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Transfer-function realization for multipliers of the Arveson space

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Abstract

An interesting and recently much studied generalization of the classical Schur class is the class of contractive operator-valued multipliers for the reproducing kernel Hilbert space $\mathcal{H}(k_d)$ on the unit ball $\mathbb{B}^d \subset \mathbb{C}^d$, where k_d is the positive kernel $k_d(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = 1/(1 - \langle \boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle)$ on $\mathbb{B}^d$. We study this space from the point of view of realization theory and functional models of de Branges–Rovnyak type. We highlight features which depart from the classical univariate case: coisometric realizations have only partial uniqueness properties, the nonuniqueness can be described explicitly, and this description assumes a particularly concrete form in the functional-model context.

Keywords: Operator-valued functions; Schur multiplier

1. Introduction

Let $\mathcal{U}$ and $\mathcal{Y}$ be two Hilbert spaces and let $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ be the space of all bounded linear operators between $\mathcal{U}$ and $\mathcal{Y}$. We also let $H^2_{\mathcal{U}}$ be the standard Hardy space of the $\mathcal{U}$-valued holomorphic functions on the unit disk $\mathbb{D}$. The operator-valued version of the classical Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ is defined to be the set of all holomorphic, contractive $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions on $\mathbb{D}$. The following equivalent characterizations of the Schur class are well known.

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Theorem 1.1. *Let $S:\mathbb{D}\to\mathcal{L}(\mathcal{U},\mathcal{Y})$ be given. Then the following are equivalent*:

(1) $S\in\mathcal{S}(\mathcal{U},\mathcal{Y})$, *i.e., S is holomorphic on $\mathbb{D}$ with $\|S(\lambda)\|\leqslant 1$ for all $\lambda\in\mathbb{D}$.*

(1′) *The multiplication operator $M_S: f(z)\mapsto S(z)\cdot f(z)$ is a contraction from $H^2_{\mathcal{U}}$ into $H^2_{\mathcal{Y}}$: $\|M_S\|_{\text{op}}\leqslant 1$.*

(2) *The associated kernel function*

$$K_S(\lambda,\zeta)=\frac{I_{\mathcal{Y}}-S(\lambda)S(\zeta)^*}{1-\lambda\bar{\zeta}} \tag{1.1}$$

is a positive kernel on $\mathbb{D}\times\mathbb{D}$, i.e., there exists an operator-valued function $H:\mathbb{D}\to\mathcal{L}(\mathcal{H},\mathcal{Y})$ for some auxiliary Hilbert space $\mathcal{H}$ so that

$$K_S(\lambda,\zeta)=H(\lambda)H(\zeta)^*. \tag{1.2}$$

(3) *There is an auxiliary Hilbert space $\mathcal{X}$ and a unitary connecting operator*

$$U=\begin{bmatrix} A & B\\ C & D\end{bmatrix}:\begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix}\to\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}$$

so that $S(\lambda)$ can be expressed as

$$S(\lambda)=D+\lambda C(I-\lambda A)^{-1}B. \tag{1.3}$$

(4) *$S(\lambda)$ has a realization as in* (1.3) *where the connecting operator U is any one of* (i) *isometric,* (ii) *coisometric, or* (iii) *contractive.*

We remark that the proof that the coisometric version of (4) implies (2) in Theorem 1.1 is particularly transparent: if $S(\lambda)$ has the form (1.3) with $U=\left[\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right]$ coisometric, a simple calculation reveals that (1.2) holds with $H(\lambda)=C(I-\lambda A)^{-1}$, i.e.,

$$K_S(\lambda,\zeta)=C(I-\lambda A)^{-1}\bigl(I-\bar{\zeta}A^*\bigr)^{-1}C^* := K_{C,A}(\lambda,\zeta). \tag{1.4}$$

Among all the possible classes for the connecting operator U (i.e., unitary, isometric, coisometric or simply contractive), the class of coisometric ones is particularly prominent due to its connection with functional-model realizations using the de Branges–Rovnyak reproducing kernel Hilbert space $\mathcal{H}(K_S)$ associated with the positive kernel K_S given by (1.1). We recall (see the original work of Aronszajn [3]) that any positive kernel $(\lambda,\zeta)\mapsto k(\lambda,\zeta)\in\mathcal{L}(\mathcal{Y})$ on a set $\Omega\times\Omega$ (so $\lambda,\zeta\in\Omega$) gives rise to a reproducing kernel Hilbert space (RKHS) $\mathcal{H}(k)$ consisting of $\mathcal{Y}$-valued functions on Ω with the defining property: for each $\zeta\in\Omega$ and $y\in\mathcal{Y}$, the $\mathcal{Y}$-valued function $(k_\zeta y)(\lambda):=k(\lambda,\zeta)y$ is in $\mathcal{H}(k)$ and has the reproducing property

$$\langle f,k_\zeta y\rangle_{\mathcal{H}(k)}=\bigl\langle f(\zeta),y\bigr\rangle_{\mathcal{Y}}\quad\text{for all } y\in\mathcal{Y},\ f\in\mathcal{H}(k).$$

We remark that the Hardy space $H^2_{\mathcal{Y}}$ is the RKHS associated with the Szegö kernel $k_{\text{Sz}}(\lambda,\zeta)=(1-\lambda\bar{\zeta})^{-1}I_{\mathcal{Y}}$ positive on $\mathbb{D}\times\mathbb{D}$ where $\mathbb{D}$ is the unit disk. Applying Aronszajn's construction to the positive kernel K_S on $\mathbb{D}$ for a Schur-class function S as in (1.4) gives the reproducing kernel Hilbert space $\mathcal{H}(K_S)$, the de Branges–Rovnyak space associated with S. Then we have the following concrete, functional-model realization for the Schur-class function S [16,17].

Theorem 1.2. *Suppose that $S\in\mathcal{S}(\mathcal{U},\mathcal{Y})$ and let $\mathcal{H}(K_S)$ be the associated de Branges–Rovnyak model space. Then the connecting operator*

$$U=\begin{bmatrix} A & B\\ C & D\end{bmatrix}:\begin{bmatrix}\mathcal{H}(K_S)\\ \mathcal{U}\end{bmatrix}\to\begin{bmatrix}\mathcal{H}(K_S)\\ \mathcal{Y}\end{bmatrix}$$

with the entries defined by

$$A: f(\lambda) \mapsto \frac{f(\lambda) - f(0)}{\lambda}, \quad C: f \mapsto f(0) \quad \text{for } f \in \mathcal{H}(K_S), \tag{1.5}$$

$$B: u \mapsto \frac{S(\lambda) - S(0)}{\lambda} u, \qquad D: u \mapsto S(0)u \quad \text{for } u \in \mathcal{U}, \tag{1.6}$$

provides a coisometric realization of $S(\lambda)$*, i.e.,* U *is coisometric as an operator from* $\left[\begin{smallmatrix} \mathcal{H}(K_S) \\ \mathcal{U} \end{smallmatrix}\right]$ *to* $\left[\begin{smallmatrix} \mathcal{H}(K_S) \\ \mathcal{Y} \end{smallmatrix}\right]$ *and we recover* $S(\lambda)$ *via the formula* (1.3).

The de Branges–Rovnyak functional-model realization is *closely outer-connected* in the sense that the pair (C, A) is *observable*, i.e., that

$$C(I - zA)^{-1}x = 0 \quad \text{for all } z \in \mathbb{D} \quad \Longrightarrow \quad x = 0.$$

Observability of the pair (C, A) is a minimality condition under which the coisometric realization is essentially unique: *every coisometric closely outer-connected realization of an* $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ *is unitarily equivalent to the de Branges–Rovnyak functional-model realization*. It can also be shown that, if (C, A) is observable and if $U = \left[\begin{smallmatrix} A & B \\ C & S(0) \end{smallmatrix}\right]$ provides a coisometric realization for the $S(\lambda)$, then the operator B is already uniquely determined by C, A and S (see Remark 3.3 below).

A multivariable generalization of the Szegö kernel much studied of late (see [5,6]) is the positive kernel

$$k_d(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = \frac{1}{1 - \langle \boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle}$$

on $\mathbb{B}^d \times \mathbb{B}^d$ where $\mathbb{B}^d = \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d\colon \langle \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle < 1\}$ is the unit ball of the d-dimensional Euclidean space $\mathbb{C}^d$. By

$$\langle \boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle = \langle \boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle_{\mathbb{C}^d} = \sum_{j=1}^{d} \lambda_j \bar{\zeta}_j \quad \text{for } \boldsymbol{\lambda}, \boldsymbol{\zeta} \in \mathbb{C}^d$$

we mean the standard inner product in $\mathbb{C}^d$. The associated RKHS $\mathcal{H}(k_d)$ obtained via Aronszajn's construction is a natural multivariable analogue of the Hardy space H^2 of the unit disk and coincides with H^2 if $d = 1$.

