}^2<\infty\right\}. \tag{1.1}$$

Here and in what follows, we use standard multivariable notations: for multi-integers $\mathbf{n}=(n_1,\dots,n_d)\in\mathbb{Z}_+^d$ and points $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_d)\in\mathbb{C}^d$ we set

$$|\mathbf{n}|=n_1+n_2+\dots+n_d,\qquad \mathbf{n}!=n_1!n_2!\dots n_d!,\qquad \boldsymbol{\lambda}^{\mathbf{n}}=\lambda_1^{n_1}\lambda_2^{n_2}\dots\lambda_d^{n_d}. \tag{1.2}$$

By $\mathcal{L}(\mathcal{U},\mathcal{Y})$ we denote the space of all bounded linear operators between Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$. The space of multipliers $\mathcal{M}_d(\mathcal{U},\mathcal{Y})$ is defined as the space of all $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued analytic functions S on $\mathbb{B}^d$ such that the induced multiplication operator

$$M_S: f(\boldsymbol{\lambda})\to S(\boldsymbol{\lambda})\cdot f(\boldsymbol{\lambda}) \tag{1.3}$$

maps $\mathcal{H}_{\mathcal{U}}(k_d)$ into $\mathcal{H}_{\mathcal{Y}}(k_d)$. It follows by the closed graph theorem that for every $S\in\mathcal{M}_d(\mathcal{U},\mathcal{Y})$, the operator M_S is bounded. We shall pay particular attention to the unit ball of $\mathcal{M}_d(\mathcal{U},\mathcal{Y})$, denoted by

$$\mathcal{S}_d(\mathcal{U},\mathcal{Y})=\bigl\{S\in\mathcal{M}_d(\mathcal{U},\mathcal{Y})\colon \|M_S\|_{\mathrm{op}}\leqslant 1\bigr\}.$$

Since $\mathcal{S}_1(\mathcal{U},\mathcal{Y})$ collapses to the classical Schur class (of holomorphic, contractive $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued functions on $\mathbb{D}$), we refer to $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ as a generalized (d-variable) *Schur class*. Characterizations of $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ in terms of realizations originate to [1,11]. We recall this result in the form presented in [7].

Theorem 1.1. *Let S be an $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function defined on $\mathbb{B}^d$. The following are equivalent*:

(1) *S belongs to $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$.*
(2) *The kernel*

$$K_S(\boldsymbol{\lambda},\boldsymbol{\zeta})=\frac{I_{\mathcal{Y}}-S(\boldsymbol{\lambda})S(\boldsymbol{\zeta})^*}{1-\langle\boldsymbol{\lambda},\boldsymbol{\zeta}\rangle} \tag{1.4}$$

is positive on $\mathbb{B}^d\times\mathbb{B}^d$, i.e., there exists an operator-valued function $H:\mathbb{B}^d\to\mathcal{L}(\mathcal{H},\mathcal{Y})$ for an auxiliary Hilbert space $\mathcal{H}$ so that $K_S(\boldsymbol{\lambda},\boldsymbol{\zeta})=H(\boldsymbol{\lambda})H(\boldsymbol{\zeta})^*$.

(3) *There exist a Hilbert space $\mathcal{X}$ and a unitary connecting operator* (*or colligation*) $\mathbf{U}$ *of the form*

$$\mathbf{U}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}=\begin{bmatrix}A_1 & B_1\\ \vdots & \vdots\\ A_d & B_d\\ C & D\end{bmatrix}:\begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix}\to\begin{bmatrix}\mathcal{X}^d\\ \mathcal{Y}\end{bmatrix} \tag{1.5}$$

so that $S(\boldsymbol{\lambda})$ can be realized in the form

$$\begin{aligned} S(\boldsymbol{\lambda}) &= D + C(I_{\mathcal{X}} - \lambda_1 A_1 - \cdots - \lambda_d A_d)^{-1}(\lambda_1 B_1 + \cdots + \lambda_d B_d) \\ &= D + C\big(I - Z(\boldsymbol{\lambda})A\big)^{-1} Z(\boldsymbol{\lambda}) B \end{aligned} \tag{1.6}$$

where we set

$$Z(\boldsymbol{\lambda}) = [\,\lambda_1 I_{\mathcal{X}} \quad \ldots \quad \lambda_d I_{\mathcal{X}}\,], \qquad A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}. \tag{1.7}$$

(4) *There exist a Hilbert space $\mathcal{X}$ and a contractive connecting operator* $\mathbf{U}$ *of the form* (1.5) *so that $S(\boldsymbol{\lambda})$ can be realized in the form* (1.6).

In analogy with the univariate case, a realization of the form (1.6) is called *coisometric*, *isometric*, *unitary* or *contractive* if the operator $\mathbf{U}$ is respectively, coisometric, isometric, unitary or just contractive. It turns out that a more useful analogue of "coisometric realization" appearing in the classical univariate case is not that the whole connecting operator $\mathbf{U}^*$ be isometric, but rather that $\mathbf{U}^*$ be isometric on a certain subspace of $\mathcal{X}^d \oplus \mathcal{Y}$.

Definition 1.2. A realization (1.6) of $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is called *weakly coisometric* if the adjoint $\mathbf{U}^* : \mathcal{X}^d \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{U}$ of the connecting operator is contractive and isometric on the subspace

$$\begin{bmatrix} \mathcal{D} \\ \mathcal{Y} \end{bmatrix} \subset \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix}$$

where

$$\mathcal{D} := \overline{\operatorname{span}}\big\{Z(\boldsymbol{\zeta})^*\big(I_{\mathcal{X}} - A^* Z(\boldsymbol{\zeta})^*\big)^{-1} C^* y \colon \boldsymbol{\zeta} \in \mathbb{B}^d,\ y \in \mathcal{Y}\big\} \subset \mathcal{X}^d. \tag{1.8}$$

Weakly coisometric realizations for $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ can be constructed in certain canonical way as follows. Upon applying Aronszajn's construction to the kernel K_S defined as in (1.4) (which is positive on $\mathbb{B}^d$ by Theorem 1.1), one gets the de Branges–Rovnyak space $\mathcal{H}(K_S)$. A weakly coisometric realization for S with the state space equal to $\mathcal{H}(K_S)$ (and output operator C equal to evaluation at zero on $\mathcal{H}(K_S)$) will be called a *generalized functional-model realization*. Here we use the term *generalized* functional-model realization since it may be the case that the state space $\mathcal{H}(K_S)$ is not even invariant under the adjoints $M_{\lambda_1}^*, \ldots, M_{\lambda_d}^*$ of the multiplication operators $M_{\lambda_j} : f(\boldsymbol{\lambda}) \mapsto \lambda_j \cdot f(\boldsymbol{\lambda})$ ($j = 1, \ldots, d$) on $\mathcal{H}_{\mathcal{Y}}(k_d)$ and hence one cannot take the state operators $A_1, \ldots, A_d$ to be given by $A_j = M_{\lambda_j}^*$ as one would expect from the classical case. As it was shown in [7], any function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ admits a generalized functional-model realization. In the univariate case, this collapses to the well-known de Branges–Rovnyak functional-model realization [17,18]. Another parallel to the univariate case is that *any* observable weakly coisometric realization of a Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is unitarily equivalent to some generalized functional-model realization (observability is a minimality condition that is fulfilled automatically for every generalized functional-model realization). However, in contrast to the univariate case, this realization is not unique in general (even up to unitary equivalence); moreover, a function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ may admit generalized functional-model realizations with the same state space operators $A_1, \ldots, A_d$ and different input operators B_j's. A curious fact is that none of the generalized functional-model realizations for S may be coisometric.

In this paper we study another issue not present in the univariate classical case, namely the distinction between *commutative realizations* (where the state space operators $A_1, \ldots, A_d$ in (1.6) commute with each other) versus general realizations. Commutative realization is a natural notion that appears for example in model theory for commuting row contractions [14]: the characteristic function of a commuting row contraction $(T_1, \ldots, T_d)$ is, by definition, a Schur-class function that admits a unitary commutative realization with the state space operators $T_1, \ldots, T_d$. It turns out that not every $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ can be identified as a characteristic function of a commutative row contraction; thus not every $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ admits a commutative unitary realization. Some more delicate arguments based on backward-shift invariance in $\mathcal{H}_{\mathcal{Y}}(k_d)$ show that not every $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ admits a commutative weakly coisometric realization (see Theorem 3.5 below); more surprisingly, there are Schur-class functions that do not admit even *contractive* commutative realizations (see Example 3.4 below). If the Schur-class function admits a commutative weakly coisometric realization, then the associated de Branges–Rovnyak space $\mathcal{H}(K_S)$ is invariant for the backward shift operators $M_{\lambda_j}^*$

and one can arrange for a generalized functional-model realization with the additional property that the state operators $A_1, \dots, A_d$ are given by $A_j = M^*_{\lambda_j}|_{\mathcal{H}(K_S)}$ for $j = 1, \dots, d$; we say that such a realization is a (non-generalized) *functional-model realization*. The operators $B_1, \dots, B_d$ are not defined uniquely by $S(\boldsymbol{\lambda})$, $\mathbf{A} = (A_1, \dots, A_d)$ and C (this is yet another distinction from the univariate case); however the nonuniqueness can be described in an explicit way. Furthermore, any observable, commutative, weakly coisometric realization for a given S is unitarily equivalent to exactly one functional-model realization (Theorem 3.6).

Inner functions, i.e., Schur-class multipliers $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ for which the associated multiplication operator is a partial isometry, are special in that an inner function necessarily has a commutative weakly coisometric realization (see Theorem 3.5 below). Inner functions also play a special role as representers for (forward) shift-invariant subspaces of $\mathcal{H}_{\mathcal{Y}}(k_d)$; for the case $d = 1$ this is the classical Beurling–Lax–Halmos theorem [13,21,22] while the case for general d appears more recently in the work of Arveson [4,5] and of McCullough and Trent [23] (for the general framework of a complete Nevanlinna–Pick kernel). Here we use our realization-theoretic characterization of inner multipliers to present a new proof of the $\mathcal{H}_{\mathcal{Y}}(k_d)$-Beurling–Lax theorem. The idea in this approach is to represent the shift-invariant subspace $\mathcal{M}$ as the set of all $\mathcal{H}_{\mathcal{Y}}(k_d)$-solutions of fairly general set of homogeneous interpolation conditions, and then to construct a realization $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ for $S(\boldsymbol{\lambda})$ from the operators defining the homogeneous interpolation conditions. For the case $d = 1$, this approach can be found in [9] for the rational case and in [10] for the non-rational case, done there in the more complicated context where the shift-invariant subspace $\mathcal{M}$ is merely contained in the $\mathcal{Y}$-valued L^2 space over the unit circle $\mathbb{T}$ and is not necessarily contained in the Hardy space $\mathcal{H}_{\mathcal{Y}}(k_1) = H^2_{\mathcal{Y}}$. We also use our analysis of the nonuniqueness of the input operator B in weakly coisometric realizations to characterize the nonuniqueness in the choice of inner-function representer S for a given shift-invariant subspace $\mathcal{M}$ (see Theorem 5.5).

A more general version of the $\mathcal{H}_{\mathcal{Y}}(k_d)$-Beurling–Lax theorem, where the subspace $\mathcal{M}$ is only contractively included in $\mathcal{H}_{\mathcal{Y}}(k_d)$ and the representer is not necessarily an inner Schur-class multiplier, appears in the work of de Branges–Rovnyak [17,18] for the case $d = 1$ and of the authors [6] for the case of general d. The realization produced by our approach here (working with $\mathcal{M}^\perp$ rather than directly with $\mathcal{M}$) is more explicit for the situation where $\mathcal{M}$ is presented as the solution set for a homogeneous interpolation problem. Extensions of these ideas to a noncommutative-variable Fock-space setting appear in [8].

The paper is organized as follows. After the present introduction, Section 2 recalls needed preliminaries from our earlier papers [6,7] concerning weakly coisometric realizations (see Definition 1.2 above). Section 3 collects the results concerning such realizations where the collection of state operators $A_1, \dots, A_d$ is commutative. Section 4 specializes the general theory to the case of inner functions. The final Section 5 discusses connections with characteristic functions and operator-model theory for commutative row contractions, a topic of recent work of Bhattacharyya, Eschmeier, Sarkar and Popescu [14–16,27,28], where some extensions to more general settings are also addressed.