For $\mathcal{Y}$ an auxiliary Hilbert space, we consider the tensor product Hilbert space $\mathcal{H}_{\mathcal{Y}}(k_d) := \mathcal{H}(k_d) \otimes \mathcal{Y}$ whose elements can be viewed as $\mathcal{Y}$-valued functions in $\mathcal{H}(k_d)$. The space of multipliers $\mathcal{M}_d(\mathcal{U}, \mathcal{Y})$ is defined as the space of all $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued analytic functions S on $\mathbb{B}^d$ such that the induced multiplication operator

$$M_S: f(\boldsymbol{\lambda}) \to S(\boldsymbol{\lambda}) \cdot f(\boldsymbol{\lambda}) \tag{1.7}$$

maps $\mathcal{H}_{\mathcal{U}}(k_d)$ into $\mathcal{H}_{\mathcal{Y}}(k_d)$. It follows by the closed graph theorem that for every $S \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y})$, the operator M_S is bounded. We shall pay particular attention to the unit ball of $\mathcal{M}_d(\mathcal{U}, \mathcal{Y})$, denoted by

$$\mathcal{S}_d(\mathcal{U}, \mathcal{Y}) = \bigl\{S \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y})\colon \|M_S\|_{\text{op}} \leqslant 1\bigr\}.$$

Since $\mathcal{S}_1(\mathcal{U}, \mathcal{Y})$ collapses to the classical Schur class (by the equivalence (1) $\Leftrightarrow$ (1′) in Theorem 1.1), we refer to $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ as a generalized (d-variable) *Schur class*. The following result appears in [1,10] and is the precise analogue of Theorem 1.1 for the multivariable case. Note that there is no analogue of condition (1) in Theorem 1.1 and condition (1) in Theorem 1.3 is the analogue of condition (1′) in Theorem 1.1.

Theorem 1.3. *Let S be an $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function defined on $\mathbb{B}^d$. The following are equivalent*:

(1) *S belongs to $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$.*
(2) *The kernel*

$$K_S(\boldsymbol{\lambda},\boldsymbol{\zeta}) = \frac{I_{\mathcal{Y}} - S(\boldsymbol{\lambda})S(\boldsymbol{\zeta})^*}{1-\langle \boldsymbol{\lambda},\boldsymbol{\zeta}\rangle} \tag{1.8}$$

is positive on $\mathbb{B}^d \times \mathbb{B}^d$.
(3) *There exists a Hilbert space $\mathcal{X}$ and a unitary connecting operator* (*or colligation*) $\mathbf{U}$ *of the form*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix} \tag{1.9}$$

so that $S(\boldsymbol{\lambda})$ can be realized in the form

$$\begin{aligned} S(\boldsymbol{\lambda}) &= D + C(I_{\mathcal{X}} - \lambda_1 A_1 - \cdots - \lambda_d A_d)^{-1}(\lambda_1 B_1 + \cdots + \lambda_d B_d) \\ &= D + C\big(I - Z(\boldsymbol{\lambda})A\big)^{-1} Z(\boldsymbol{\lambda})B \end{aligned} \tag{1.10}$$

where we set

$$Z(\boldsymbol{\lambda}) = [\,\lambda_1 I_{\mathcal{X}} \quad \cdots \quad \lambda_d I_{\mathcal{X}}\,], \qquad A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}. \tag{1.11}$$

(4) *There exist a Hilbert space $\mathcal{X}$ and a contractive connecting operator* $\mathbf{U}$ *of the form* (1.9) *so that $S(\boldsymbol{\lambda})$ can be realized in the form* (1.10).

Although statement (4) in Theorem 1.3 concerning contractive realizations does not appear in [1,10], its equivalence to statements (1)–(3) is quite obvious. Indeed, implication (3) ⇒ (4) is trivial; on the other hand, a straightforward calculation (see e.g., [2, Lemma 2.2]) shows that for S of the form (1.10),

$$\begin{aligned} K_S(\boldsymbol{\lambda},\boldsymbol{\zeta}) &= C\big(I_{\mathcal{X}} - Z(\boldsymbol{\lambda})A\big)^{-1}\big(I_{\mathcal{X}} - A^*Z(\boldsymbol{\zeta})^*\big)^{-1}C^* \\ &\quad + \big[\,C(I - Z(\boldsymbol{\lambda})A)^{-1}Z(\boldsymbol{\lambda}) \quad I\,\big] \frac{I - \mathbf{U}\mathbf{U}^*}{1-\langle \boldsymbol{\lambda},\boldsymbol{\zeta}\rangle} \begin{bmatrix} Z(\boldsymbol{\zeta})^*(I_{\mathcal{X}} - A^*Z(\boldsymbol{\zeta})^*)^{-1}C^* \\ I \end{bmatrix} \end{aligned} \tag{1.12}$$

where $\mathbf{U}$ is defined in (1.9). Thus, if $\mathbf{U}$ is a contraction, the kernel $K_S(\boldsymbol{\lambda},\boldsymbol{\zeta})$ is positive on $\mathbb{B}^d \times \mathbb{B}^d$ which proves implication (4) ⇒ (2) in Theorem 1.3.

In analogy with the univariate case, a realization of the form (1.10) is called *coisometric, isometric, unitary* or *contractive* if the operator $\mathbf{U}$ is respectively, coisometric, isometric, unitary or just contractive. It turns out that a more useful analogue of "coisometric realization" appearing in the classical univariate case is not that the whole connecting operator $\mathbf{U}^*$ be isometric, but rather that $\mathbf{U}^*$ be isometric on a certain canonical subspace of $\mathcal{X}^d \oplus \mathcal{Y}$.

Definition 1.4. A realization (1.10) of $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is called *weakly coisometric* if the adjoint $\mathbf{U}^* : \mathcal{X}^d \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{U}$ of the connecting operator is contractive and isometric on the subspace $\left[\begin{smallmatrix} \mathcal{D}_{C,A} \\ \mathcal{Y} \end{smallmatrix}\right] \subset \left[\begin{smallmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{smallmatrix}\right]$ where

$$\mathcal{D} = \mathcal{D}_{C,A} := \overline{\operatorname{span}}\left\{Z(\boldsymbol{\zeta})^*\left(I - A^* Z(\boldsymbol{\zeta})^*\right)^{-1} C^* y \colon \boldsymbol{\zeta} \in \mathbb{B}^d,\ y \in \mathcal{Y}\right\} \subset \mathcal{X}^d. \tag{1.13}$$

The notion of weakly coisometric realizations has been introduced in [10]. It does not appear in the single-variable context for a simple reason that if the pair (C, A) is observable, then a weakly coisometric realization is automatically coisometric (see [10, p. 100] and also Remark 3.3 below). The following intrinsic kernel characterization as to when a given contractive realization is a weakly coisometric realization turns out to be a convenient tool for our current purposes. Equality (1.14) below is the multivariable analogue of equality (1.4).

Proposition 1.5. *A contractive realization* (1.10) *of* $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ *is weakly coisometric if and only if the kernel* $K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta})$ *associated to* S *via* (1.8) *can alternatively be written as*

$$K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_{C,\mathbf{A}}(\boldsymbol{\lambda}, \boldsymbol{\zeta}), \tag{1.14}$$

where

$$K_{C,\mathbf{A}}(\boldsymbol{\lambda}, \boldsymbol{\zeta}) := C(I - \lambda_1 A_1 - \cdots - \lambda_d A_d)^{-1}\left(I - \overline{\zeta_1} A_1^* - \cdots - \overline{\zeta_d} A_d^*\right)^{-1} C^*. \tag{1.15}$$

Proof. Let $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be the connecting operator of a contractive realization of $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. It is readily seen from the formula (1.12), that equality (1.14) holds if and only if the operator $\mathbf{U}^*$ is isometric on the space

$$\mathcal{M} := \overline{\operatorname{span}}\left\{\begin{bmatrix} Z(\boldsymbol{\zeta})^*(I_{\mathcal{X}} - A^* Z(\boldsymbol{\zeta})^*)^{-1} C^* \\ I \end{bmatrix} y \colon \boldsymbol{\zeta} \in \mathbb{B}^d,\ y \in \mathcal{Y}\right\} \subset \mathcal{X}^d \oplus \mathcal{Y}.$$

By setting $\boldsymbol{\zeta} = 0$ in the last formula, we see that $\left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right] \in \mathcal{M}$ for all $y \in \mathcal{Y}$ and thus $\mathcal{M}$ splits in the form $\mathcal{M} = \left[\begin{smallmatrix} \mathcal{D} \\ \mathcal{Y} \end{smallmatrix}\right]$ where $\mathcal{D}$ is defined in (1.13). The rest follows by Definition 1.4. □

The present paper analyzes a number of finer structural issues surrounding a Schur-class function $S(\boldsymbol{\lambda})$ and its associated positive kernel (1.8). We analyze when equality (1.14) holds in both a realization and a purely function-theoretic context. We analyze the problem of realizing a kernel of the form $K_{C,\mathbf{A}}(\boldsymbol{\lambda}, \boldsymbol{\zeta})$ as $K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta})$ for a Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ (Theorems 2.2 and 2.11) and we analyze the nonuniqueness of the input operator B inherent in a weakly coisometric (as well as coisometric or unitary) realization of a given Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ using a given output pair $(C, \mathbf{A})$ which is observable in an appropriate multivariable sense (Theorems 2.4 and 2.7). Upon applying Aronszajn's construction to the kernel K_S associated with a Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ (which is positive on $\mathbb{B}^d$ by Theorem 1.3), one gets the de Branges–Rovnyak space $\mathcal{H}(K_S)$ that can serve as the state space for a weakly coisometric realization for S. A weakly coisometric realization for S with the state space equal to $\mathcal{H}(K_S)$ and with the output operator C equal to evaluation at zero on $\mathcal{H}(K_S)$ will be called a *generalized functional-model realization.*[1]

[1] The term (not necessarily generalized) *functional-model realization* is explained below.

Our earlier paper [7] focuses on the structure of reproducing kernel Hilbert spaces $\mathcal{H}(K_{C,\mathbf{A}})$ with reproducing kernel $K_{C,\mathbf{A}}$ of the form (1.15). Such spaces can be viewed as the range of an observability operator associated with a state-output multidimensional linear system of the form

$$\Sigma: \quad \begin{cases} x(\mathbf{n}) = A_1 x(\sigma_1(\mathbf{n})) + \cdots + A_d x(\sigma_d(\mathbf{n})), \\ y(\mathbf{n}) = Cx(\mathbf{n}) \end{cases}$$

where

$$\sigma_k(\mathbf{n}) = \sigma_k\big((n_1, \ldots, n_d)\big) = (n_1, \ldots, n_{k-1}, n_k + 1, n_{k+1}, \ldots, n_d)$$

for $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$. Also discussed in [7] are connections with noncommutative analogues of these objects, where the reproducing kernel Hilbert space is of the noncommutative type discussed in [11] consisting of formal power series with vector coefficients and where the system has evolution along a free semigroup rather than along $\mathbb{Z}_+^d$. The paper [7] also serves as a resource for the present paper, since, once one has established the equality (1.14), results concerning $\mathcal{H}(K_{C,\mathbf{A}})$ from [7] immediately yield the corresponding result for the space $\mathcal{H}(K_S)$.

We reserve the term (non-generalized) *functional-model realization* for the case where $\mathcal{H}(K_S)$ is invariant under the adjoints $M_{\lambda_j}^*$ of the multiplication operators $M_{\lambda_j} : f(\boldsymbol{\lambda}) \mapsto \lambda_j f(\boldsymbol{\lambda})$ on $\mathcal{H}_{\mathcal{Y}}(k_d)$ and the state-space operators $\mathbf{A} = (A_1, \ldots, A_d)$ in the realization are taken to be $A_j = M_{\lambda_j}^*|_{\mathcal{H}(K_S)}$; the characteristic function $S_{\mathbf{T}}(\boldsymbol{\lambda})$ for a commuting row contraction $\mathbf{T} = (T_1, \ldots, T_d)$ (see [13–15]) as well as inner functions (Schur-class multipliers S for which the associated multiplication operator $M_S : f(\boldsymbol{\lambda}) \mapsto S(\boldsymbol{\lambda}) \cdot f(\boldsymbol{\lambda})$ is a partial isometry) are of this type. We discuss the special features of this case (where $\mathcal{H}(K_S)$ is invariant under $M_{\lambda_j}^*$ for $j = 1, \ldots, d$ and where $S(\boldsymbol{\lambda})$ has a realization with commuting state-space operators $A_1, \ldots, A_d$) in our separate paper [9].

The paper is organized as follows. Section 2 develops the ideas surrounding observable weakly coisometric realizations and the quantification of the nonuniqueness of the input operator in such realizations. In Section 3 we show that any Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ admits a generalized functional-model realization and that any observable weakly coisometric realization of S is unitarily equivalent to some generalized functional-model realization. Preliminary results of this latter type appear in the paper of Alpay, Dijksma and Rovnyak [2]. In Section 4 we introduce a general setting for the overlapping spaces appearing prominently in the work of de Branges and Rovnyak [16,17] and indicate how special cases of these spaces appear in Sections 2 and 3 in connection with the nonuniqueness of the input operator in observable weakly coisometric realizations.

In our followup paper [8], we develop the noncommutative theory parallel to the results of the present paper. In this setting, the Schur-class function S becomes a formal power series in noncommuting indeterminates inducing a contractive multiplication operator between Fock–Hilbert spaces consisting of formal power series with vector coefficients. Such a Schur-class multiplier induces a kernel $K_S(z, w)$ in noncommuting indeterminates $z = (z_1, \ldots, z_d)$ and $w = (w_1, \ldots, w_d)$ which is a noncommutative positive kernel in the sense of [11]. The associated noncommutative formal reproducing kernel Hilbert space $\mathcal{H}(K_S)$ is a noncommutative analogue of the space $\mathcal{H}(K_S)$ studied here (where elements of the space are functions of commuting variables $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$) and is an alternative multivariable generalization of the classical case [16,17]. For this setting the analogy with the classical case turns out to be more compelling than for the case of several commuting variables presented here.

2. Weakly coisometric realizations

Weakly coisometric realizations of Schur-class functions are closely related to range spaces of observability operators studied in [7]. Let $\mathbf{A} = (A_1, \dots, A_d)$ be a d-tuple of operators in $\mathcal{L}(\mathcal{X})$. If $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, then the pair $(C, \mathbf{A})$ is said to be an *output pair*. Such an output pair is said to be *contractive* if

$$A_1^*A_1 + \cdots + A_d^*A_d + C^*C \leqslant I_{\mathcal{X}},$$

to be *isometric* if equality holds in the above relation, and to be *output-stable* if the associated observability operator

$$\mathcal{O}_{C,\mathbf{A}} : x \mapsto C(I - \lambda_1 A_1 - \cdots - \lambda_d A_d)^{-1} x \tag{2.1}$$

maps $\mathcal{X}$ into $\mathcal{H}_{\mathcal{Y}}(k_d)$. As it was shown in [7], any contractive pair $(C, \mathbf{A})$ is output stable and, moreover, the corresponding observability operator $\mathcal{O}_{C,\mathbf{A}} : \mathcal{X} \to \mathcal{H}_{\mathcal{Y}}(k_d)$ is a contraction. An output stable pair $(C, \mathbf{A})$ is called *observable* if the observability operator $\mathcal{O}_{C,\mathbf{A}}$ is injective, i.e.,

$$C(I - \lambda_1 A_1 - \cdots - \lambda_d A_d)^{-1} x \equiv 0 \quad \Longrightarrow \quad x = 0.$$

The following result from [7] gives the close connection between spaces of the form $\mathcal{H}(K_{C,\mathbf{A}})$ and ranges of observability operators.

Theorem 2.1. *Let* $(C, \mathbf{A})$ *be a contractive pair with* $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *and with associated positive kernel* $K_{C,\mathbf{A}}$ *given by* (1.15) *and the observability operator* $\mathcal{O}_{C,\mathbf{A}}$ *given by* (2.1). *Then*:

(1) *The reproducing kernel Hilbert space* $\mathcal{H}(K_{C,\mathbf{A}})$ *is characterized as*

$$\mathcal{H}(K_{C,\mathbf{A}}) = \operatorname{Ran} \mathcal{O}_{C,\mathbf{A}}$$

with the lifted norm given by $\|\mathcal{O}_{C,\mathbf{A}} x\|_{\mathcal{H}(K_{C,\mathbf{A}})} = \|Qx\|_{\mathcal{X}}$, *where* Q *is the orthogonal projection onto* $(\mathrm{Ker}, \mathcal{O}_{C,\mathbf{A}})^{\perp}$.

(2) *The operator* $\mathcal{O}_{C,\mathbf{A}}$ *is a contraction of* $\mathcal{X}$ *into* $\mathcal{H}(K_{C,\mathbf{A}})$. *It is an isometry if and only if the pair* $(C, \mathbf{A})$ *is observable.*

(3) *There exist operators* $T_1, \dots, T_d \in \mathcal{L}(\mathcal{H}(K_{C,\mathbf{A}}))$ *such that relations*

$$f(\boldsymbol{\lambda}) - f(0) = \sum_{j=1}^{d} \lambda_j (T_j f)(\boldsymbol{\lambda}) \quad \bigl(\boldsymbol{\lambda} \in \mathbb{B}^d\bigr)$$

and

$$\sum_{j=1}^{d} \|T_j f\|^2_{\mathcal{H}(K_{C,\mathbf{A}})} \leqslant \|f\|^2_{\mathcal{H}(K_{C,\mathbf{A}})} - \|f(0)\|^2_{\mathcal{Y}}$$

hold for every function $f \in \mathcal{H}(K_{C,\mathbf{A}})$.

Proposition 1.5 and Theorem 2.1 assert that every weakly coisometric realization of a Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ identifies the corresponding de Branges–Rovnyak space $\mathcal{H}(K_S)$ as the range space of the observability operator corresponding to a contractive pair $(C, \mathbf{A})$. The next proposition shows that the reverse identification is also possible.