2. Weakly coisometric realizations

Weakly coisometric realizations of Schur-class functions are closely related to range spaces of observability operators appearing in the context of Fornasini–Marchesini-type linear systems with evolution along the integer lattice $\mathbb{Z}^d$. Let $\mathbf{A} = (A_1, \dots, A_d)$ be a d-tuple of operators in $\mathcal{L}(\mathcal{X})$. If $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, then the pair $(C, \mathbf{A})$ is said to be an *output pair*. Such an output pair is said to be *contractive* if

$$A_1^* A_1 + \dots + A_d^* A_d + C^* C \leqslant I_{\mathcal{X}}, \tag{2.1}$$

to be *isometric* if equality holds in the above relation, and to be *output-stable* if the associated observability operator

$$\mathcal{O}_{C,\mathbf{A}} : x \mapsto C\big(I_{\mathcal{X}} - Z(\boldsymbol{\lambda})A\big)^{-1} x = C(I - \lambda_1 A_1 - \dots - \lambda_d A_d)^{-1} x \tag{2.2}$$

(where $Z(\boldsymbol{\lambda})$ and A are defined as in (1.7)) maps $\mathcal{X}$ into $\mathcal{H}_{\mathcal{Y}}(k_d)$. As it was shown in [6], any contractive pair $(C, \mathbf{A})$ is output stable and moreover, the corresponding observability operator $\mathcal{O}_{C,\mathbf{A}} : \mathcal{X} \to \mathcal{H}_{\mathcal{Y}}(k_d)$ is a contraction. An output stable pair $(C, \mathbf{A})$ is called *observable* if the observability operator $\mathcal{O}_{C,\mathbf{A}}$ is injective, i.e.,

$$C\big(I_{\mathcal{X}} - Z(\boldsymbol{\lambda})A\big)^{-1} x \equiv 0 \quad \Rightarrow \quad x = 0.$$

Given an output stable pair $(C, \mathbf{A})$, the kernel

$$K_{C,\mathbf{A}}(\boldsymbol{\lambda},\boldsymbol{\zeta}) := C\big(I_{\mathcal{X}} - Z(\boldsymbol{\lambda})A\big)^{-1}\big(I_{\mathcal{X}} - A^*Z(\boldsymbol{\zeta})^*\big)^{-1}C^*. \tag{2.3}$$

is positive on $\mathbb{B}^d \times \mathbb{B}^d$; let $\mathcal{H}(K_{C,\mathbf{A}})$ denote the associated RKHS. We recall (see [3]) that any positive kernel $(\lambda,\zeta) \mapsto K(\lambda,\zeta) \in \mathcal{L}(\mathcal{Y})$ on a set $\Omega \times \Omega$ (so $\lambda,\zeta \in \Omega$) gives rise to a RKHS $\mathcal{H}(K)$ consisting of $\mathcal{Y}$-valued functions on Ω with the defining property: for each $\zeta \in \Omega$ and $y \in \mathcal{Y}$, the $\mathcal{Y}$-valued function $K_\zeta y(\lambda) := K(\lambda,\zeta)y$ is in $\mathcal{H}(K)$ and has the reproducing property

$$\langle f, K_\zeta y\rangle_{\mathcal{H}(K)} = \big\langle f(\zeta), y\big\rangle_{\mathcal{Y}} \quad \text{for all } y \in \mathcal{Y},\ f \in \mathcal{H}(K).$$

The following result from [6] gives the close connection between spaces of the form $\mathcal{H}(K_{C,\mathbf{A}})$ and ranges of observability operators.

Theorem 2.1. *(See [6, Theorem 3.20].) Let $(C,\mathbf{A})$ be a contractive pair with $C \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ and with associated positive kernel $K_{C,\mathbf{A}}$ given by* (2.3) *and the observability operator $\mathcal{O}_{C,\mathbf{A}}$ given by* (2.2)*. Then*:

(1) *The reproducing kernel Hilbert space $\mathcal{H}(K_{C,\mathbf{A}})$ is characterized as*

$$\mathcal{H}(K_{C,\mathbf{A}}) = \operatorname{Ran}\mathcal{O}_{C,\mathbf{A}}$$

with the lifted norm given by $\|\mathcal{O}_{C,\mathbf{A}}x\|_{\mathcal{H}(K_{C,\mathbf{A}})} = \|Qx\|_{\mathcal{X}}$ where Q is the orthogonal projection onto $(\operatorname{Ker}\mathcal{O}_{C,\mathbf{A}})^\perp$.

(2) *The operator $\mathcal{O}_{C,\mathbf{A}}$ is a contraction of $\mathcal{X}$ into $\mathcal{H}(K_{C,\mathbf{A}})$. It is an isometry if and only if the pair $(C,\mathbf{A})$ is observable.*

(3) *The space $\mathcal{H}(K_{C,\mathbf{A}})$ is contractively included in the Arveson space $\mathcal{H}_{\mathcal{Y}}(k_d)$; it is isometrically included in $\mathcal{H}_{\mathcal{Y}}(k_d)$ if and only if $\mathcal{O}_{C,\mathbf{A}}$ (as an operator from $\mathcal{X}$ into $\mathcal{H}_{\mathcal{Y}}(k_d)$) is a partial isometry.*

If S is realized as in (1.6) and $\mathbf{U}$ is the connecting operator given by (1.5), then the associated kernels K_S and $K_{C,\mathbf{A}}$ (defined in (1.4) and (2.3), respectively) are related by the following easily verified identity:

$$K_S(\boldsymbol{\lambda},\boldsymbol{\zeta}) =: K_{C,\mathbf{A}}(\boldsymbol{\lambda},\boldsymbol{\zeta}) + \begin{bmatrix} C(I - Z(\boldsymbol{\lambda})A)^{-1}Z(\boldsymbol{\lambda}) & I \end{bmatrix} \frac{I - \mathbf{U}\mathbf{U}^*}{1 - \langle\boldsymbol{\lambda},\boldsymbol{\zeta}\rangle} \begin{bmatrix} Z(\boldsymbol{\zeta})^*(I_{\mathcal{X}} - A^*Z(\boldsymbol{\zeta})^*)^{-1}C^* \\ I \end{bmatrix} \tag{2.4}$$

and then it is easily shown (see Proposition 1.5 in [7] for details) that the second term on the right vanishes if and only if $\mathbf{U}^*$ is isometric on the space $\mathcal{D} \oplus \mathcal{Y}$ defined as in Definition 1.2. This observation leads us to the following intrinsic kernel characterization as to when a given contractive realization is weakly coisometric.

Proposition 2.2. *A contractive realization* (1.6) *of an $S \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$ is weakly coisometric if and only if the kernel $K_S(\boldsymbol{\lambda},\boldsymbol{\zeta})$ associated with S via* (1.4) *can alternatively be written as*

$$K_S(\boldsymbol{\lambda},\boldsymbol{\zeta}) = K_{C,\mathbf{A}}(\boldsymbol{\lambda},\boldsymbol{\zeta}) \tag{2.5}$$

where $K_{C,\mathbf{A}}$ is given by (2.3).

Proposition 2.2 states that once a contractive realization $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ of S is such that (2.5) holds, then this realization is weakly coisometric. The next result asserts that equality (2.5) itself guarantees the existence of weakly coisometric realizations for S with preassigned C and $\mathbf{A} = (A_1,\dots,A_d)$.

Theorem 2.3. *(See [7, Theorem 2.4].) Suppose that a Schur-class function $S \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$ and a contractive pair $(C,\mathbf{A})$ are such that* (2.5) *holds and let $D := S(0)$. Then there exists an operator $B : \mathcal{U} \to \mathcal{X}^d$ so that the operator $\mathbf{U}$ of the form* (1.5) *is weakly coisometric and S can be realized as in* (1.6).

The pair $(C,\mathbf{A})$ for a weakly coisometric realization can be constructed in a certain canonical way. Recall that the de Branges–Rovnyak space $\mathcal{H}(K_S)$ associated with $S \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$ is the reproducing kernel Hilbert space with reproducing kernel K_S defined as in (1.4).

Theorem 2.4. *(See [7, Theorem 3.20].) Let $S \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$ and let $\mathcal{H}(K_S)$ be the associated de Branges–Rovnyak space. Then*:

(1) *There exist bounded operators $A_j : \mathcal{H}(K_S) \to \mathcal{H}(K_S)$ such that*

$$f(\boldsymbol{\lambda}) - f(0) = \sum_{j=1}^{d} \lambda_j (A_j f)(\boldsymbol{\lambda}) \quad \text{for every } f \in \mathcal{H}(K_S) \text{ and } \boldsymbol{\lambda} \in \mathbb{B}^d, \tag{2.6}$$

and

$$\sum_{j=1}^{d} \|A_j f\|^2_{\mathcal{H}(K_S)} \leqslant \|f\|^2_{\mathcal{H}(K_S)} - \|f(0)\|^2_{\mathcal{Y}}. \tag{2.7}$$

(2) *There is a weakly coisometric realization* (1.6) *for S with state space $\mathcal{X}$ equal to $\mathcal{H}(K_S)$ with the state operators $A_1, \dots, A_d$ from part* (1) *and the operator $C : \mathcal{H}(K_S) \to \mathcal{Y}$ defined by*

$$Cf = f(0) \quad \text{for all } f \in \mathcal{H}(K_S). \tag{2.8}$$

Equality (2.6) means that the operator tuple $\mathbf{A} = (A_1, \dots, A_d)$ solves *the Gleason problem* [19] for $\mathcal{H}(K_S)$. Let us say that $\mathbf{A}$ *is a contractive solution of the Gleason problem* if in addition relation (2.7) holds for every $f \in \mathcal{H}(K_S)$ or, equivalently, if the pair $(C, \mathbf{A})$ is contractive where $C : \mathcal{H}(K_S) \to \mathcal{Y}$ is defined as in (2.8). Theorem 2.4 shows that any contractive solution $\mathbf{A} = (A_1, \dots, A_d)$ of the Gleason problem for $\mathcal{H}(K_S)$ gives rise to a weakly coisometric realization for $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ (not unique, in general). Let us call any such weakly coisometric realization a *generalized functional-model* realization of $S(\boldsymbol{\lambda})$. We note that any generalized functional-model realization of S is observable and that the formula

$$K_S(\cdot, \boldsymbol{\zeta})y = \big(I - A^* Z(\boldsymbol{\zeta})^*\big)^{-1} C^* y \quad \big(y \in \mathcal{Y},\ \zeta \in \mathbb{B}^d\big) \tag{2.9}$$

is valid for any generalized functional-model realization. Furthermore, if

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}(K_S) \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(K_S)^d \\ \mathcal{Y} \end{bmatrix} \tag{2.10}$$

is a generalized functional model realization for an $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, then the space $\mathcal{D}$ introduced in (1.8) can be described in the following explicit functional form:

$$\mathcal{D} = \overline{\operatorname{span}}\big\{Z(\boldsymbol{\zeta})^* K_S(\cdot, \boldsymbol{\zeta})y\colon\ \boldsymbol{\zeta} \in \mathbb{B}^d,\ y \in \mathcal{Y}\big\}. \tag{2.11}$$

Then a simple calculation shows that $\mathcal{D}^\perp = \mathcal{H}(K_S)^d \ominus \mathcal{D}$ can be characterized in similar terms as

$$\mathcal{D}^\perp = \big\{h \in \mathcal{H}(K_S)^d\colon\ Z(\boldsymbol{\lambda})h(\boldsymbol{\lambda}) \equiv 0\big\}. \tag{2.12}$$

3. Realizations with commutative state-space operators

The class of Schur-class functions admitting unitary realizations of the form (1.6) with commutative state-space tuple $\mathbf{A} = (A_1, \dots, A_d)$ is a natural object appearing in the model theory for commutative row contractions (see [14]): the characteristic function of a commutative row contraction (see formula (6.1) below) is a Schur-class function of this type (subject to an additional normalization). In the commutative context, a key role is played by the commuting d-tuple $\mathbf{M}_{\boldsymbol{\lambda}} := (M_{\lambda_1}, \dots, M_{\lambda_d})$ consisting of operators of multiplication by the coordinate functions of $\mathbb{C}^d$ which will be called *the shift* (operator-tuple) of $\mathcal{H}_{\mathcal{Y}}(k_d)$, whereas the commuting d-tuple $\mathbf{M}^*_{\boldsymbol{\lambda}} := (M^*_{\lambda_1}, \dots, M^*_{\lambda_d})$ consisting of the adjoints of M_{λ_j}'s (in the metric of $\mathcal{H}_{\mathcal{Y}}(k_d)$) will be referred to as to the *backward shift*. By the characterization (1.1) and in notation (1.2), the monomials $\frac{\mathbf{n}!}{|\mathbf{n}|!}\boldsymbol{\lambda}^{\mathbf{n}}$ form an orthonormal basis in $\mathcal{H}(k_d)$ and then a simple calculation shows that

$$M^*_{\lambda_j}\boldsymbol{\lambda}^{\mathbf{m}} = \frac{m_j}{|\mathbf{m}|}\boldsymbol{\lambda}^{\mathbf{m}-e_j} \quad (m_j \geqslant 1) \quad \text{and} \quad M^*_{\lambda_j}\boldsymbol{\lambda}^{\mathbf{m}} = 0 \quad (m_j = 0) \tag{3.1}$$

where $\mathbf{m} = (m_1, \dots, m_d)$ and e_j is the jth standard coordinate vector of $\mathbb{C}^d$. Some properties of the shift tuple $\mathbf{M}^*_{\boldsymbol{\lambda}}$ needed in the sequel are listed below (for the proof, see e.g. [6, Proposition 3.12]). In the formulation and in what follows, we use multivariable power notation

$$\mathbf{A}^{\mathbf{n}} := A_1^{n_1} A_2^{n_2} \dots A_d^{n_d}$$

for any d-tuple $\mathbf{A} = (A_1, \dots, A_d)$ of commuting operators on a space $\mathcal{X}$ and any $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$.