Theorem 2.2. *Let* $(C, \mathbf{A})$ *with* $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *be a contractive pair. Then there exist an input space* $\mathcal{U}$ *and an* $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ *such that*

$$K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_{C,\mathbf{A}}(\boldsymbol{\lambda}, \boldsymbol{\zeta}). \tag{2.2}$$

Proof. Choose a Hilbert space $\mathcal{U}$ with

$$\dim \mathcal{U} \geqslant \operatorname{rank}\left(\begin{bmatrix} I_{\mathcal{X}^d} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix}\begin{bmatrix} A^* & C^* \end{bmatrix}\right)$$

and let $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]: \mathcal{U} \to \mathcal{X}^d \oplus \mathcal{Y}$ be a solution of the Cholesky factorization problem

$$\begin{bmatrix} B \\ D \end{bmatrix}\begin{bmatrix} B^* & D^* \end{bmatrix} = \begin{bmatrix} I_{\mathcal{X}^d} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix}\begin{bmatrix} A^* & C^* \end{bmatrix}.$$

Then $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is coisometric. Let $S(\boldsymbol{\lambda})$ be given by the realization formula (1.10). Then $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and Proposition 1.5 guarantees (2.2) as wanted. □

Theorem 2.2 shows that every range space $\operatorname{Ran} \mathcal{O}_{C,\mathbf{A}} = \mathcal{H}(K_{C,\mathbf{A}})$ associated with a contractive pair $(C, \mathbf{A})$ can be considered as the de Branges–Rovnyak space $\mathcal{H}(K_S)$ for an appropriately chosen Schur-class function S, which we will call *a representer of* $\mathcal{H}(K_{C,\mathbf{A}})$. A description of all representers for a given $\mathcal{H}(K_{C,\mathbf{A}})$ will be given below in Theorem 2.11.

Now we discuss equality (2.2) independently of the realization context. With a given contractive pair $(C, \mathbf{A})$ with $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function S defined on $\mathbb{B}^d$ we associate the operator

$$V = \begin{bmatrix} A_V & B_V \\ C_V & D_V \end{bmatrix} : \begin{bmatrix} \mathcal{D} \\ \mathcal{Y} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \tag{2.3}$$

(where the space $\mathcal{D}$ is defined in (1.13)) with the entries given by

$$A_V = A^*|_{\mathcal{D}}, \qquad B_V = C^*, \qquad D_V = S(0)^*, \tag{2.4}$$

and where C_V is uniquely determined by linearity and continuity by its action on a generic generating vector for $\mathcal{D}$:

$$C_V : Z(\boldsymbol{\zeta})^*\big(I - A^* Z(\boldsymbol{\zeta})^*\big)^{-1} C^* y \mapsto \big(S(\boldsymbol{\zeta})^* - S(0)^*\big) y \quad \text{for } \boldsymbol{\zeta} \in \mathbb{B}^d,\ y \in \mathcal{Y}. \tag{2.5}$$

Lemma 2.3. *Let* $(C, \mathbf{A})$ *be a contractive pair and let* S *be an* $\mathcal{L}(\mathcal{U}, \mathcal{Y})$*-valued function defined on* $\mathbb{B}^d$. *Then* (2.2) *holds* (*and therefore also* S *belongs to* $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$) *if and only if the operator* V *defined in* (2.3)–(2.5) *is an isometry from* $\mathcal{D} \oplus \mathcal{Y}$ *onto*

$$\mathcal{R}_V := \overline{\operatorname{span}}\left\{\begin{bmatrix} (I - A^* Z(\boldsymbol{\zeta})^*)^{-1} C^* y \\ S(\boldsymbol{\zeta})^* y \end{bmatrix} : \boldsymbol{\zeta} \in \mathbb{B}^d,\ y \in \mathcal{Y}\right\} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}. \tag{2.6}$$

Proof. Let $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and let equality (2.2) hold, i.e., let

$$\frac{I_{\mathcal{Y}} - S(\boldsymbol{\lambda}) S(\boldsymbol{\zeta})^*}{1 - \langle \boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle} = C\big(I - Z(\boldsymbol{\lambda}) A\big)^{-1}\big(I - A^* Z(\boldsymbol{\zeta})^*\big)^{-1} C^*,$$

which can be written equivalently (due to the formula (1.11) for $Z(\boldsymbol{\lambda})$) as

$$\begin{aligned} &C\big(I - Z(\boldsymbol{\lambda}) A^*\big)^{-1} Z(\boldsymbol{\lambda}) Z(\boldsymbol{\zeta})^*\big(I - A^* Z(\boldsymbol{\zeta})^*\big)^{-1} C^* + I_{\mathcal{Y}} \\ &\quad = C\big(I - Z(\boldsymbol{\lambda}) A\big)^{-1}\big(I - A^* Z(\boldsymbol{\zeta})^*\big)^{-1} C^* + S(\boldsymbol{\lambda}) S(\boldsymbol{\zeta})^*. \end{aligned} \tag{2.7}$$

It follows from the latter identity that the map

$$V' : \begin{bmatrix} Z(\boldsymbol{\zeta})^*(I - A^*Z(\boldsymbol{\zeta})^*)^{-1}C^* \\ I_{\mathcal{Y}} \end{bmatrix} y \mapsto \begin{bmatrix} (I - A^*Z(\boldsymbol{\zeta})^*)^{-1}C^* \\ S(\boldsymbol{\zeta})^* \end{bmatrix} y \tag{2.8}$$

can be extended by linearity and continuity to an isometry (still denoted by V') from the subspace

$$\mathcal{D}_V := \overline{\operatorname{span}} \left\{ \begin{bmatrix} Z(\boldsymbol{\zeta})^*(I - A^*Z(\boldsymbol{\zeta})^*)^{-1}C^* \\ I_{\mathcal{Y}} \end{bmatrix} y \colon \boldsymbol{\zeta} \in \mathbb{B}^d \text{ and } y \in \mathcal{Y} \right\}$$

onto the subspace $\mathcal{R}_V$ given in (2.6). Note that the setting $\boldsymbol{\zeta} = 0 \in \mathbb{B}^d$ in the formula $\left[\begin{smallmatrix} Z(\boldsymbol{\zeta})^*(I-A^*Z(\boldsymbol{\zeta})^*)^{-1}C^*y \\ y \end{smallmatrix}\right]$ for a generic generator of $\mathcal{D}_V$ shows that $\{0\} \oplus \mathcal{Y} \subset \mathcal{D}_V$ and hence we actually have $\mathcal{D}_V = \mathcal{D} \oplus \mathcal{Y}$ where $\mathcal{D}$ is defined as in (1.13). Just as in the proof of Proposition 1.5, setting $\boldsymbol{\zeta} = 0$ in the formula (2.8) for the action of V' implies that

$$V' : \begin{bmatrix} 0 \\ y \end{bmatrix} \mapsto \begin{bmatrix} C^* \\ S(0)^* \end{bmatrix} y \quad \text{for every } y \in \mathcal{Y}. \tag{2.9}$$

Write V' in the block-matrix form $V' = \left[\begin{smallmatrix} A'_V & B'_V \\ C'_V & D'_V \end{smallmatrix}\right]$ conformal with (2.3) and define A_V, B_V, C_V, D_V as in (2.4) and (2.5). We conclude from (2.9) that $B'_V = C^* = B_V$, $D'_V = S(0)^* = D_V$. Then (2.8) implies that C'_V satisfies

$$\begin{aligned} C'_V\big(Z(\boldsymbol{\zeta})^*\big(I - A^*Z(\boldsymbol{\zeta})^*\big)^{-1}C^*y\big) &= S(\boldsymbol{\zeta})^*y - S(0)^*y \\ &= C_V\big(Z(\boldsymbol{\zeta})^*\big(I - A^*Z(\boldsymbol{\zeta})^*\big)^{-1}y\big) \end{aligned}$$

and hence $C'_V = C_V$. Similarly,

$$\begin{aligned} A'_V\big(Z(\boldsymbol{\zeta})^*\big(I - A^*Z(\boldsymbol{\zeta})^*\big)^{-1}C^*y\big) &= \big(I - A^*Z(\boldsymbol{\zeta})^*\big)^{-1}C^*y - C^*y \\ &= A^*\big(Z(\boldsymbol{\zeta})^*\big(I - A^*Z(\boldsymbol{\zeta})^*\big)^{-1}C^*y\big) \end{aligned}$$

and we conclude that $A'_V = A^*|_{\mathcal{D}} = A_V$. Thus, $V' = V$ and therefore V is an isometry.

Conversely, if V defined in (2.3)–(2.5) is isometric, then for two generic generators

$$f = \begin{bmatrix} Z(\boldsymbol{\zeta})^*(I - A^*Z(\boldsymbol{\zeta})^*)^{-1}C^*y \\ y \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} Z(\boldsymbol{\lambda})^*(I - A^*Z(\boldsymbol{\lambda})^*)^{-1}C^*y' \\ y' \end{bmatrix}$$

in $\mathcal{D}_V = \mathcal{D} \oplus \mathcal{Y}$, we have

$$\langle f, g\rangle_{\mathcal{X}^d \oplus \mathcal{Y}} = \langle Vf, Vg\rangle_{\mathcal{X} \oplus \mathcal{U}}. \tag{2.10}$$

Note that

$$\begin{aligned} \langle f, g\rangle_{\mathcal{X}^d \oplus \mathcal{Y}} &= \big\langle \big[C\big(I - Z(\boldsymbol{\lambda})A\big)^{-1}Z(\boldsymbol{\lambda})Z(\boldsymbol{\zeta})^*\big(I - A^*Z(\boldsymbol{\zeta})^*\big)^{-1}C^* + I\big]y, y'\big\rangle_{\mathcal{Y}} \\ &= \big\langle \big[\langle \boldsymbol{\lambda}, \boldsymbol{\zeta}\rangle C\big(I - Z(\boldsymbol{\lambda})A\big)^{-1}\big(I - A^*Z(\boldsymbol{\zeta})^*\big)^{-1}C^* + I\big]y, y'\big\rangle_{\mathcal{Y}} \end{aligned}$$

and

$$\begin{aligned} \langle Vf, Vg\rangle_{\mathcal{X} \oplus \mathcal{U}} &= \left\langle \begin{bmatrix} (I - A^*Z(\boldsymbol{\zeta})^*)^{-1}C^* \\ S(\boldsymbol{\zeta})^* \end{bmatrix} y, \begin{bmatrix} (I - A^*Z(\boldsymbol{\lambda})^*)^{-1}C^* \\ S(\boldsymbol{\lambda})^* \end{bmatrix} y' \right\rangle_{\mathcal{X} \oplus \mathcal{U}} \\ &= \big\langle \big[C\big(I - Z(\boldsymbol{\lambda})A\big)^{-1}\big(I - A^*Z(\boldsymbol{\zeta})^*\big)^{-1}C^* + S(\boldsymbol{\lambda})S(\boldsymbol{\zeta})^*\big]y, y'\big\rangle_{\mathcal{Y}}. \end{aligned}$$

Substituting the two latter equalities into (2.10) and taking into account that y and y' are arbitrary vectors in $\mathcal{Y}$, we get (2.7), which is equivalent to (2.2). □

Proposition 1.5 states that once a contractive realization $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ of S is such that (1.14) holds, then this realization is weakly coisometric. Our next result asserts that equality (1.14) itself guarantees the existence of weakly coisometric realizations for S with preassigned C and $\mathbf{A} = (A_1, \dots, A_d)$.

Theorem 2.4. *Suppose that a Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and a contractive pair $(C, \mathbf{A})$ are such that* (1.14) *holds and let $D := S(0)$. Then there exist operators $B_j : \mathcal{U} \to \mathcal{X}$ for $j = 1, \dots, d$ so that the operator $\mathbf{U}$ of the form* (1.9) *is weakly coisometric and S can be realized as in* (1.10).

Proof. We are given C, A, $D = S(0)$ and $S(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathbb{B}^d$ and seek $B : \mathcal{U} \to \mathcal{X}^d$ so that

$$C\big(I - Z(\boldsymbol{\lambda})A\big)^{-1} Z(\boldsymbol{\lambda})B + D = S(\boldsymbol{\lambda}),$$

or, in adjoint form with $\boldsymbol{\zeta}$ in place of $\boldsymbol{\lambda}$,

$$B^* Z(\boldsymbol{\zeta})^* \big(I - A^* Z(\boldsymbol{\zeta})^*\big)^{-1} C^* + D^* = S(\boldsymbol{\zeta})^*.$$

The latter equality is equivalent to

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} Z(\boldsymbol{\zeta})^*(I - A^* Z(\boldsymbol{\zeta})^*)^{-1} C^* \\ I \end{bmatrix} = \begin{bmatrix} (I - A^* Z(\boldsymbol{\zeta})^*)^{-1} C^* \\ S(\boldsymbol{\zeta})^* \end{bmatrix}, \tag{2.11}$$

since the identity

$$A^* Z(\boldsymbol{\zeta})^* \big(I - A^* Z(\boldsymbol{\zeta})^*\big)^{-1} C^* + C^* = \big(I - A^* Z(\boldsymbol{\zeta})^*\big)^{-1} C^*$$

expressing equality of the top components in (2.11) holds true automatically. On the other hand, since assumption (1.14) holds, Lemma 2.3 applies and the operator $V : \mathcal{D} \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{U}$ defined in (2.3)–(2.5) is isometric and satisfies a similar equality (2.8) (with $V' = V$). It follows that any choice of $B = \left[\begin{smallmatrix} B_1 \\ \vdots \\ B_d \end{smallmatrix}\right]$ such that $\mathbf{U}^* = \left[\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right]$ is a contractive extension of V from $\mathcal{D} \oplus \mathcal{Y}$ to the whole of $\mathcal{X}^d \oplus \mathcal{Y}$ gives rise to a weakly coisometric realization $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ for $S(\boldsymbol{\lambda})$.

Our completion problem (construction of B subject to (2.11) and that $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be contractive) can now be reformulated as follows: *Find an operator $B : \mathcal{U} \to \mathcal{X}^d$ so that*

(1) *the operator matrix $\left[\begin{smallmatrix} A^* & C^* \\ B^* & S(0)^* \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ is a contraction, and*
(2) $B^*|_{\mathcal{D}} = C_V$, *where $C_V : \mathcal{D} \to \mathcal{U}$ is given by* (2.5).

This is a contractive matrix-completion problem with linear side-constraint (2). We convert this problem to a standard matrix-completion problem as follows. Let $\mathcal{D}^{\perp} := \mathcal{X}^d \ominus \mathcal{D}$ and define operators

$$T_{11} : \mathcal{D}^{\perp} \to \mathcal{X}, \qquad T_{12} : \mathcal{D} \oplus \mathcal{Y} \to \mathcal{X}, \qquad T_{22} : \mathcal{D} \oplus \mathcal{Y} \to \mathcal{U}$$

by

$$T_{11} = A^*|_{\mathcal{D}^{\perp}}, \qquad T_{12} = \begin{bmatrix} A^*|_{\mathcal{D}} & C^* \end{bmatrix}, \qquad T_{22} = \begin{bmatrix} C_V & S(0)^* \end{bmatrix}. \tag{2.12}$$

Then our extension problem can be reformulated again as follows.

Problem 2.5. Find an operator X from $\mathcal{D}^\perp$ to $\mathcal{U}$ so that the block operator matrix

$$\mathbf{U}^* = \begin{bmatrix} T_{11} & T_{12} \\ X & T_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{D}^\perp \\ \mathcal{D} \oplus \mathcal{Y} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \tag{2.13}$$

is a contraction.

This is a standard matrix-completion problem handled by the result of Parrott [20]: *Problem* 2.5 *has a solution* X *if and only if the obvious necessary conditions hold*:

$$\left\| [\, T_{11} \quad T_{12} \,] \right\| \leqslant 1, \qquad \left\| \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix} \right\| \leqslant 1. \tag{2.14}$$

Making use of the definitions of T_{11}, T_{12}, T_{22} from (2.12), we get more explicitly

$$[\, T_{11} \quad T_{12} \,] = [\, A^* \quad C^* \,], \qquad \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix} = \begin{bmatrix} A^*|_{\mathcal{D}} & C^* \\ C_V & S(0)^* \end{bmatrix} = \begin{bmatrix} A_V & B_V \\ C_V & D_V \end{bmatrix} = V, \tag{2.15}$$

where we use the identification

$$\begin{bmatrix} \mathcal{D}^\perp \\ \mathcal{D} \oplus \mathcal{Y} \end{bmatrix} \cong \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix}$$

in the first expression. Thus the first expression in (2.15) is contractive by our assumption that $(C, \mathbf{A})$ is a contractive pair while the second expression collapses to V which is isometric. We conclude that the necessary conditions (2.14) are satisfied and hence, by the result of [20], there exists a solution X to Problem 2.5. To complete the proof of Theorem 2.4, we set

$$B = \begin{bmatrix} X^* \\ C_V^* \end{bmatrix} : \mathcal{U} \to \begin{bmatrix} \mathcal{D}^\perp \\ \mathcal{D} \end{bmatrix} \cong \mathcal{X}^d, \tag{2.16}$$

where X is any solution of the matrix-completion problem (2.13). Note that the isometry property of V then gives that the resulting colligation $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is weakly coisometric. □

Remark 2.6. Every $X \in \mathcal{L}(\mathcal{D}^\perp, \mathcal{U})$ leading to a contractive (isometric or unitary) $\mathbf{U}^*$ in (2.13), gives rise via formula (2.16) to a weakly coisometric (respectively, coisometric or unitary) realization of S of the form

$$\mathbf{U} = \begin{bmatrix} A & ? \\ C & S(0) \end{bmatrix}. \tag{2.17}$$

Applying well-known descriptions [4,18,20,21] of all X's solving contractive, isometric and unitary completion problems (2.13) one can get all weakly coisometric, coisometric or unitary realizations for S of the form (2.17) as follows. Let T_{11}, T_{12}, T_{22} be as in (2.12). Since $[\, T_{11} \quad T_{12} \,]$ is a contraction, there is a unique $G_1 : \overline{\mathrm{Ran}}(I - T_{12}T_{12}^*)^{1/2} \to \mathcal{D}^\perp$ so that

$$G_1\big(I - T_{12}T_{12}^*\big)^{1/2} = T_{11}^*, \qquad \operatorname{Ker} G_1^* = \operatorname{Ker} T_{11}. \tag{2.18}$$