Proposition 3.1. *Let $\mathbf{M}_{\boldsymbol{\lambda}}^*$ be the d-tuple of backward shifts on $\mathcal{H}_{\mathcal{Y}}(k_d)$ and let*

$$G : f \mapsto f(0) \quad \bigl(f \in \mathcal{H}_{\mathcal{Y}}(k_d)\bigr) \tag{3.2}$$

be the operator of evaluation at the origin. Then:

(1) *For every $f \in \mathcal{H}_{\mathcal{Y}}(k_d)$ and every $\boldsymbol{\lambda} \in \mathbb{B}^d$ we have*

$$f(\boldsymbol{\lambda}) - f(0) = \sum_{j=1}^{d} \lambda_j \bigl(M_{\lambda_j}^* f\bigr)(\boldsymbol{\lambda}). \tag{3.3}$$

(2) *The pair $(G, \mathbf{M}_{\boldsymbol{\lambda}}^*)$ is isometric and the associated observability operator is the identity operator*

$$\mathcal{O}_{G,\mathbf{M}_{\boldsymbol{\lambda}}^*} = I_{\mathcal{H}_{\mathcal{Y}}(k_d)}. \tag{3.4}$$

(3) *The d-tuple $\mathbf{M}_{\boldsymbol{\lambda}}^*$ is strongly stable, that is,*

$$\lim_{N\to\infty} \sum_{\mathbf{n}\in\mathbb{Z}_+^d:\ |\mathbf{n}|=N} \frac{N!}{\mathbf{n}!} \bigl\| \bigl(\mathbf{M}_{\boldsymbol{\lambda}}^*\bigr)^{\mathbf{n}} f \bigr\|_{\mathcal{H}_{\mathcal{Y}}(k_d)}^2 = 0 \quad \text{for every } f \in \mathcal{H}_{\mathcal{Y}}(k_d). \tag{3.5}$$

We will also need the commutative analogue of Theorem 2.1 (see [6, Theorem 3.15] for the proof).

Theorem 3.2. *Let $(C, \mathbf{A})$ be a contractive pair such that $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and the d-tuple $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{L}(\mathcal{X})^d$ is commutative. Let $K_{C,\mathbf{A}}$ be the associated kernel given by* (2.3). *Then:*

(1) *The reproducing kernel Hilbert space $\mathcal{H}(K_{C,\mathbf{A}})$ is invariant under $M_{\lambda_j}^*$ for $j = 1, \dots, d$ ($\mathbf{M}_{\boldsymbol{\lambda}}^*$-invariant) and the difference-quotient inequality*

$$\sum_{j=1}^{d} \bigl\| M_{\lambda_j}^* f \bigr\|_{\mathcal{H}(K_{C,\mathbf{A}})}^2 \leqslant \|f\|_{\mathcal{H}(K_{C,\mathbf{A}})}^2 - \bigl\|f(0)\bigr\|_{\mathcal{Y}}^2$$

holds for every $f \in \mathcal{H}(K_{C,\mathbf{A}})$.

(2) *The space $\mathcal{H}(K_{C,\mathbf{A}})$ is contractively included in $\mathcal{H}_{\mathcal{Y}}(k_d)$. The inclusion is isometric exactly when the pair $(C, \mathbf{A})$ is isometric*

$$I_{\mathcal{X}} - A_1^* A_1 - \dots - A_d^* A_d = C^* C, \tag{3.6}$$

and $\mathbf{A}$ is strongly stable

$$\lim_{N\to\infty} \sum_{\mathbf{n}\in\mathbb{Z}_+^d:\ |\mathbf{n}|=N} \frac{N}{\mathbf{n}!} \bigl\| \mathbf{A}^{\mathbf{n}} x \bigr\|_{\mathcal{X}}^2 = 0 \quad \text{for all } x \in \mathcal{X}. \tag{3.7}$$

If one drops the requirement of the connecting operator $\mathbf{U}$ being contractive, constructing a commutative realization is not an issue not only for Schur-class functions, but even for functions from $\mathcal{H}(k_d) \otimes \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Indeed, for an $S \in \mathcal{H}(k_d) \otimes \mathcal{L}(\mathcal{U}, \mathcal{Y})$, let

$$C = G, \qquad D = S(0), \qquad A_j = M_{\lambda_j}^*, \qquad B_j = M_{\lambda_j}^* M_S|_{\mathcal{U}} \quad (j = 1, \dots, d)$$

where $M_S|_{\mathcal{U}} : u \mapsto S(\boldsymbol{\lambda})u$ and $G : \mathcal{H}_{\mathcal{Y}}(k_d) \to \mathcal{Y}$ is given by (3.2). Pick a vector $u \in \mathcal{U}$ and note that an account of equality (3.4) and equality (3.3) applied to $f(\boldsymbol{\lambda}) = S(\boldsymbol{\lambda})u$,

$$\begin{aligned}\big(D+C\big(I-Z(\boldsymbol{\lambda})A\big)^{-1}Z(\boldsymbol{\lambda})B\big)u &= S(0)u+\mathcal{O}_{G,\mathbf{M}_{\boldsymbol{\lambda}}^*}\big(Z(\boldsymbol{\lambda})Bu\big)\\ &= S(0)u+Z(\boldsymbol{\lambda})Bu\\ &= S(0)u+\sum_{j=1}^{d}\lambda_j\big(M_{\lambda_j}^*Su\big)(\boldsymbol{\lambda})\\ &= S(0)u+S(\boldsymbol{\lambda})u-S(0)u=S(\boldsymbol{\lambda})u\end{aligned}$$

and thus, $\mathbf{U}=\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]$ is a realization for S. This realization is commutative and observable. However, it is not contractive: a simple calculation based again on identity (3.3) shows that for $\mathbf{U}$ as above and

$$g=\begin{bmatrix} f\\ u\end{bmatrix}\in\begin{bmatrix}\mathcal{H}_{\mathcal{Y}}(k_d)\\ \mathcal{U}\end{bmatrix},$$

we have

$$\|g\|^2-\|\mathbf{U}g\|^2=\|f\|^2_{\mathcal{H}_{\mathcal{Y}}(k_d)}+\|u\|^2_{\mathcal{U}}-\|f+Su\|^2_{\mathcal{H}_{\mathcal{Y}}(k_d)}$$

which cannot be nonnegative for all $f\in\mathcal{H}_{\mathcal{Y}}(k_d)$ and $u\in\mathcal{U}$ unless $S(\boldsymbol{\lambda})\equiv 0$.

Since our primary object of interest are Schur-class functions for which norm-constrained (contractive, unitary and all intermediate) realizations do exist (by Theorem 1.1), it is natural to construct commutative realizations of the same types. Note that Theorem 1.1 and the more specific Theorem 2.4 give no clue as to when and how one can achieve such a realization of a given $S\in\mathcal{S}_d(\mathcal{U},\mathcal{Y})$. The next proposition shows that there are Schur-class functions which do not have a commutative contractive realization.

Proposition 3.3. *Let $S\in\mathcal{S}_d(\mathcal{U},\mathcal{U})$ be such that the associated de Branges–Rovnyak space $\mathcal{H}(K_S)$ is finite-dimensional and is not $\mathbf{M}_{\boldsymbol{\lambda}}^*$-invariant. Then S does not have a commutative contractive realization.*

Proof. Assume that S admits a commutative realization (1.6) with a contractive $\mathbf{U}=\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]$. Since $\mathbf{U}$ is contractive, the formula (2.4) for K_S can be written in the form

$$K_S(\boldsymbol{\lambda},\boldsymbol{\zeta})=K_{C,\mathbf{A}}(\boldsymbol{\lambda},\boldsymbol{\zeta})+\frac{S_1(\boldsymbol{\lambda})S_1(\boldsymbol{\zeta})^*}{1-\langle\boldsymbol{\lambda},\boldsymbol{\zeta}\rangle}$$

where $S_1\in\mathcal{S}_d(\mathcal{F},\mathcal{U})$ is a Schur-class function with an appropriately chosen coefficient space $\mathcal{F}$ (the explicit formula for S_1 is not that important). If $S_1\not\equiv 0$, then $\mathcal{H}(K_S)$ contains the space $S_1\mathcal{H}_{\mathcal{F}}(k_d)$ and therefore is infinite dimensional which contradicts one of the assumptions about $\mathcal{H}(K_S)$. If $S_1\equiv 0$, then $K_S(\boldsymbol{\lambda},\boldsymbol{\zeta})=K_{C,\mathbf{A}}(\boldsymbol{\lambda},\boldsymbol{\zeta})$ and therefore $\mathcal{H}(K_S)=\mathcal{H}(K_{C,\mathbf{A}})$. Since the tuple $\mathbf{A}$ is commutative, the space $\mathcal{H}(K_S)=\mathcal{H}(K_{C,\mathbf{A}})$ is $\mathbf{M}_{\boldsymbol{\lambda}}^*$-invariant (by Theorem 3.2) which contradicts another assumption about $\mathcal{H}(K_S)$. $\square$

Example 3.4. For a concrete example of a Schur-class function satisfying the assumptions in Proposition 3.3, let

$$S(\lambda_1,\lambda_2)=\frac{1}{4-\lambda_1\lambda_2}\begin{bmatrix}2\sqrt{3}\lambda_1 & \sqrt{3}\lambda_2^2 & 2-2\lambda_1\lambda_2 & -3\lambda_2\\ \sqrt{3}\lambda_1^2 & 2\sqrt{3}\lambda_2 & -3\lambda_1 & 2-2\lambda_1\lambda_2\end{bmatrix}.$$

A straightforward calculation gives

$$K_S(\boldsymbol{\lambda},\boldsymbol{\zeta}):=\frac{I_2-S(\boldsymbol{\lambda})S(\boldsymbol{\zeta})^*}{1-\lambda_1\bar{\zeta}_1-\lambda_2\bar{\zeta}_2}=\frac{3}{(4-\lambda_1\lambda_2)(4-\bar{\zeta}_1\bar{\zeta}_2)}\begin{bmatrix}2 & \lambda_2\\ \lambda_1 & 2\end{bmatrix}\begin{bmatrix}2 & \bar{\zeta}_1\\ \bar{\zeta}_2 & 2\end{bmatrix}.$$

Thus the kernel $K_S(\boldsymbol{\lambda},\boldsymbol{\zeta})$ is positive on $\mathbb{B}^2\times\mathbb{B}^2$ and $S\in\mathcal{S}_2(\mathbb{C}^4,\mathbb{C}^2)$. The associated de Branges–Rovnyak space $\mathcal{H}(K_S)$ is spanned by rational functions

$$f_1(\boldsymbol{\lambda})=\frac{4}{4-\lambda_1\lambda_2}\begin{bmatrix}2\\ \lambda_1\end{bmatrix}\quad\text{and}\quad f_2(\boldsymbol{\lambda})=\frac{4}{4-\lambda_1\lambda_2}\begin{bmatrix}\lambda_2\\ 2\end{bmatrix}.$$

Furthermore, since by (3.1) we have

$$M_{\lambda_1}^*\big(\lambda_1^{n_1}\lambda_2^{n_2}\big)=\frac{n_1}{n_1+n_2}\lambda_1^{n_1-1}\lambda_2^{n_2},$$

it follows that

$$M_{\lambda_1}^*\left(\frac{4\lambda_1}{4-\lambda_1\lambda_2}\right)=M_{\lambda_1}^*\left(\sum_{j=0}^{\infty}\frac{\lambda_1^{j+1}\lambda_2^j}{4^j}\right)=\sum_{j=0}^{\infty}\frac{j+1}{2j+1}\left(\frac{\lambda_1\lambda_2}{4}\right)^j. \tag{3.8}$$

The latter function is rational if and only if the single-variable function $F(\lambda)=\sum_{j=0}^{\infty}\frac{j+1}{2j+1}\lambda^j$ is rational. By the well-known Kronecker theorem, F in turn is rational if and only if the associated infinite Hankel matrix

$$\mathbb{H}=[s_{i+j}]_{i,j=0}^{\infty}\quad\text{where } s_k=\frac{k+1}{2k+1}$$

has finite rank. However one can check that the finite Hankel matrices $\mathbb{H}_n=[s_{i+j}]_{i,j=0}^{n}$ have full rank for all $n=0,1,2,\dots$ and hence $F(\lambda)$ is not rational. We conclude that the function on the right-hand side in (3.8) is not rational. Now it follows that $M_{\lambda_1}^* f_1$ does not belong to $\mathcal{H}(K_S)$. Therefore $\mathcal{H}(K_S)$ is not invariant under $M_{\lambda_1}^*$ and since $\dim\mathcal{H}(K_S)=2<\infty$, the function S does not admit contractive commutative realizations by Proposition 3.3.

A characterization of which Schur-class functions do admit contractive commutative realizations will be given in Theorem 3.10 below. The next result gives a characterization of Schur-class functions that admit weakly coisometric commutative realizations.

Theorem 3.5. *A Schur-class function $S\in\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ admits a commutative weakly coisometric realization if and only if the following conditions hold*:

(1) *the associated de Branges–Rovnyak space $\mathcal{H}(K_S)$ is $\mathbf{M}_{\boldsymbol{\lambda}}^*$-invariant, and*
(2) *the inequality*

$$\sum_{j=1}^{d}\left\|M_{\lambda_j}^* f\right\|_{\mathcal{H}(K_S)}^2\leqslant\|f\|_{\mathcal{H}(K_S)}^2-\left\|f(0)\right\|_{\mathcal{Y}}^2\quad \textit{holds for all } f\in\mathcal{H}(K_S). \tag{3.9}$$

Proof. To prove necessity, suppose that $S\in\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ admits a weakly coisometric realization (1.6). As noted in Proposition 2.2, it follows that $\mathcal{H}(K_S)=\mathcal{H}(K_{C,\mathbf{A}})$. Since $\mathbf{A}$ is commutative, Theorem 3.2 implies that the space $\mathcal{H}(K_S)=\mathcal{H}(K_{C,\mathbf{A}})$ is $\mathbf{M}_{\boldsymbol{\lambda}}^*$-invariant with inequality (3.9) holding.