Since $\left[\begin{smallmatrix} T_{12} \\ T_{22} \end{smallmatrix}\right] = V$ is an isometry, there exists a unique partial isometry

$$G_2 : \overline{\mathrm{Ran}}\big(I - T_{12}^*T_{12}\big)^{1/2} = (\operatorname{Ker} T_{22})^\perp \to \mathcal{U}$$

so that

$$G_2\big(I - T_{12}^*T_{12}\big)^{1/2} = T_{22}, \qquad \operatorname{Ker} G_2^* = \operatorname{Ker} T_{22}^*. \tag{2.19}$$

The latter equality can be considered as the polar decomposition of T_{22}. Note that

$$I_{\mathcal{U}} - G_2 G_2^* = P_{\operatorname{Ker} T_{22}^*}. \tag{2.20}$$

(Here $P_{\operatorname{Ker} T_{22}^*}$ denotes the orthogonal projection onto $\operatorname{Ker} T_{22}^*$.) From the formula for T_{22} in (2.12) combined with the formula (2.5) for the action of C_V on a generic generating vectors of $\mathcal{D}$, we see that

$$\overline{\operatorname{Ran}}\, T_{22} = \overline{\operatorname{span}}\{S(\boldsymbol{\zeta})^* y\colon\ \boldsymbol{\zeta} \in \mathbb{B}^d,\ y \in \mathcal{Y}\}$$

and hence

$$\operatorname{Ker} T_{22}^* = (\overline{\operatorname{Ran}}\, T_{22})^{\perp} = \{u \in \mathcal{U}\colon\ S(\boldsymbol{\lambda})u \equiv 0\} =: \mathcal{U}_S^0. \tag{2.21}$$

Theorem 2.7. *Suppose that $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and a contractive pair $(C, \mathbf{A})$ are such that $K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_{C,\mathbf{A}}(\boldsymbol{\lambda}, \boldsymbol{\zeta})$. Let $\mathcal{D} \subset \mathcal{X}^d$, C_V, T_{11}, T_{12} and T_{22} be as in* (1.13), (2.5), (2.12) *with G_1, G_2 constructed as in* (2.18), (2.19) *and the subspace $\mathcal{U}_S^0$ given as in* (2.21). *Then*:

(1) *A realization $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & S(0)\end{smallmatrix}\right]$ of S is weakly coisometric if and only if B is of the form*

$$B = \begin{bmatrix} X^* \\ C_V^* \end{bmatrix} \quad \text{where } X = -G_2 T_{12}^* G_1^* + Q\big(I_{\mathcal{D}^{\perp}} - G_1 G_1^*\big)^{1/2} \tag{2.22}$$

and where $Q : \overline{\operatorname{Ran}}(I_{\mathcal{D}^{\perp}} - G_1 G_1^*)^{1/2} \to \mathcal{U}_S^0$ is a contraction.

(2) *S admits a coisometric realization $\mathbf{U}$ of the form* (2.17) *if and only if*

$$\dim \overline{\operatorname{Ran}}\big(I_{\mathcal{D}^{\perp}} - G_1 G_1^*\big)^{1/2} \leqslant \dim \mathcal{U}_S^0. \tag{2.23}$$

In this case, a realization $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & S(0)\end{smallmatrix}\right]$ of S is coisometric if and only if B is of the form (2.22) *for some isometric $Q : \overline{\operatorname{Ran}}(I_{\mathcal{D}^{\perp}} - G_1 G_1^*)^{1/2} \to \mathcal{U}_S^0$.*

(3) *S admits a unitary realization $\mathbf{U}$ of the form* (2.17) *if and only if $(C, \mathbf{A})$ is an isometric pair, i.e.*

$$A_1^* A_1 + \cdots + A_d^* A_d + C^* C = I_{\mathcal{X}}, \tag{2.24}$$

and

$$\dim\big(\operatorname{Ker} A^* \cap \mathcal{D}^{\perp}\big) = \dim \mathcal{U}_S^0. \tag{2.25}$$

In this case, a realization $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & S(0)\end{smallmatrix}\right]$ of S is unitary if and only if B is of the form (2.22) *for some unitary $Q : \overline{\operatorname{Ran}}(I_{\mathcal{D}^{\perp}} - G_1 G_1^*)^{1/2} \to \mathcal{U}_S^0$.*

Proof. Problem 2.5 is equivalent to the following positive completion problem: *find X such that*

$$\begin{bmatrix} I & 0 & T_{11}^* & X^* \\ 0 & I & T_{12}^* & T_{22}^* \\ T_{11} & T_{12} & I & 0 \\ X & T_{22} & 0 & I \end{bmatrix} \geqslant 0. \tag{2.26}$$

Substituting expressions (2.18) and (2.19) for T_{11}^* and T_{22} into (2.26) and taking the Schur complement to the principal (positive semidefinite) block $\left[\begin{smallmatrix} I & T_{12}^* \\ T_{12} & I\end{smallmatrix}\right]$ we get (upon invoking (2.20) and (2.21))

$$\begin{bmatrix} I_{\mathcal{D}^{\perp}} - G_1 G_1^* & X^* + G_1 T_{12} G_2^* \\ X + G_2 T_{12}^* G_1^* & P_{\mathcal{U}_S^0} \end{bmatrix} \geqslant 0 \tag{2.27}$$

which is equivalent to (2.26). It follows from (2.27) and (2.20) that X is a solution of the contractive completion problem (2.13) (and therefore it leads via formula (2.16) to a weakly coisometric realization of S) if and only if it is of the form

$$X = -G_2 T_{12}^* G_1^* + Q\big(I_{\mathcal{D}^\perp} - G_1 G_1^*\big)^{1/2} \tag{2.28}$$

for some contraction $Q : \overline{\mathrm{Ran}}(I_{\mathcal{D}^\perp} - G_1 G_1^*)^{1/2} \to \mathcal{U}_S^0$ which on account of Remark 2.6 completes the proof of the first statement in the theorem.

Note that a contractive $\mathbf{U}^*$ of the form (2.13) is an isometry if and only if

$$T_{11}^* T_{11} + X^* X = I_{\mathcal{D}^\perp}. \tag{2.29}$$

To simplify the latter relation we need the following two equalities:

$$G_2^* Q\big(I_{\mathcal{D}^\perp} - G_1 G_1^*\big)^{1/2} = 0 \quad \text{and} \quad G_1 T_{12}|_{\mathrm{Ker}\, G_2} = 0. \tag{2.30}$$

The first equality holds true since $\mathrm{Ran}\, Q \subset \mathcal{U}_S^0 = \mathrm{Ker}\, T_{22}^* = \mathrm{Ker}\, G_2^*$, by (2.21) and (2.19). To verify the second equality, take a vector $x \in \mathrm{Ker}\, G_2$ in the form

$$x = \big(I - T_{12}^* T_{12}\big)^{1/2} y \quad \text{where } y \in \mathrm{Ker}\, T_{22}.$$

Then, by (2.18),

$$G_1 T_{12} x = G_1 T_{12}\big(I - T_{12}^* T_{12}\big)^{1/2} y = G_1\big(I - T_{12} T_{12}^*\big)^{1/2} T_{12} y = T_{11}^* T_{12} y. \tag{2.31}$$

Since $[\, T_{11} \quad T_{12}\,]$ is a contraction,

$$\big\| T_{11}^* T_{12} y\big\|^2 + \big\| T_{12}^* T_{12} y\big\|^2 \leqslant \| T_{12} y\|^2$$

and since $\left[\begin{smallmatrix} T_{12} \\ T_{22} \end{smallmatrix}\right]$ is an isometry and $T_{22} y = 0$, we have

$$\| T_{12} y\| = \big\| T_{12}^* T_{12} y\big\| = \| y\|.$$

Combining the two latter relations we conclude that $T_{11}^* T_{12} y = 0$ and now the second relation in (2.30) follows from (2.31). Making use of (2.28) and of the first relation in (2.30), we get

$$X^* X = \big(I - G_1 G_1^*\big)^{1/2} Q^* Q\big(I - G_1 G_1^*\big)^{1/2} + G_1 T_{12} G_2^* G_2 T_{12}^* G_1^*$$

which being substituted along with (2.18) into (2.29) allows us to write (2.29) equivalently as

$$\big(I - G_1 G_1^*\big)^{1/2}\big(I - Q^* Q\big)\big(I - G_1 G_1^*\big)^{1/2} = -G_1 T_{12}\big(I - G_2^* G_2\big) T_{12}^* G_1^*.$$

Since $I - G_2^* G_2$ is equal to the orthogonal projection onto $\mathrm{Ker}\, G_2$, the expression on the right-hand side equals zero and thus, (2.29) is equivalent to

$$\big(I_{\mathcal{D}^\perp} - G_1 G_1^*\big)^{1/2}\big(I - Q^* Q\big)\big(I_{\mathcal{D}^\perp} - G_1 G_1^*\big)^{1/2} = 0$$

which means that Q is isometric. The latter may occur if and only if condition (2.23) holds. This completes the proof of the second statement in the theorem.