To prove sufficiency, suppose that $S\in\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ is such that $\mathcal{H}(K_S)$ is $\mathbf{M}_{\boldsymbol{\lambda}}^*$-invariant with (3.9) holding. Define operators $A_1,\dots,A_d:\mathcal{H}(K_S)\to\mathcal{H}(K_S)$, $C:\mathcal{H}(K_S)\to\mathcal{Y}$ and $D:\mathcal{U}\to\mathcal{Y}$ by

$$A_j=M_{\lambda_j}^*\big|_{\mathcal{H}(K_S)}\quad (j=1,\dots,d),\qquad C:f\to f(0),\qquad D=S(0). \tag{3.10}$$

Formula (3.3) tells us that the operators $M_{\lambda_1}^*,\dots,M_{\lambda_d}^*$ solve the Gleason problem for $\mathcal{H}_{\mathcal{Y}}(k_d)$. In particular, the restriction of this formula to $f\in\mathcal{H}(K_S)$ can be written in terms of the operators (3.10) in the form (2.6), which means that $A_1,\dots,A_d$ solve the Gleason problem for $\mathcal{H}(K_S)$. Then we apply Theorem 2.4 (part (2)) to conclude that there is a choice of $B:\mathcal{U}\to\mathcal{H}(K_S)^d$ with $\mathbf{U}$ of the form (2.10) weakly coisometric so that $S(\boldsymbol{\lambda})=D+C(I-Z(\boldsymbol{\lambda})A)^{-1}Z(\boldsymbol{\lambda})B$. This completes the proof. $\square$

Note that the proof of Theorem 3.5 obtains a realization for $S\in\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ of a special form under the assumption that $\mathcal{H}(K_S)$ is $\mathbf{M}_{\boldsymbol{\lambda}}^*$-invariant: the state space $\mathcal{X}$ is taken to be the de Branges–Rovnyak space $\mathcal{H}(K_S)$ and the operators $\mathbf{A}=(A_1,\dots,A_d)$, C, D are given by (3.10); only the operators $B_j:\mathcal{U}\to\mathcal{H}(K_S)$ remain to be determined. We shall say that *any* contractive realization of a given Schur-class function S of this form (i.e., with $\mathcal{X}=\mathcal{H}(K_S)$ and A,C,D given by (3.10)) is a *functional-model realization* of S. It is readily seen that any functional-model realization is also a generalized functional-model realization; in particular, it is weakly coisometric and observable.

Let us recall that two colligations

$$\mathbf{U}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}:\mathcal{X}\oplus\mathcal{U}\to\mathcal{X}^d\oplus\mathcal{Y}\quad\text{and}\quad\widetilde{\mathbf{U}}=\begin{bmatrix}\widetilde{A} & \widetilde{B}\\ \widetilde{C} & \widetilde{D}\end{bmatrix}:\widetilde{\mathcal{X}}\oplus\mathcal{U}\to\widetilde{\mathcal{X}}^d\oplus\mathcal{Y}$$

are said to be *unitarily equivalent* if there is a unitary operator $U : \mathcal{X} \to \widetilde{\mathcal{X}}$ such that

$$\begin{bmatrix} \bigoplus_{k=1}^{d} U & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}.$$

As it was shown in [7], any observable weakly coisometric realization of a Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is unitarily equivalent to some generalized functional-model realization. An analogous result concerning the universality of functional-model realizations among commutative realizations is more specific.

Theorem 3.6. *Suppose that $S(\boldsymbol{\lambda}) \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is a Schur-class function that admits functional-model realizations. Then any commutative, observable, weakly coisometric realization of S is unitarily equivalent to exactly one functional-model realization of S.*

Proof. Let $S(\boldsymbol{\lambda}) = D + \widetilde{C}(I_{\widetilde{\mathcal{X}}} - Z(\boldsymbol{\lambda})\widetilde{A})^{-1}Z(\boldsymbol{\lambda})\widetilde{B}$ be a commutative, observable, weakly coisometric realization of S. Then $K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_{\widetilde{C},\widetilde{\mathbf{A}}}(\boldsymbol{\lambda}, \boldsymbol{\zeta})$ by Proposition 2.2. Define operators A_j's and C as in (3.10). Since S admits functional-model realizations (that contain A_j's and C and are weakly coisometric), then we have also $K_{C,\mathbf{A}}(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta})$. Therefore $K_{C,\mathbf{A}} = K_{\widetilde{C},\widetilde{\mathbf{A}}}$. Since the pairs $(\widetilde{C}, \widetilde{\mathbf{A}})$ and $(C, \mathbf{A})$ are observable, the latter equality implies (see [6, Theorem 3.17]) that there exists a unitary operator $U : \mathcal{H}(K_S) \to \widetilde{\mathcal{X}}$ such that

$$C = \widetilde{C}U \quad \text{and} \quad A_j = U^*\widetilde{A}_jU \quad \text{for } j = 1, \dots, d.$$

Now we let $B_j := U^*\widetilde{B}_j : \mathcal{Y} \to \mathcal{H}(K_S)$ for $j = 1, \dots, d$ which is the unique choice that guarantees the realization $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ (with A and B defined as in (1.7)) to be unitarily equivalent to the original realization $\widetilde{\mathbf{U}} = \left[\begin{smallmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & D \end{smallmatrix}\right]$ and it is functional-model realization due to the canonical choice of C and A_j's. □

Corollary 3.7. *Let $\mathbf{U}' = \left[\begin{smallmatrix} A' & B' \\ C' & D \end{smallmatrix}\right]$ and $\mathbf{U}'' = \left[\begin{smallmatrix} A'' & B'' \\ C'' & D \end{smallmatrix}\right]$ be two observable commutative weakly coisometric realizations of a Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. Then the pairs $(C', \mathbf{A}')$ and $(C'', \mathbf{A}'')$ are unitarily equivalent.*

Proof. By Theorem 3.6, the pairs $(C', \mathbf{A}')$ and $(C'', \mathbf{A}'')$ are both unitarily equivalent to the canonical pair $(C, \mathbf{A})$ with C and $\mathbf{A} = (A_1, \dots, A_d)$ defined as in (3.10). Hence $(C', \mathbf{A}')$ and $(C'', \mathbf{A}'')$ are unitarily equivalent to each other. □

Remark 3.8. It was pointed out in [7] and justified by examples (e.g., [7, Example 3.5]) that a Schur-class function may have many weakly coisometric observable realizations with associated output pairs $(C, \mathbf{A})$ not unitarily equivalent. Theorem 3.6 above shows that if $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ admits a commutative weakly coisometric realization, then the output pair $(C, \mathbf{A})$ of *any* commutative weakly coisometric observable realization is uniquely defined up to unitary equivalence. The example below shows that in the latter case, S may also admit many *noncommutative* observable weakly coisometric realizations with output pairs not unitarily equivalent. This example is of certain interest because of Theorem 4.4 below showing that this situation is not relevant if S is an inner multiplier.

Example 3.9. Take the matrices

$$C = \begin{bmatrix} \tfrac{1}{2} & 0 & 0 \end{bmatrix}, \qquad A_{0,1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_{0,2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B_{0,1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad B_{0,2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 & \tfrac{\sqrt{3}}{2} \end{bmatrix} \tag{3.11}$$

so that the 7×8 matrix $\mathbf{U}_0 = \left[\begin{smallmatrix} A_{0,1} & B_{0,1} \\ A_{0,2} & B_{0,2} \\ C & D \end{smallmatrix}\right]$ is coisometric. Then the characteristic function of the colligation $\mathbf{U}_0$,

$$S(\boldsymbol{\lambda}) = D + C(I - \lambda_1 A_{0,1} - \lambda_2 A_{0,2})^{-1}(\lambda_1 B_{0,1} + \lambda_2 B_{0,2}) \tag{3.12}$$

belongs to the Schur class $\mathcal{S}_2(\mathbb{C}^5, \mathbb{C})$. It is readily seen that

$$C(I-\lambda_1 A_{0,1}-\lambda_2 A_{0,2})^{-1}=\frac{1}{2}\cdot[\,1 \quad \lambda_1 \quad \lambda_2\,] \tag{3.13}$$

which being substituted along with (3.11) into (3.12) gives the explicit formula

$$S(\boldsymbol{\lambda})=\frac{1}{2}\cdot[\,\lambda_1^2 \quad \lambda_1\lambda_2 \quad \lambda_1\lambda_2 \quad \lambda_2^2 \quad \sqrt{3}\,]. \tag{3.14}$$

It is readily seen that the pair $(C,\mathbf{A}_0)$ is observable (where we let $\mathbf{A}_0=(A_{0,1},A_{0,2})$) and thus, representation (3.12) is a coisometric (and therefore, also weakly coisometric) observable realization of the function $S\in\mathcal{S}_2(\mathbb{C}^5,\mathbb{C})$ given by (3.14). Then we also have

$$K_S(\boldsymbol{\lambda},\boldsymbol{\zeta})=C(I-\lambda_1 A_{0,1}-\lambda_2 A_{0,2})^{-1}\big(I-\bar{\zeta}_1 A_{0,1}^*-\bar{\zeta}_2 A_{0,2}^*\big)^{-1}C^*=K_{C,\mathbf{A}_0}(\boldsymbol{\lambda},\boldsymbol{\zeta}). \tag{3.15}$$

Now let us consider the matrices $A_{\gamma,1}=\left[\begin{smallmatrix}0&1&0\\0&0&0\\\gamma&0&0\end{smallmatrix}\right]$ and $A_{\gamma,2}=\left[\begin{smallmatrix}0&0&1\\-\gamma&0&0\\0&0&0\end{smallmatrix}\right]$ where $\gamma\in\mathbb{C}$ is a parameter, and note that

$$C(I-\lambda_1 A_{\gamma,1}-\lambda_2 A_{\gamma,2})^{-1}=\frac{1}{2}\cdot[\,1 \quad \lambda_1 \quad \lambda_2\,]$$

for every γ. In particular, the pair $(C,\mathbf{A}_\gamma)$ is observable for every γ. The latter equality together with (3.15) gives

$$K_S(\boldsymbol{\lambda},\boldsymbol{\zeta})=K_{C,\mathbf{A}_\gamma}(\boldsymbol{\lambda},\boldsymbol{\zeta}). \tag{3.16}$$

Now pick any γ so that $|\gamma|<\sqrt{\frac{3}{8}}$. As it is easily seen, the latter inequality is equivalent to the pair $(C,\mathbf{A}_\gamma)$ being contractive. Thus, we have a Schur-class function S and a contractive pair $(C,\mathbf{A}_\gamma)$ such that equality (3.16) holds. Then by Theorem 2.3, there exist operators $B_{\gamma,1}$ and $B_{\gamma,2}$ so that the operator $\mathbf{U}_\gamma=\left[\begin{smallmatrix}A_{\gamma,1}&B_{\gamma,1}\\A_{\gamma,2}&B_{\gamma,2}\\C&D\end{smallmatrix}\right]$ is weakly coisometric and S can be realized as

$$S(\boldsymbol{\lambda})=D+C(I-\lambda_1 A_{\gamma,1}-\lambda_2 A_{\gamma,2})^{-1}(\lambda_1 B_{\gamma,1}+\lambda_2 B_{\gamma,2}).$$

It remains to note that the pairs $(C,\mathbf{A}_\gamma)$ and $(C,\mathbf{A}_{\gamma'})$ are not unitarily equivalent (which is shown by another elementary calculation) unless $\gamma=\gamma'$.

We conclude this section with characterizing Schur-class functions that admit contractive commutative realizations.

Theorem 3.10. *A Schur-class function $S\in\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ admits a contractive commutative realization if and only if it can be extended to a Schur-class function*

$$\widehat{S}(\boldsymbol{\lambda})=[\,S(\boldsymbol{\lambda}) \quad \widetilde{S}(\boldsymbol{\lambda})\,]\in\mathcal{S}_d(\mathcal{U}\oplus\mathcal{F},\mathcal{Y}) \tag{3.17}$$

such that the de Branges–Rovnyak space $\mathcal{H}(K_{\widehat{S}})$ is $\mathbf{M}_{\boldsymbol{\lambda}}^*$-invariant and the inequality

$$\sum_{j=1}^d \big\|M_{\lambda_j}^* f\big\|^2_{\mathcal{H}(K_{\widehat{S}})}\leqslant\|f\|^2_{\mathcal{H}(K_{\widehat{S}})}-\big\|f(0)\big\|^2_{\mathcal{Y}} \tag{3.18}$$

holds for every $f\in\mathcal{H}(K_{\widehat{S}})$.

Proof. Let S admit a contractive commutative realization of the form (1.6). Extend the connecting operator $\mathbf{U}$ of the form (1.5) to a coisometric operator

$$\widehat{\mathbf{U}}=\begin{bmatrix}A & B & \widetilde{B}\\ C & D & \widetilde{D}\end{bmatrix}:\begin{bmatrix}\mathcal{X}\\ \mathcal{U}\oplus\mathcal{F}\end{bmatrix}\to\begin{bmatrix}\mathcal{X}^d\\ \mathcal{Y}\end{bmatrix}. \tag{3.19}$$

The function

$$\widehat{S}(\boldsymbol{\lambda})=[\,D \quad \widetilde{D}\,]+C\big(I-Z(\boldsymbol{\lambda})A\big)^{-1}Z(\boldsymbol{\lambda})[\,B \quad \widetilde{B}\,] \tag{3.20}$$

is an extension of S in the sense of (3.17). The latter realization is coisometric and commutative; thus $\mathbf{M}_{\boldsymbol{\lambda}}^*$-invariance of $\mathcal{H}(\widehat{S})$ and inequality (3.18) hold by Theorem 3.5.