Finally, for $\mathbf{U}^*$ to be unitary it is necessary that $[\, T_{11} \quad T_{12}\,]$ is a coisometry, which on account of (2.15) can be written as $A^* A + C^* C = I_{\mathcal{X}}$ and is equivalent to (2.24). In this case the operator G_1 defined in (2.18) is a partial isometry and

$$I_{\mathcal{D}^\perp} - G_1 G_1^* = P_{\mathrm{Ker}\, T_{11}}.$$

Then the parametrization formula (2.28) for all solutions X of the contractive completion problem takes the form

$$X = G_2 T_{12}^* G_1^* + Q \tag{2.32}$$

where $Q : \operatorname{Ker} T_{11} \to \mathcal{U}_S^0$ is a contraction. A contraction $\mathbf{U}^*$ of the form (2.13) is an isometry if and only if

$$T_{11}^* T_{11} + X^* X = I_{\mathcal{D}^\perp} \quad \text{and} \quad X X^* + T_{22} T_{22}^* = I_{\mathcal{U}}.$$

Substituting (2.18) and (2.32) into the latter equalities we write them equivalently as

$$I_{\operatorname{Ker} T_{11}} - Q^* Q = 0 \quad \text{and} \quad I_{\mathcal{U}_S^0} - Q Q^* = 0$$

which means that Q must be unitary. The latter may occur if and only if

$$\dim \operatorname{Ker} T_{11} = \dim \mathcal{U}_S^0.$$

Since $\operatorname{Ker} T_{11} = \operatorname{Ker} A^*|_{\mathcal{D}^\perp} = \operatorname{Ker} A^* \cap \mathcal{D}^\perp$, the last condition is equivalent to (2.25). □

As a corollary we obtain the following uniqueness result.

Corollary 2.8. *Suppose that $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and a contractive pair $(C, \mathbf{A})$ are such that $K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_{C,\mathbf{A}}(\boldsymbol{\lambda}, \boldsymbol{\zeta})$. Let $\mathcal{D} \subset \mathcal{X}^d$, T_{11}, T_{12} be as in* (1.13), (2.12) *with G_1, constructed as in* (2.18)*, and the subspace $\mathcal{U}_S^0$ given as in* (2.21)*. Then*:

(1) *S admits a unique weakly coisometric realization $\mathbf{U}$ of the form* (2.17) *if and only if either*

$$G_1 G_1^* = I_{\mathcal{D}^\perp} \quad \textit{or} \quad \mathcal{U}_S^0 = \{0\}. \tag{2.33}$$

(2) *If $G_1 G_1^* = I_{\mathcal{D}^\perp}$, then this unique realization is also coisometric and it is unitary if $(C, \mathbf{A})$ is an isometric pair and both conditions in* (2.33) *are satisfied.*
(3) *In either case, this unique realization is obtained via formula* (2.22) *applied to $X = -G_2 T_{12}^* G_1^*$.*

The second condition in (2.33) is much easier to be verified. We display uniqueness caused by this condition as a separate statement.

Corollary 2.9. *Let $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and let $(C, \mathbf{A})$ be a contractive pair such that $K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_{C,\mathbf{A}}(\boldsymbol{\lambda}, \boldsymbol{\zeta})$. Suppose that $\mathcal{U}_S^0 = \{0\}$, i.e., that*

$$S(\boldsymbol{\lambda}) u \equiv 0 \quad \Longrightarrow \quad u = 0. \tag{2.34}$$

Then S admits a unique weakly coisometric realization $\mathbf{U}$ of the form (1.10) *consistent with the preassigned choice of output pair $(C, \mathbf{A})$. Moreover*:

(1) *This realization is coisometric if and only if $G_1 G_1^* = I_{\mathcal{D}^\perp}$, where G_1 is defined in* (2.18).
(2) *This realization is unitary if and only if $(C, \mathbf{A})$ is an isometric pair and* $\operatorname{Ker} A^* \cap \mathcal{D}^\perp = \{0\}$.

The case when S satisfies condition (2.34) is generic in the following sense: if the subspace $\mathcal{U}_S^0$ is not trivial, we represent $\mathcal{U}$ as $(\mathcal{U}_S^0)^\perp \oplus \mathcal{U}_S^0$ and write $S(\boldsymbol{\lambda})$ with respect to this decomposition as

$$S(\boldsymbol{\lambda}) = \begin{bmatrix} \widetilde{S}(\boldsymbol{\lambda}) & 0 \end{bmatrix}.$$

Then $\widetilde{S} \in \mathcal{S}_d((\mathcal{U}_S^0)^\perp, \mathcal{Y})$ satisfies the condition (2.34) and besides, $K_{C,\mathbf{A}}(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_{\widetilde{S}}(\boldsymbol{\lambda}, \boldsymbol{\zeta})$. Suppose that we are given a contractive pair $(C, \mathbf{A})$ such that $K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_{C,\mathbf{A}}(\boldsymbol{\lambda}, \boldsymbol{\zeta})$ and we let

$$\widetilde{S}(\boldsymbol{\lambda}) = \widetilde{D} + C\big(I - Z(\boldsymbol{\lambda})A\big)^{-1} Z(\boldsymbol{\lambda})\widetilde{B}$$

be the unique weakly coisometric realization of $\widetilde{S}$ consistent with $(C, \mathbf{A})$ and $\widetilde{D} = \widetilde{S}(0)$. Then every weakly coisometric realization for S consistent with $(C, \mathbf{A})$ and $\widetilde{S}(0)$ is of the form (1.10) with

$$D = [\,\widetilde{D} \quad 0\,] \quad \text{and} \quad B = \big[\,\widetilde{B} \quad B^0\,\big],$$

where $B^0 : \mathcal{U}_S^0 \to \mathcal{X}^d$ is an operator subject to the sole constraint that the operator

$$\mathbf{U} = \begin{bmatrix} A & \widetilde{B} & B^0 \\ C & \widetilde{D} & 0 \end{bmatrix} \tag{2.35}$$

be a contraction. This operator B^0 is responsible for nonuniqueness of weakly coisometric realizations compatible with a given contractive pair $(C, \mathbf{A})$; it is also clear that if $\dim \mathcal{U}_S^0$ is large enough, $\mathbf{U}$ of the form (2.35) can be arranged to be coisometric. We can look at this from another point of view as follows.

Proposition 2.10. *If $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ admits a weakly coisometric realization, then there exists a Hilbert space $\mathcal{F}$ and a partial isometry $W : \mathcal{F} \to \mathcal{U}$ so that the function $S_W(z) = S(z)W \in \mathcal{S}_d(\mathcal{F}, \mathcal{Y})$ admits a coisometric realization. If in addition condition* (2.24) *is satisfied, then $\mathcal{F}$ and W can be chosen so that S_W admits a unitary realization.*

Proof. It suffices to pick $\mathcal{F} = (\mathcal{U}_S^0)^\perp \oplus \overline{\operatorname{Ran}}(I_{\mathcal{D}^\perp} - G_1 G_1^*)^{1/2}$ and to define the partial isometry $W : \mathcal{F} \to \mathcal{U}$ by $Wf = f$ if $f \in (\mathcal{U}_S^0)^\perp$ and $Wf = 0$ if $f \in \overline{\operatorname{Ran}}(I_{\mathcal{D}^\perp} - G_1 G_1^*)^{1/2}$. □

The analysis in the proofs of Theorems 2.4 and 2.7 can be slightly modified to get a description of all Schur-class representers of a contractive pair $(C, \mathbf{A})$.

Theorem 2.11. *Let $(C, \mathbf{A})$ be a contractive pair with $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, let $\mathcal{D}$ be the subspace of $\mathcal{X}^d$ given by* (1.13) *and let*

$$T := \big[\,A^*|_{\mathcal{D}} \quad C^*\,\big] : \mathcal{D} \oplus \mathcal{Y} \to \mathcal{X}. \tag{2.36}$$

(1) *Given a Hilbert space $\mathcal{U}$, there exists an $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ such that*

$$K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_{C,\mathbf{A}}(\boldsymbol{\lambda}, \boldsymbol{\zeta}) \tag{2.37}$$

if and only if

$$\dim \mathcal{U} \geqslant \dim \overline{\operatorname{Ran}}\big(I - T^*T\big)^{1/2}. \tag{2.38}$$

(2) *If* (2.38) *is satisfied, then all $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ for which* (2.37) *holds are described by the formula*

$$S(\boldsymbol{\lambda}) = \big[\,C(I - Z(\boldsymbol{\lambda})A)^{-1} Z(\boldsymbol{\lambda}) \quad I_{\mathcal{Y}}\,\big]\big(I - T^*T\big)^{1/2} G^*, \tag{2.39}$$

where G is an isometry from $\overline{\operatorname{Ran}}(I - T^*T)^{1/2}$ onto $\operatorname{Ran} G \subset \mathcal{U}$.