Conversely, if S can be extended to a Schur-class function $\widehat{S}$ with associated de Branges–Rovnyak space $\mathcal{H}(K_{\widehat{S}})$ invariant under $\mathbf{M}_{\boldsymbol{\lambda}}^*$ and satisfying property (3.18), we consider a weakly coisometric commutative realization (3.20) of $\widehat{S}$ (which exists by Theorem 3.5) and restrict the input space to $\mathcal{U}$. This gives a contractive commutative realization for S. □

4. Realization for inner multipliers

The de Branges–Rovnyak space $\mathcal{H}(K_S)$ defined for $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ originally as the RKHS with reproducing kernel K_S of the form (1.4) can be alternatively characterized via general complementation theory as the range space $\mathcal{H}(K_S) = \operatorname{Ran}(I - M_S M_S^*)^{1/2}$ with the range norm

$$\big\| \big(I - M_S M_S^*\big)^{1/2} f \big\|_{\mathcal{H}(K_S)} = \|Q_1 f\|_{\mathcal{H}_{\mathcal{Y}}(k_d)} \quad \big(f \in \mathcal{H}_{\mathcal{Y}}(k_d)\big) \tag{4.1}$$

where $M_S : \mathcal{H}_{\mathcal{U}}(k_d) \to \mathcal{H}_{\mathcal{Y}}(k_d)$ is the multiplication operator defined in (1.3) and Q_1 is the orthogonal projection of $\mathcal{H}_{\mathcal{Y}}(k_d)$ onto $(\operatorname{Ker}(I - M_S M_S^*)^{1/2})^{\perp}$. The space complementary to $\mathcal{H}(K_S)$ is the space $\mathcal{H}(\mathbb{M}_S)$ with reproducing kernel

$$\mathbb{M}_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = \frac{S(\boldsymbol{\lambda}) S(\boldsymbol{\zeta})^*}{1 - \langle \boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle} = k_d(\boldsymbol{\lambda}, \boldsymbol{\zeta}) I_{\mathcal{Y}} - K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta})$$

which, in turn, can be characterized as the range space $\mathcal{H}(\mathbb{M}_S) = \operatorname{Ran} M_S$ with the lifted norm

$$\|M_S g\|_{\mathcal{H}(\mathbb{M}_S)} = \|Q_2 g\|_{\mathcal{H}_{\mathcal{U}}(k_d)} \quad \text{for all } g \in \mathcal{H}_{\mathcal{U}}(k_d) \tag{4.2}$$

where Q_2 is the orthogonal projection of $\mathcal{H}_{\mathcal{U}}(k_d)$ onto $(\operatorname{Ker} M_S)^{\perp}$. The spaces $\mathcal{H}(K_S)$ and $\mathcal{H}(\mathbb{M}_S)$ are contractively included in $\mathcal{H}_{\mathcal{Y}}(k_d)$; the case when they are *isometrically* included in $\mathcal{H}_{\mathcal{Y}}(k_d)$ is of special interest. Let us recall that a Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is said to be an *inner multiplier* (called *inner sequence* in [5]) if the multiplication operator $M_S : \mathcal{H}_{\mathcal{U}}(k_d) \to \mathcal{H}_{\mathcal{Y}}(k_d)$ is a partial isometry.

Proposition 4.1. *Let $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. The following are equivalent*:

(1) *S is inner.*
(2) *$\mathcal{H}(K_S)$ is contained in $\mathcal{H}_{\mathcal{Y}}(k_d)$ isometrically.*
(3) *$\mathcal{H}(K_S) = (\operatorname{Ran} M_S)^{\perp}$ isometrically.*

In this case, $M_S M_S^*$ and $I_{\mathcal{H}_{\mathcal{Y}}(k_d)} - M_S M_S^*$ are the orthogonal projections onto the closed subspaces $\operatorname{Ran} M_S$ *and* $(\operatorname{Ran} M_S)^{\perp}$ *of $\mathcal{H}_{\mathcal{Y}}(k_d)$, respectively.*

Proof. The multiplier S being inner is equivalent to $\operatorname{Ran} M_S$ and $\operatorname{Ran}(I - M_S M_S^*)^{1/2}$ being closed subspaces of $\mathcal{H}_{\mathcal{Y}}(k_d)$ such that $M_S M_S^*$ and $I - M_S M_S^* = (I - M_S M_S^*)^{1/2}$ are the orthogonal projections onto $\operatorname{Ran} M_S$ and $\operatorname{Ran}(I - M_S M_S^*)$, respectively. In this case the lifted-norm formulas (4.1) and (4.2) lead to isometric inclusions of $\mathcal{H}(K_S)$ and of $\mathcal{H}(\mathbb{M}_S)$ in $\mathcal{H}_{\mathcal{Y}}(k_d)$. The fact that $P_{\mathcal{H}(K_S)} = I - P_{\mathcal{H}(\mathbb{M}_S)}$ tells us that $\mathcal{H}(K_S)$ and $\mathcal{H}(\mathbb{M}_S)$ are orthogonal complements of each other. □

In this section we focus on realization theory for inner multipliers. First we will show that an inner multiplier always admits a commutative weakly coisometric realization (Theorem 4.2). Then we show that, in contrast to general contractive multipliers (see Example 3.9), an inner multiplier cannot have a noncommutative observable weakly coisometric realization (Theorem 4.4). Finally, Theorem 4.7 discusses coisometric and unitary functional-model realizations for an inner multiplier. We start with a realization characterization of inner multipliers.

Theorem 4.2. *An $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function S defined on $\mathbb{B}^d$ is an inner multiplier if and only if it admits a weakly coisometric realization* (1.6) *where*

(1) *the d-tuple* $\mathbf{A} = (A_1, \dots, A_d)$ *of the state space operators is commutative and is strongly stable* (*i.e.,* (3.7) *holds*), *and*
(2) *the output pair* $(C, \mathbf{A})$ *is isometric.*

Proof. Suppose that S admits a realization (1.6) with $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ weakly coisometric with $\mathbf{A}$ commutative and strongly stable and with (3.6) holding. By Proposition 2.2 we know that $K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_{C,\mathbf{A}}(\boldsymbol{\lambda}, \boldsymbol{\zeta})$. Combining this equality with Theorem 3.2 (part (2)), we conclude that the space $\mathcal{H}(K_S) = \mathcal{H}(K_{C,\mathbf{A}})$ is included isometrically in $\mathcal{H}_{\mathcal{Y}}(k_d)$. Therefore S is inner by Proposition 4.1.

Conversely, suppose that S is inner. Then, according to Proposition 4.1, $\mathcal{H}(K_S)$ is isometrically equal to the orthogonal complement of $\operatorname{Ran} M_S$. As $\operatorname{Ran} M_S$ is invariant under $\mathbf{M}_{\boldsymbol{\lambda}}$, it follows that $\mathcal{H}(K_S) = (\operatorname{Ran} M_S)^{\perp}$ is $\mathbf{M}_{\boldsymbol{\lambda}}^*$-invariant. Hence Theorem 3.5 applies; we let $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be any weakly coisometric functional-model realization for S, that is with $\mathbf{A} = (A_1, \dots, A_d)$, C and D defined as in (3.10). Then $\mathbf{A}$ is commutative since $\mathbf{M}_{\boldsymbol{\lambda}}^*$ is commutative. As has been already observed, $\mathcal{H}(K_S) = (\operatorname{Ran} M_S)^{\perp}$ is contained in $\mathcal{H}_{\mathcal{Y}}(k_d)$ isometrically. Therefore $\mathbf{A} = \mathbf{M}_{\boldsymbol{\lambda}}^*|_{\mathcal{H}(K_S)}$ is strongly stable since $\mathbf{M}_{\boldsymbol{\lambda}}^*$ is strongly stable on $\mathcal{H}_{\mathcal{Y}}(k_d)$ by Proposition 3.1 (part (3)). By part (2) in the same proposition, the pair $(G, \mathbf{M}_{\boldsymbol{\lambda}}^*)$ is isometric, i.e.,

$$I_{\mathcal{H}_{\mathcal{Y}}(k_d)} - M_{\lambda_1} M_{\lambda_1}^* - \cdots - M_{\lambda_d} M_{\lambda_d}^* = G^* G. \tag{4.3}$$

Since G and C are the operators of evaluation at the origin on $\mathcal{H}_{\mathcal{Y}}(k_d)$ and on $\mathcal{H}(K_S)$, respectively, we have $C = G|_{\mathcal{H}(K_S)}$. Then the restriction operator equality (4.3) to $\mathcal{H}(K_S)$ can be expressed in terms of $\mathbf{A}$ and C as

$$I_{\mathcal{H}(K_S)} - A_1^* A_1 - \cdots - A_d^* A_d = C^* C$$

which means that the pair $(C, \mathbf{A})$ is isometric. □

The next theorem is a variant of Theorem 3.10 for the inner case; the proof is much the same as that of Theorem 3.10 and hence will be omitted.

Theorem 4.3. *A Schur-class function* $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ *admits a contractive, commutative realization of the form* (1.6) *with* $\mathbf{A} = (A_1, \dots, A_d)$ *strongly stable and* $(C, \mathbf{A})$ *isometric if and only if* S *can be extended to an inner multiplier*

$$\widehat{S}(\boldsymbol{\lambda}) = [\, S(\boldsymbol{\lambda}) \quad \widetilde{S}(\boldsymbol{\lambda}) \,] \in \mathcal{S}_d(\mathcal{U} \oplus \mathcal{F}, \mathcal{Y}).$$

If S is inner, then, as we have seen in the proof of Theorem 4.2, any functional-model realization for S yields a commutative observable weakly coisometric realization for S. We now show the converse.

Theorem 4.4. *If* $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ *is inner, then any observable weakly coisometric realization of* S *is commutative.*

Proof. Let (1.6) be an observable weakly coisometric realization of the inner function S. Then $K_S = K_{C,\mathbf{A}}$ (by Proposition 2.2) and therefore, since S is inner, the space $\mathcal{H}(K_{C,\mathbf{A}})$ is isometrically included into $\mathcal{H}_{\mathcal{Y}}(k_d)$. By Theorem 2.1 (part (3)), the observability operator $\mathcal{O}_{C,\mathbf{A}} : \mathcal{X} \to \mathcal{H}_{\mathcal{Y}}(k_d)$ is a partial isometry. Since the pair $(C, \mathbf{A})$ is observable, $\mathcal{O}_{C,\mathbf{A}}$ is in fact an isometry. As $\mathcal{H}(K_{C,\mathbf{A}})$ is isometrically included in $\mathcal{H}_{\mathcal{Y}}(k_d)$, it follows that $\mathcal{O}_{C,\mathbf{A}}$ is unitary when considered as an operator from $\mathcal{X}$ to $\mathcal{H}(K_{C,\mathbf{A}}) = \mathcal{H}(K_S)$. Define the operators $T_1, \dots, T_d$ on $\mathcal{H}(K_{C,\mathbf{A}})$ and the operator $G : \mathcal{H}(K_{C,\mathbf{A}}) \to \mathcal{Y}$ by

$$T_j \mathcal{O}_{C,\mathbf{A}} x = \mathcal{O}_{C,\mathbf{A}} A_j x \quad (j = 1, \dots, d), \qquad G \mathcal{O}_{C,\mathbf{A}} x = C x \quad \text{for } x \in \mathcal{X}. \tag{4.4}$$

Then for the generic element $f = \mathcal{O}_{C,\mathbf{A}} x$ of $\mathcal{H}(K_S) = \mathcal{H}(K_{C,\mathbf{A}}) = \operatorname{Ran} \mathcal{O}_{C,\mathbf{A}}$, we have

$$f(\boldsymbol{\lambda}) = C\big(I - Z(\boldsymbol{\lambda}) A\big)^{-1} x, \qquad f(0) = C x = G \mathcal{O}_{C,\mathbf{A}} x = G f$$

and therefore,

$$\begin{aligned} f(\boldsymbol{\lambda}) - f(0) &= C\big(I - Z(\boldsymbol{\lambda}) A\big)^{-1} x - C x \\ &= C\big(I - Z(\boldsymbol{\lambda}) A\big)^{-1} Z(\boldsymbol{\lambda}) A x \end{aligned}$$

$$
\begin{aligned}
&= C\big(I - Z(\boldsymbol{\lambda})A\big)^{-1}\left(\sum_{j=1}^{d} \lambda_j A_j x\right) \\
&= \sum_{j=1}^{d} \lambda_j \cdot (\mathcal{O}_{C,\mathbf{A}} A_j x)(\boldsymbol{\lambda}) \\
&= \sum_{j=1}^{d} \lambda_j \cdot (T_j \mathcal{O}_{C,\mathbf{A}} x)(\boldsymbol{\lambda}) = \sum_{j=1}^{d} \lambda_j \cdot (T_j f)(\boldsymbol{\lambda})
\end{aligned}
$$

which means that the d-tuple $\mathbf{T} = (T_1, \dots, T_d)$ solves the Gleason problem on $\mathcal{H}(K_{C,\mathbf{A}})$ and that G is simply the operator of evaluation at the origin. Since the pair $(C, \mathbf{A})$ is contractive and $\mathcal{O}_{C,\mathbf{A}}$ is isometric, it follows from (4.4) that the pair $(G, \mathbf{T})$ is also contractive. Now we recall a uniqueness result from [6, Theorem 3.22]: *if $\mathcal{M}$ is a backward-shift invariant subspace of $\mathcal{H}_{\mathcal{Y}}(k_d)$ isometrically included in $\mathcal{H}_{\mathcal{Y}}(k_d)$, then the d-tuple $\mathbf{M}^*_{\boldsymbol{\lambda}}|_{\mathcal{M}} = (M^*_{\lambda_1}|_{\mathcal{M}}, \dots, M^*_{\lambda_d}|_{\mathcal{M}})$ is the only contractive solution of the Gleason problem on $\mathcal{M}$*. By this result applied to $\mathcal{M} = \mathcal{H}(K_{C,\mathbf{A}}) = \mathcal{H}(K_S) = (S \cdot \mathcal{H}_{\mathcal{U}}(k_d))^{\perp}$ (which is backward-shift invariant and isometrically included in $\mathcal{H}_{\mathcal{Y}}(k_d)$ since S is inner) we conclude that $T_j = M^*_{\lambda_j}|_{\mathcal{H}(K_{C,\mathbf{A}})}$ for $j = 1, \dots, d$. In particular, the tuple $\mathbf{T}$ is commutative and therefore the original state space tuple $\mathbf{A}$ is necessarily commutative. □

By the result of [20], any inner function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ has nontangential boundary values $S(\omega)$ which are partial isometries of some fixed rank for almost all ω on the $(2d-1)$-dimensional sphere $\mathbb{S}^{2d-1} := \partial\mathbb{B}^d$. Let us say that the inner function S is a *full-range inner function* if the boundary-value function $S(\omega)$ has coisometric values for almost all $\omega \in \mathbb{S}^{2d-1}$. Then we have the following extension of Theorem 4.4 to contractive realizations.

Theorem 4.5. *If $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is a full-range inner function* (*in particular, if S is a nonzero inner function in $\mathcal{S}_d(\mathcal{U}, \mathbb{C})$*), *then any contractive realization of S is commutative.*

Proof. Assume that $S(\boldsymbol{\lambda})$ is a full-range inner function of the form $S(\boldsymbol{\lambda}) = D + C(I - Z(\boldsymbol{\lambda})A)^{-1}Z(\boldsymbol{\lambda})B$ for a contractive connecting operator $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathcal{X} \oplus \mathcal{U} \to \mathcal{X}^d \oplus \mathcal{Y}$. Extend $\mathbf{U}$ to a coisometric operator $\widehat{\mathbf{U}}$ as in (3.19) and consider its characteristic function $\widehat{S}$ (see (3.20)) which extends S in the sense of (3.17). Since $\widehat{\mathbf{U}}$ is a contraction, $\widehat{S}$ is in $\mathcal{S}_d(\mathcal{U} \oplus \mathcal{F}, \mathcal{Y})$. By (3.17),

$$
I_{\mathcal{Y}} - \widehat{S}(\omega)\widehat{S}(\omega)^* = I_{\mathcal{Y}} - S(\omega)S(\omega)^* - \widetilde{S}(\omega)\widetilde{S}(\omega)^* \geqslant 0
$$

for almost all $\omega \in \mathbb{S}^{2d-1}$. Since by assumption S is a full-range inner, its boundary values are coisometries and hence $\widetilde{S}(\omega) = 0$ for almost all $\omega \in \mathbb{S}^{2d-1}$. Therefore $\widetilde{S} \equiv 0$ and hence $\widehat{S} = [\,S \quad 0\,]$ is also inner. The formula (3.20) then gives an observable coisometric (and therefore also weakly coisometric) realization of the inner function $\widehat{S}(\boldsymbol{\lambda})$. By Theorem 4.4 it follows that this realization is necessarily commutative, i.e., $\mathbf{A} = (A_1, \dots, A_d)$ is commutative. Hence the original realization $\mathbf{U}$ for $S(\boldsymbol{\lambda})$ is commutative as asserted. □

Remark 4.6. In general it is difficult to tell from the boundary values alone when a given Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is inner. The result of the preceding paragraph suggests the following open question: *If $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is such that* (1) *the boundary values of S on $\mathbb{S}^{2d-1}$ are partially isometric of fixed rank* (*or, as a special case, coisometric*) *a.e., and* (2) *every observable contractive realization of S is commutative, does it then follow that S is an inner function*?

Theorems 4.4 and 4.5 combined with Theorem 3.6 imply that any observable weakly coisometric realization of S if S is inner, or even any observable contractive realization of S if S is full-range inner, of the form (1.6) is commutative with operators $A_1, \dots, A_d, C$ uniquely defined (up to simultaneous unitary equivalence) and with D given by formula (3.10). The nonuniqueness caused by possible different choices of $B_1, \dots, B_d : \mathcal{U} \mapsto \mathcal{H}(K_S)$ can be described explicitly. This was done in [7, Theorem 2.7] in the context of general contractive multipliers. For inner multipliers the corresponding results are much more explicit.

Theorem 4.7. *Let $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ be inner, let $\mathbf{A} = (A_1, \dots, A_d)$, C, D be given as in* (3.10)*, let*

$$\mathcal{U}_S^0 := \{u \in \mathcal{U}\colon\ S(\boldsymbol{\lambda})u \equiv 0\} \subset \mathcal{U} \tag{4.5}$$

and let the subspaces $\mathcal{D}$ and $\mathcal{D}^\perp \subset \mathcal{H}(K_S)^d$ be defined as in (2.11) *and* (2.12)*, respectively. Let $N : \mathcal{D} \to \mathcal{U}$ be the operator defined by*

$$N : Z(\boldsymbol{\zeta})^* K_S(\cdot, \boldsymbol{\zeta}) y \to S(\boldsymbol{\zeta})^* y - S(0)^* y. \tag{4.6}$$

Then

(1) *A realization $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & S(0) \end{smallmatrix}\right]$ of S is weakly coisometric if and only if B is of the form*

$$B^* = [\,X \quad N\,] \tag{4.7}$$

where N is given by (4.6) *and X is a contraction from $\mathcal{D}^\perp$ into $\mathcal{U}_S^0$.*

(2) *S admits a coisometric* (*unitary*) *functional-model realization if and only if* $\dim \mathcal{D}^\perp \leqslant \dim \mathcal{U}_S^0$ (*respectively* $\dim \mathcal{D}^\perp = \dim \mathcal{U}_S^0$). *In this case, a realization $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & S(0) \end{smallmatrix}\right]$ of S is coisometric* (*unitary*) *if and only if B is of the form* (4.7) *for some isometric* (*respectively unitary*) $X : \mathcal{D}^\perp \to \mathcal{U}_S^0$.

(3) *S admits a unique weakly coisometric functional-model realization if and only if $\mathcal{U}_S^0 = \{0\}$. In this case, the operator B is defined by*

$$B^*\big|_{\mathcal{D}} : Z(\boldsymbol{\zeta})^* K_S(\cdot, \boldsymbol{\zeta}) y \to S(\boldsymbol{\zeta})^* y - S(0)^* y \quad \textit{and} \quad B^*\big|_{\mathcal{D}^\perp} = 0.$$

This unique weakly coisometric functional-model realization is never coisometric.

Proof. Let $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ be inner, let $\mathbf{A} = (A_1, \dots, A_d)$, C, D be given as in (3.10) and let $B : \mathcal{U} \to (\mathcal{H}(K_S))^d$ be any operator so that S can be realized in the form (1.6) and $\mathbf{U}$ as in (1.5) is contractive. Then taking adjoints in (1.6) gives

$$B^* Z(\boldsymbol{\zeta})^* \big(I - A^* Z(\boldsymbol{\zeta})^*\big)^{-1} C^* = S(\boldsymbol{\zeta})^* - D^*$$

which, on account of (2.9), can be written equivalently as

$$B^* Z(\boldsymbol{\zeta})^* K_S(\cdot, \boldsymbol{\zeta}) y = S(\boldsymbol{\zeta})^* y - S(0)^* y \quad \big(\boldsymbol{\zeta} \in \mathbb{B}^d,\ y \in \mathcal{Y}\big).$$

Comparing the latter formula with (4.6) gives $B^*|_{\mathcal{D}} = N$. Write B^* in the form (4.7) with $X = B^*|_{\mathcal{D}^\perp} : \mathcal{D}^\perp \to \mathcal{U}$. Next we note the explicit formulas for the adjoints A_j^*'s

$$A_j^* = \mathcal{P}_{\mathcal{H}(K_S)} M_{\lambda_j}|_{\mathcal{H}(K_S)} \quad (j = 1, \dots, d) \tag{4.8}$$

(where $\mathcal{P}_{\mathcal{H}(K_S)}$ stands for the orthogonal projection of $\mathcal{H}_{\mathcal{Y}}(k_d)$ onto $\mathcal{H}(K_S)$) which are not available in the case of general (noninner) Schur-class functions. Indeed, since $\mathcal{H}(K_S)$ is isometrically included in $\mathcal{H}_{\mathcal{Y}}(k_d)$, we have for every $h, g \in \mathcal{H}(K_S)$,

$$\begin{aligned}
\big\langle h, A_j^* g\big\rangle_{\mathcal{H}(K_S)} &= \langle A_j h, g\rangle_{\mathcal{H}(K_S)} \\
&= \big\langle M_{\lambda_j}^* h, g\big\rangle_{\mathcal{H}(K_S)} \\
&= \big\langle M_{\lambda_j}^* h, g\big\rangle_{\mathcal{H}_{\mathcal{Y}}(k_d)} \\
&= \langle h, M_{\lambda_j} g\rangle_{\mathcal{H}_{\mathcal{Y}}(k_d)} = \langle h, \mathcal{P}_{\mathcal{H}(K_S)} M_{\lambda_j} g\rangle_{\mathcal{H}(K_S)}
\end{aligned}$$

and (4.8) follows. As a consequence of (4.8) we get

$$A^*\big|_{\mathcal{D}^\perp} = 0. \tag{4.9}$$

Indeed, if $h = \begin{bmatrix} h_1 \\ \vdots \\ h_d \end{bmatrix} \in \mathcal{D}^\perp$, it holds that $Z(\boldsymbol{\lambda})h(\boldsymbol{\lambda}) \equiv 0$ (by the characterization of $\mathcal{D}^\perp$ in (2.12)) and then

$$A^* h = \sum_{j=1}^d A_j^* h_j = \mathcal{P}_{\mathcal{H}(K_S)} M_{\lambda_j} h_j = \mathcal{P}_{\mathcal{H}(K_S)}(Zh) = 0.$$

Now we define the operators $T_1 : \mathcal{D} \oplus \mathcal{Y} \to \mathcal{H}(K_S)$ and $T_2 : \mathcal{D} \oplus \mathcal{Y} \to \mathcal{U}$ by

$$T_1 = [\, A^*|_{\mathcal{D}} \quad C^* \,] \quad \text{and} \quad T_2 = [\, B^*|_{\mathcal{D}} \quad S(0)^* \,] \tag{4.10}$$

and combining the two latter formulas with (4.7) and (4.9), we may write the adjoint of the connecting operator $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ as

$$\mathbf{U}^* = \begin{bmatrix} 0 & T_1 \\ X & T_2 \end{bmatrix} : \begin{bmatrix} \mathcal{D}^\perp \\ \mathcal{D} \oplus \mathcal{Y} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(K_S) \\ \mathcal{U} \end{bmatrix}. \tag{4.11}$$

In the latter formula we have identified $\left[\begin{smallmatrix} \mathcal{D}^\perp \\ \mathcal{D} \oplus \mathcal{Y} \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} \mathcal{H}(K_S)^d \\ \mathcal{Y} \end{smallmatrix}\right]$. Every X such that the matrix in (4.11) is contractive leads to a contractive functional-model realization for S (due to canonical choice (3.10) of C and $\mathbf{A}$) which is automatically weakly coisometric. Therefore, the restriction of $\mathbf{U}^*$ to the space $\mathcal{D} \oplus \mathcal{Y}$ (that is, the operator $\left[\begin{smallmatrix} T_1 \\ T_2 \end{smallmatrix}\right]$) is isometric

$$T_1^* T_1 + T_2^* T_2 = I_{\mathcal{D} \oplus \mathcal{Y}}. \tag{4.12}$$

Since the pair $(C, \mathbf{A})$ is isometric, it follows from (4.9) and the formula for T_1 in (4.10) that T_1 is coisometric

$$T_1 T_1^* = A^* A + C^* C = I_{\mathcal{H}(K_S)}. \tag{4.13}$$

Then we also have $T_1 T_2^* T_2 T_1^* = T_1 (I - T_1^* T_1) T_1^* = I - I = 0$, so that

$$T_1 T_2^* = 0. \tag{4.14}$$

Now we invoke (4.11) and make use of (4.12)–(4.14) to write the block-matrix formulas

$$I - \mathbf{U}\mathbf{U}^* = \begin{bmatrix} I - X^* X & -X^* T_2 \\ -T_2^* X & 0 \end{bmatrix} \tag{4.15}$$

and

$$I - \mathbf{U}^*\mathbf{U} = \begin{bmatrix} 0 & 0 \\ 0 & I - X X^* - T_2 T_2^* \end{bmatrix}. \tag{4.16}$$

From the formula for T_2 in (4.10) combined with the formula (4.6) for the action of $B^*|_{\mathcal{D}}$ on a generic generator of $\mathcal{D}$, we see that

$$\overline{\operatorname{Ran}}\, T_2 = \overline{\operatorname{span}} \bigl\{ S(\boldsymbol{\zeta})^* y \colon\ \boldsymbol{\zeta} \in \mathbb{B}^d,\ y \in \mathcal{Y} \bigr\}$$

and hence

$$\operatorname{Ker} T_2^* = (\overline{\operatorname{Ran}}\, T_2)^\perp = \bigl\{ u \in \mathcal{U} \colon\ S(\boldsymbol{\lambda}) u \equiv 0 \bigr\} =: \mathcal{U}_S^0. \tag{4.17}$$

Now it follows from (4.15) that $\mathbf{U}^*$ of the form (4.11) is contractive (isometric) if and only if X is a contraction (an isometry) from $\mathcal{D}^\perp$ into (onto) $\mathcal{U}_S^0$. Then (4.16) implies that $\mathbf{U}$ is unitary if and only if $X : \mathcal{D}^\perp \to \mathcal{U}_S^0$ is unitary. The corresponding B of the form (4.7) leads to weakly coisometric, coisometric and unitary realizations for S. This completes the proof. □

5. Beurling–Lax representation theorem for shift-invariant subspaces

The Beurling–Lax theorem for the context of the Arveson space $\mathcal{H}_{\mathcal{Y}}(k_d)$ asserts that any closed $\mathbf{M}_{\boldsymbol{\lambda}}$-invariant subspace $\mathcal{M}$ of $\mathcal{H}_{\mathcal{Y}}(k_d)$ can be represented in the form

$$\mathcal{M} = S \cdot \mathcal{H}_{\mathcal{U}}(k_d) \tag{5.1}$$

for some inner multiplier $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and an appropriately chosen coefficient space $\mathcal{U}$ (see [13,21,22] for the classical case $d = 1$ and [5,6,23] for the case of general d). We shall call any such S a *representer of* $\mathcal{M}$. Here we present a realization-theoretic proof of the $\mathcal{H}_{\mathcal{Y}}(k_d)$-Beurling–Lax theorem as an application of Theorem 4.2 (see [9,10] for an illustration of this approach for the case $d = 1$). We first need some preliminaries.

Suppose that $\mathbf{A} = (A_1, \ldots, A_d)$ is a commutative d-tuple of bounded, linear operators on the Hilbert space $\mathcal{X}$ and that $(C, \mathbf{A})$ is an output stable pair. We define a *left-tangential functional calculus* $f \to (C^* f)^{\wedge L}(\mathbf{A}^*)$ on $\mathcal{H}_{\mathcal{Y}}(k_d)$ by

$$\big(C^* f\big)^{\wedge L}\big(\mathbf{A}^*\big) = \sum_{\mathbf{n}\in\mathbb{Z}_+^d} \mathbf{A}^{*\mathbf{n}} C^* f_{\mathbf{n}} \quad \text{if } f = \sum_{\mathbf{n}\in\mathbb{Z}_+^d} f_{\mathbf{n}} \boldsymbol{\lambda}^{\mathbf{n}} \in \mathcal{H}_{\mathcal{Y}}(k_d). \tag{5.2}$$

The computation

$$\left\langle \sum_{\mathbf{n}\in\mathbb{Z}_+^d} \mathbf{A}^{*\mathbf{n}} C^* f_{\mathbf{n}}, x \right\rangle_{\mathcal{X}} = \sum_{\mathbf{n}\in\mathbb{Z}_+^d} \langle f_{\mathbf{n}}, C\mathbf{A}^{\mathbf{n}} x\rangle_{\mathcal{Y}} = \sum_{\mathbf{n}\in\mathbb{Z}_+^d} \frac{\mathbf{n}!}{|\mathbf{n}|!} \left\langle f_{\mathbf{n}}, \frac{|\mathbf{n}|!}{\mathbf{n}!} C\mathbf{A}^{\mathbf{n}} x \right\rangle_{\mathcal{Y}} = \langle f, \mathcal{O}_{C,\mathbf{A}} x\rangle_{\mathcal{H}_{\mathcal{Y}}(k_d)}$$

shows that the output-stability of the pair $(C, \mathbf{A})$ is exactly what is needed to verify that the infinite series in the definition (5.2) of $(C^* f)^{\wedge L}(\mathbf{A}^*)$ converges in the weak topology on $\mathcal{X}$. In fact the left-tangential evaluation with operator argument $f \to (C^* f)^{\wedge}(\mathbf{A}^*)$ amounts to the adjoint of the observability operator

$$\big(C^* f\big)^{\wedge L}\big(\mathbf{A}^*\big) = (\mathcal{O}_{C,\mathbf{A}})^* f \quad \text{for } f \in \mathcal{H}_{\mathcal{Y}}(k_d). \tag{5.3}$$

Given an output-stable pair $(C, \mathbf{A})$, define a subspace $\mathcal{M}_{\mathbf{A}^*, C^*} \subset \mathcal{H}_{\mathcal{Y}}(k_d)$ by

$$\mathcal{M}_{\mathbf{A}^*, C^*} = \big\{ f \in H_{\mathcal{Y}}(k_d) \colon \big(C^* f\big)^{\wedge L}\big(\mathbf{A}^*\big) = 0 \big\}. \tag{5.4}$$

An easy computation (using that $\mathbf{A}$ is commutative) shows that

$$\big(C^*[M_{\lambda_j} f]\big)^{\wedge L}\big(\mathbf{A}^*\big) = A_j^* \big(C^* f\big)^{\wedge L}\big(\mathbf{A}^*\big).$$

Hence any subspace $\mathcal{M} \subset \mathcal{H}_{\mathcal{Y}}(k_d)$ of the form $\mathcal{M} = \mathcal{M}_{\mathbf{A}^*, C^*}$ as in (5.4) is $\mathbf{M}_{\boldsymbol{\lambda}}$-invariant. We now obtain the converse.

Theorem 5.1. *Suppose that $\mathcal{M}$ is a closed subspace of $\mathcal{H}_{\mathcal{Y}}(k_d)$ which is $\mathbf{M}_{\boldsymbol{\lambda}}$-invariant (i.e., $\mathcal{M}$ is invariant under $M_{\lambda_j} : f(\boldsymbol{\lambda}) \mapsto \lambda_j f(\boldsymbol{\lambda})$ for $j = 1, \ldots, d$). Then there are a Hilbert space $\mathcal{X}$, a commutative d-tuple of operators $\mathbf{A} = (A_1, \ldots, A_d)$ on $\mathcal{X}$ and an operator $C : \mathcal{X} \to \mathcal{Y}$ so that*

(1) $\mathbf{A}$ *is commutative, i.e., $A_i A_j = A_j A_i$ for $1 \leqslant i, j \leqslant d$,*
(2) $\mathbf{A}$ *is strongly stable, i.e., $\mathbf{A}$ satisfies* (3.7)*, and*
(3) *the subspace $\mathcal{M}$ has the form $\mathcal{M}_{\mathbf{A}^*, C^*}$ as in* (5.4)*.*

Moreover, one choice of state space $\mathcal{X}$ and operators $A_j : \mathcal{X} \to \mathcal{X}$ and $C : \mathcal{X} \to \mathcal{Y}$ is

$$\mathcal{X} = \mathcal{M}^{\perp}, \qquad A_j = M_{\lambda_j}^*\big|_{\mathcal{M}^{\perp}} \quad \text{for } j = 1, \ldots, d, \qquad C : f \to f(0) \quad \text{for } f \in \mathcal{M}^{\perp}. \tag{5.5}$$

Proof. Define $\mathcal{X}$, $\mathbf{A} = (A_1, \ldots, A_d)$ and C as in (5.5). We note that $\mathcal{M}^{\perp}$ is contained in $\mathcal{H}_{\mathcal{Y}}(k_d)$ isometrically and that $C = G|_{\mathcal{M}^{\perp}}$, $A_j = M_{\lambda_j}^*|_{\mathcal{M}^{\perp}}$ where $\mathcal{O}_{G, \mathbf{M}_{\boldsymbol{\lambda}}^*}$ is the identity on $\mathcal{H}_{\mathcal{Y}}(k_d)$ (see part (2) of Proposition 3.1). Hence, in particular, $\operatorname{Ran} \mathcal{O}_{C,\mathbf{A}} = \mathcal{M}^{\perp}$. Taking orthogonal complements then gives

$$\operatorname{Ker}(\mathcal{O}_{C,\mathbf{A}})^* = \big(\mathcal{M}^{\perp}\big)^{\perp} = \mathcal{M}$$

which in turn is equivalent to the characterization (5.4) for $\mathcal{M}$. $\square$

We now construct an inner multiplier solving a homogeneous interpolation problem via realization theory.

Theorem 5.2. *Suppose that $(C, \mathbf{A})$ is an isometric output-stable pair, with $\mathbf{A}$ commutative and strongly stable. Let $\mathcal{M} = \mathcal{M}_{\mathbf{A}^*, C^*} \subset \mathcal{H}_{\mathcal{Y}}(k_d)$ be given by* (5.4)*. Then there is an input space $\mathcal{U}$ and an inner Schur multiplier $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ so that $\mathcal{M} = \operatorname{Ran} M_S$. One such S is given by*

$$S(\boldsymbol{\lambda}) = D + C\big(I - Z(\boldsymbol{\lambda})A\big)^{-1} Z(\boldsymbol{\lambda}) B$$

where $A_1, \dots, A_d$ and C come from the given output pair $(C, \mathbf{A})$, and where $B_1, \dots, B_d, D$ are chosen so that the colligation $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{smallmatrix}\right]$ is weakly coisometric. In particular, one achieves a coisometric realization $\mathbf{U}$ by choosing the input space $\mathcal{U}$ and $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ so as to solve the Cholesky factorization problem:

$$\begin{bmatrix} B \\ D \end{bmatrix} [\, B^* \quad D^* \,] = \begin{bmatrix} I_{\mathcal{X}^d} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} [\, A^* \quad C^* \,]. \tag{5.6}$$

Remark 5.3. Note that the model output pair $(C, \mathbf{A})$ (5.5) appearing in Theorem 5.1 is an isometric pair. In practice, however, one may be given a subspace of the form $\mathcal{M}_{\mathbf{A}^*,C}$ with $\mathbf{A}$ commutative and strongly stable but without the pair $(C, \mathbf{A})$ being isometric. If however it is the case that $(C, \mathbf{A})$ is *exactly observable* in the sense that the *observability gramian*

$$\mathcal{G}_{C,\mathbf{A}} = \mathcal{O}_{C,\mathbf{A}}^* \mathcal{O}_{C,\mathbf{A}} = \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} \mathbf{A}^{\mathbf{n}*} C^* C \mathbf{A}^{\mathbf{n}}$$

is strictly positive definite, then the adjusted output pair $(\widetilde{C}, \widetilde{\mathbf{A}})$ given by

$$\widetilde{A}_j = H^{1/2} A_j H^{-1/2} \quad \text{for } j = 1, \dots, d, \qquad \widetilde{C} = C H^{-1/2} \quad \text{where } H := \mathcal{G}_{C,\mathbf{A}}$$

is isometric and has all the other properties of the original output pair $(C, \mathbf{A})$, namely: $\widetilde{\mathbf{A}}$ is strongly stable and $\mathcal{M} = \mathcal{M}_{\widetilde{\mathbf{A}}^*, \widetilde{C}^*}$. Hence in practice the requirement that the pair $(C, \mathbf{A})$ be isometric in Theorem 5.2 can be replaced by the condition that $(C, \mathbf{A})$ is exactly observable. A more complete discussion of this point can be found in [6].

Proof of Theorem 5.2. Define $S(\lambda)$ as in the statement of the theorem. By Theorem 4.2, S is inner. By Proposition 4.1, $(\operatorname{Ran} M_S)^\perp = \mathcal{H}(K_S)$ isometrically. As $\mathbf{U}$ is weakly coisometric, we also know that $\mathcal{H}(K_S) = \mathcal{H}(K_{C,\mathbf{A}})$ by Proposition 2.2. The space $\mathcal{H}(K_{C,\mathbf{A}})$ can in turn be identified as a set with $\operatorname{Ran} \mathcal{O}_{C,\mathbf{A}}$ (see [6, Theorem 3.14]). By hypothesis, $\mathcal{M} = \operatorname{Ker}(\mathcal{O}_{C,\mathbf{A}})^*$; hence $\operatorname{Ran} \mathcal{O}_{C,\mathbf{A}} = \mathcal{M}^\perp$. Putting all this together gives $(\operatorname{Ran} M_S)^\perp = \mathcal{M}^\perp$ and therefore $\operatorname{Ran} M_S = \mathcal{M}$ as wanted. □

Remark 5.4. The choice $(\mathbf{A}^*, C^*) = (a, (1 - a^*a)^{1/2})$ in Theorem 5.2 (with B, D chosen to solve the Cholesky factorization problem (5.6)) leads to the Blaschke factor based on the point $a = (a_1, \dots, a_d) \in \mathbb{B}^d$ appearing in [2]. These Blaschke factors are also important in the characterization of the automorphisms of the ball mapping the origin to a given point (see [29, Theorem 2.2.2]).

We next show how our analysis can be used to give a description of all Beurling–Lax representers for a given shift-invariant subspace of $\mathcal{H}_{\mathcal{Y}}(k_d)$.

Theorem 5.5. *Let $\mathcal{M}$ be a closed $\mathbf{M}_\lambda$-invariant subspace of $\mathcal{H}_{\mathcal{Y}}(k_d)$, let $\mathcal{N} = \mathcal{M}^\perp = \mathcal{H}_{\mathcal{Y}}(k_d) \ominus \mathcal{M}$ and let*

$$A_j = M_{\lambda_j}^*\big|_{\mathcal{N}} \quad (j = 1, \dots, d), \qquad C : f \to f(0) \quad (f \in \mathcal{N}).$$

Let $\mathcal{D}$ be the subspace of $\mathcal{N}^d$ given by (1.8) *and let*

$$T := [\, A^*|_{\mathcal{D}} \quad C^* \,] : \mathcal{D} \oplus \mathcal{Y} \to \mathcal{N}.$$

Then:

(1) *Given a Hilbert space $\mathcal{U}$, there exists an inner multiplier $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ satisfying* (5.1) *if and only if*

$$\dim \mathcal{U} \geqslant \dim \operatorname{Ran}\big(I - T^*T\big)^{\frac{1}{2}}. \tag{5.7}$$

(2) *If* (5.7) *is satisfied, then all $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ for which* (5.1) *holds are described by the formula*

$$S(\lambda) = [\, C(I - Z(\lambda)A)^{-1} Z(\lambda) \quad I_{\mathcal{Y}} \,] \big(I - T^*T\big)^{\frac{1}{2}} G^* \tag{5.8}$$

where G is an isometry from $\operatorname{Ran}(I - T^*T)^{\frac{1}{2}}$ *onto* $\operatorname{Ran} G \subset \mathcal{U}$.

(3) *If* $\dim \mathcal{U} = \dim \operatorname{Ran}(I - T^*T)^{1/2}$, *then the function* $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ *such that* (5.1) *holds is defined uniquely up to a constant unitary factor on the right.*

Proof. If $\mathcal{M}$ is a closed $\mathbf{M}_{\boldsymbol{\lambda}}$-invariant subspace of $\mathcal{H}_{\mathcal{Y}}(k_d)$, then $\mathcal{M}^{\perp}$ is isometrically included in $\mathcal{H}_{\mathcal{Y}}(k_d)$ and is isometrically equal to $\mathcal{H}(K_{C,\mathbf{A}})$ where $(C, \mathbf{A})$ are the model operators on $\mathcal{M}^{\perp}$ as in (5.5). On the other hand, as a consequence of Proposition 4.1, we see that the Schur multiplier S is an inner-multiplier representer for $\mathcal{M}$ if and only if $\mathcal{H}(K_S) = \mathcal{M}^{\perp}$ isometrically. Thus the problem of describing all inner-multiplier representers S for the given $\mathbf{M}_{\boldsymbol{\lambda}}$-invariant subspace $\mathcal{M}$ is equivalent to the problem: *describe all Schur-class multipliers S such that $\mathcal{H}(K_S) = \mathcal{H}(K_{C,\mathbf{A}})$ isometrically*, where $(C, \mathbf{A})$ is given by (5.5). The various conclusions of Theorem 5.5 now follow as an application of Theorem 2.11 from [7] to the more special situation here (where $\mathbf{A}$ is strongly stable and $\mathcal{H}(K_{C,\mathbf{A}})$ is contained in $\mathcal{H}_{\mathcal{Y}}(k_d)$ isometrically). □

6. Characteristic functions of commutative row contractions

In the operator model theory for commutative row contractions (see [14,15]), one is given a d-tuple of operators $\mathbf{T} = (T_1, \dots, T_d)$ on a Hilbert space $\mathcal{X}$ for which the associated block-row matrix is contractive

$$\|T\| \leqslant 1 \quad \text{where } T = [\, T_1 \quad \cdots \quad T_d \,] : \mathcal{X}^d \to \mathcal{X}.$$

Under certain conditions (that $\mathbf{T}$ be *completely non-coisometric*—see [15,28]), the associated characteristic function $\theta_{\mathbf{T}}(\boldsymbol{\lambda})$ is a complete unitary invariant for $\mathbf{T}$; recently extensions of the theory to still more general settings have appeared (see [16,27,28]) while the fully noncommutative setting is older (see [25,26]). All this theory can be viewed as multivariable analogues of the well-known now classical operator model theory of Sz.-Nagy and Foias [24]. However, unlike the fully developed theory in [24] for the classical case and unlike the case for the fully noncommutative theory (see [12,25,26]), none of the work for the multivariable commutative setting provides a characterization of which Schur-class functions arise as characteristic functions.

To define $\theta_{\mathbf{T}}$, we set $A = [\, T_1 \quad \dots \quad T_d \,]^*$ and let $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathcal{X} \oplus \mathcal{D}_T \to \mathcal{X}^d \oplus \mathcal{D}_{T^*}$ be the *Halmos unitary dilation* of A,

$$B = D_T|_{\mathcal{D}_T} : \mathcal{D}_T \to \mathcal{X}^d, \qquad C = D_{T^*} : \mathcal{X} \to \mathcal{D}_{T^*}, \qquad D = -T|_{\mathcal{D}_T} : \mathcal{D}_T \to \mathcal{D}_{T^*},$$

where

$$D_T = \left(I_{\mathcal{X}^d} - T^*T\right)^{1/2}, \qquad \mathcal{D}_T = \overline{\operatorname{Ran}}\, D_T \subset \mathcal{X}^d,$$
$$D_{T^*} = \left(I_{\mathcal{X}} - TT^*\right)^{1/2}, \qquad \mathcal{D}_{T^*} = \overline{\operatorname{Ran}}\, D_{T^*} \subset \mathcal{X},$$

and then $\theta_{\mathbf{T}}(\boldsymbol{\lambda})$ is the transfer function associated with the colligation $\mathbf{U}$,

$$\theta_{\mathbf{T}}(\boldsymbol{\lambda}) = \left[-T + D_{T^*}\left(I - Z(\boldsymbol{\lambda})T^*\right)^{-1} Z(\boldsymbol{\lambda}) D_T\right]\big|_{\mathcal{D}_T} : \mathcal{D}_T \to \mathcal{D}_{T^*}. \tag{6.1}$$

Since $\mathbf{U}$ is unitary, it follows that $\theta_{\mathbf{T}}$ is in the Schur class $\mathcal{S}_d(\mathcal{D}_T, \mathcal{D}_{T^*})$. More generally, a Schur-class function $S \in \mathcal{S}_d(\mathcal{U}.\mathcal{Y})$ is said to *coincide* with the characteristic function $\theta_{\mathbf{T}}(\boldsymbol{\lambda})$ if there are unitary identification operators

$$\alpha : \mathcal{D}_{T^*} \to \mathcal{Y}, \qquad \beta : \mathcal{D}_T \to \mathcal{U}$$

so that

$$S(\boldsymbol{\lambda}) = \alpha \theta_{\mathbf{T}}(\boldsymbol{\lambda}) \beta^*.$$

From our point of view, what is special about a Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ which coincides with a characteristic function $\theta_{\mathbf{T}}$ is that it is required to have a *commutative unitary* realization. An additional constraint follows from the fact that the unitary colligation in the construction of a characteristic function comes via the Halmos-dilation construction. The following proposition summarizes the situation. In general let us say that the Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is *pure* if

$$\left\|S(0)u\right\| = \|u\| \quad \text{for some } u \in \mathcal{U} \quad \Rightarrow \quad u = 0. \tag{6.2}$$

For the role of this notion in the characterization of characteristic functions for the classical case, see [24, p. 188].

Proposition 6.1. *A Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ coincides with a characteristic function $\theta_{\mathbf{T}}$ if and only if*

(1) *S has a realization* (1.6) *with* $\mathbf{U}$ *unitary and* $\mathbf{A}$ *commutative, and*
(2) *S is pure, i.e., S satisfies* (6.2).

Proof. We first note the following general fact: *if $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathcal{X} \oplus \mathcal{U} \to \mathcal{X}^d \oplus \mathcal{Y}$ is unitary, then the following are equivalent*:

(i) *B is injective,*
(ii) *C^* is injective,*
(iii) *$u \in \mathcal{U}$ with $\|Du\| = \|u\|$ implies that $u = 0$.*

To see this, note that the unitary property of $\mathbf{U}$ means that

$$\begin{bmatrix} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{bmatrix} = \begin{bmatrix} I_{\mathcal{X}} & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix},$$
$$\begin{bmatrix} AA^* + BB^* & AC^* + BD^* \\ CA^* + DB^* & CC^* + DD^* \end{bmatrix} = \begin{bmatrix} I_{\mathcal{X}^d} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix}. \tag{6.3}$$

From all these relations we read off

$$Bu = 0 \quad \Rightarrow \quad \|Du\| = \|u\| \quad \text{and} \quad C^*Du = 0,$$
$$\|Du\| = \|u\| \quad \Rightarrow \quad Bu = 0 \quad \text{and} \quad C^*Du = 0,$$
$$C^*y = 0 \quad \Rightarrow \quad \|D^*y\| = \|y\| = \|D(D^*y)\| \quad \text{and} \quad BD^*y = 0.$$

Hence any one of the conditions (i), (ii) or (iii) implies the remaining ones.

Suppose now that $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ coincides with a characteristic function $\theta_{\mathbf{T}}$. Then S has a realization as in (1.6) and (1.5) for a connecting operator $\mathbf{U}$ of the form

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_{\mathcal{X}^d} & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} T^* & D_T \\ D_{T^*} & -T \end{bmatrix} \begin{bmatrix} I_{\mathcal{X}} & 0 \\ 0 & \beta^* \end{bmatrix} \tag{6.4}$$

for a commutative row-contraction $\mathbf{T} = \{T_1, \ldots, T_d\}$ where $\alpha : \mathcal{D}_{T^*} \to \mathcal{Y}$ and $\beta : \mathcal{D}_T \to \mathcal{U}$ are unitary, and where $\left[\begin{smallmatrix} T^* & D_T \\ D_{T^*} & -T \end{smallmatrix}\right] : \mathcal{X} \oplus \mathcal{D}_T \to \mathcal{X}^d \oplus \mathcal{D}_{T^*}$ is the Halmos dilation of T^* discussed above. It is then obvious that $\mathbf{U}$ gives a commutative, unitary realization for S. We also read off that any one (and hence all) of the conditions (i), (ii) and (iii) hold for $\mathbf{U}$. As $D = S(0)$, the validity of condition (iii) implies that S is pure.

Conversely suppose that $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ has a commutative, unitary realization $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathcal{X} \oplus \mathcal{U} \to \mathcal{X}^d \oplus \mathcal{Y}$ and is pure. As $D = S(0)$ and S is pure, we read off that condition (iii) above holds, and hence also conditions (i) and (ii) hold for $\mathbf{U}$. From the relations (6.3) we have, in particular,

$$C^*C = I_{\mathcal{X}} - A^*A, \qquad BB^* = I_{\mathcal{X}^d} - AA^*.$$

Hence we can define unitary operators

$$\alpha : \mathcal{D}_A \to \overline{\operatorname{Ran}}\, C = \mathcal{Y}, \qquad \beta : \mathcal{D}_{A^*} \to \overline{\operatorname{Ran}}\, B^* = \mathcal{U}$$

so that

$$\alpha D_A = C \quad \text{and} \quad \beta D_{A^*} = B^*.$$

Then we also have

$$\alpha^* D \beta D_{A^*} = \alpha^* D B^* = -\alpha^* C A^* = -D_A A^* = -A^* D_{A^*}$$

from which we get

$$D = -\alpha A^* \beta^*.$$

We conclude that $\mathbf{U}$ has the form (6.4) with $T = (A_1^*, \dots, A_d^*)$, and hence S coincides with the characteristic function $\theta_{\mathbf{T}}$. $\square$

For the inner case, we can use the results on functional-model realizations obtained above to give a more intrinsic sufficient condition for a Schur-class function to be a characteristic function.

Theorem 6.2. *Suppose that $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is inner,* $\dim \mathcal{D}^\perp = \dim \mathcal{U}_S^0$ *(where the subspaces $\mathcal{U}_S^0 \subset \mathcal{U}$ and $\mathcal{D}^\perp \subset \mathcal{H}(K_S)^d$ are defined in* (4.17) *and* (1.8)*), and that S is pure. Then S coincides with the characteristic function of a $*$-strongly stable, commutative row-contraction.*

Proof. Given S as in the hypotheses, we see from Theorem 4.7 that S has a functional-model realization $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]: \mathcal{H}(K_S) \oplus \mathcal{U} \to \mathcal{H}(K_S)^d \oplus \mathcal{Y}$ such that $\mathbf{U}$ is unitary, $\mathbf{A}$ is commutative and $\mathbf{A}$ is strongly stable. Since by assumption S is pure, we can apply Proposition 6.1 to conclude that S coincides with $\theta_{\mathbf{T}}$, where $\mathbf{T} = (A_1^*, \dots, A_d^*)$. As observed above, $\mathbf{A}$ is strongly stable, i.e., $\mathbf{T}$ is $*$-strongly stable, and the theorem follows. $\square$

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