(3) *If* $\dim \mathcal{U} = \dim \overline{\operatorname{Ran}}(I - T^*T)^{1/2}$, *then the function* $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ *such that* (2.37) *holds is defined uniquely up to a constant unitary factor on the right.*

Proof. By Lemma 2.3, if there is an $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ such that (2.37) holds, then the operator V defined by (2.3)–(2.5) is an isometry. It is readily seen from (2.4), (2.5) and (2.36) that the top block row in V is equal to T while the bottom block row

$$\widetilde{T} := [\,C_V \quad D_V\,] : \mathcal{D} \oplus \mathcal{Y} \to \mathcal{U} \tag{2.40}$$

depends on $S(\boldsymbol{\lambda})$ and is not specified in the conditions of the theorem. Thus, a necessary condition for an $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ to exist so that (2.37) holds is that there exists $\widetilde{T} : \mathcal{D} \oplus \mathcal{Y} \to \mathcal{U}$ such that the operator

$$V = \begin{bmatrix} T \\ \widetilde{T} \end{bmatrix} : \begin{bmatrix} \mathcal{D} \\ \mathcal{Y} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \tag{2.41}$$

is isometric. The latter is true if and only if the condition (2.38) is satisfied (which proves the necessity part in statement (1) of the theorem) and every such $\widetilde{T}$ is necessarily of the form

$$\widetilde{T} = G\big(I - T^*T\big)^{1/2}, \tag{2.42}$$

where G is an isometry from $\overline{\operatorname{Ran}}(I - T^*T)^{1/2}$ onto $\operatorname{Ran} G \subset \mathcal{U}$. The equality

$$S(\boldsymbol{\zeta})^* y = \widetilde{T} \begin{bmatrix} Z(\boldsymbol{\zeta})^*(I_{\mathcal{X}} - \mathbf{A}^* Z(\boldsymbol{\zeta})^*)^{-1} C^* \\ I_{\mathcal{Y}} \end{bmatrix} y \quad \big(\boldsymbol{\zeta} \in \mathbb{B}^d;\ y \in \mathcal{Y}\big) \tag{2.43}$$

defines an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function $S(\boldsymbol{\zeta})$ pointwise. By setting $\boldsymbol{\zeta} = 0$ in (2.43) we get

$$S(0)^* y = \widetilde{T} \begin{bmatrix} 0 \\ I_{\mathcal{Y}} \end{bmatrix} y \quad \big(\boldsymbol{\zeta} \in \mathbb{B}^d;\ y \in \mathcal{Y}\big),$$

and therefore, the block entry D_V in (2.40) is equal to $S(0)^*$. Then it follows from (2.43) that the block entry C_V in (2.40) is defined explicitly as in the formula (2.5). Thus, the isometry V in (2.41) coincides with that in (2.3)–(2.5). Then we apply Lemma 2.3 to conclude that (2.37) holds for S defined in (2.43) and in particular, that this S belongs to $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. This completes the proof of statement (1).

Since every representer S gives rise to an isometric extension V of T as in (2.41) and since (2.42) is the general formula for the bottom component of V, it follows that the formula (2.43) gives a parametrization of all representers $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. Replacing $\widetilde{T}$ in (2.43) by its expression (2.42) and taking into account that $y \in \mathcal{Y}$ is arbitrary, we get

$$S(\boldsymbol{\zeta})^* y = G\big(I - T^*T\big)^{1/2} \begin{bmatrix} Z(\boldsymbol{\zeta})^*(I_{\mathcal{X}} - \mathbf{A}^* Z(\boldsymbol{\zeta})^*)^{-1} C^* \\ I \end{bmatrix}.$$

Taking adjoints we arrive at (2.39). The last statement of the theorem now is self-evident, since under the assumption that $\dim \mathcal{U} = \dim \overline{\operatorname{Ran}}(I - T^*T)^{1/2}$, the operator G is unitary. □

3. Generalized functional-model realizations

Constructing a weakly coisometric realization for a given $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is not an issue: by Theorem 1.3, every $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ admits even a unitary realization. However, the pair $(C, \mathbf{A})$ for a weakly coisometric realization can be constructed in a certain canonical way.

Theorem 3.1. *Let $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and let $\mathcal{H}(K_S)$ be the associated de Branges–Rovnyak space. Then*:

(1) *There exist bounded operators $A_j : \mathcal{H}(K_S) \to \mathcal{H}(K_S)$ such that*

$$f(\boldsymbol{\lambda}) - f(0) = \sum_{j=1}^{d} \lambda_j (A_j f)(\boldsymbol{\lambda}) \quad \textit{for every } f \in \mathcal{H}(K_S) \textit{ and } \boldsymbol{\lambda} \in \mathbb{B}^d, \tag{3.1}$$

and

$$\sum_{j=1}^{d} \|A_j f\|^2_{\mathcal{H}(K_S)} \leqslant \|f\|^2_{\mathcal{H}(K_S)} - \|f(0)\|^2_{\mathcal{Y}}. \tag{3.2}$$

(2) *There is a weakly coisometric realization* (1.10) *for S with state space $\mathcal{X}$ equal to $\mathcal{H}(K_S)$ with the state operators $A_1, \ldots, A_d$ from part* (1) *and the operator $C : \mathcal{H}(K_S) \to \mathcal{Y}$ defined by*

$$Cf = f(0) \quad \textit{for all } f \in \mathcal{H}(K_S). \tag{3.3}$$

Proof. Since every $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ admits a weakly coisometric realization, the associated space $\mathcal{H}(K_S)$ can be identified as the range space of the observability operator for some contractive pair. Then part (1) of the theorem follows from Theorem 2.1. Now let us assume that relations (3.1) and (3.2) hold and that C is defined as in (3.3). Then (3.2) says that the pair $(C, \mathbf{A})$ is contractive. Iteration of (3.1) says that, for each $f \in \mathcal{H}(K_S)$,

$$f(\boldsymbol{\lambda}) = \sum_{j_1=1}^{d} \lambda_{j_1} \Biggl[(A_{j_1} f)(0) + \sum_{j_2=1}^{d} \lambda_{j_2} \Biggl[(A_{j_2} A_{j_1} f)(0) + \sum_{j_3=1}^{d} \lambda_{j_3} \Biggl[(A_{j_3} A_{j_2} A_{j_1} f)(0) + \cdots + \sum_{j_k=1}^{d} \lambda_{j_k} \bigl[(A_{j_k} \cdots A_{j_2} A_{j_1} f)(0) + \cdots \bigr] \cdots \Biggr]\Biggr]\Biggr].$$

This unravels to the tautology

$$f(\boldsymbol{\lambda}) = C\bigl(I - Z(\boldsymbol{\lambda})A\bigr)^{-1} f \quad \text{for all } f \in \mathcal{H}(K_S). \tag{3.4}$$

Hence, by the reproducing property of K_S, for any $\boldsymbol{\zeta} \in \mathbb{B}^d$, $y \in \mathcal{Y}$ and $f \in \mathcal{H}(K_S)$, we have

$$\begin{aligned} \bigl\langle f, K_S(\cdot, \boldsymbol{\zeta}) y \bigr\rangle_{\mathcal{H}(K_S)} &= \bigl\langle f(\boldsymbol{\zeta}), y \bigr\rangle_{\mathcal{Y}} \\ &= \bigl\langle C\bigl(I - Z(\boldsymbol{\zeta})A\bigr)^{-1} f, y \bigr\rangle_{\mathcal{Y}} \\ &= \bigl\langle f, \bigl(I - A^* Z(\boldsymbol{\zeta})^*\bigr)^{-1} C^* y \bigr\rangle_{\mathcal{H}(K_S)} \end{aligned}$$

and we conclude that

$$K_S(\cdot, \boldsymbol{\zeta}) y = \bigl(I - A^* Z(\boldsymbol{\zeta})^*\bigr)^{-1} C^* y. \tag{3.5}$$

Hence, for all $\boldsymbol{\lambda}, \boldsymbol{\zeta} \in \mathbb{B}^d$ and $y, y' \in \mathcal{Y}$ we have

$$\begin{aligned} \bigl\langle K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) y, y' \bigr\rangle_{\mathcal{Y}} &= \bigl\langle K_S(\cdot, \boldsymbol{\zeta}) y, K_S(\cdot, \boldsymbol{\lambda}) y' \bigr\rangle_{\mathcal{H}(K_S)} \\ &= \bigl\langle \bigl(I - A^* Z(\boldsymbol{\zeta})^*\bigr)^{-1} C^* y, \bigl(I - A^* Z(\boldsymbol{\lambda})^*\bigr)^{-1} C^* y' \bigr\rangle_{\mathcal{H}(K_S)} \end{aligned}$$

$$= \langle C(I - Z(\lambda)A)^{-1}(I - A^*Z(\zeta)^*)^{-1}C^*y, y'\rangle_{\mathcal{Y}}$$
$$= \langle K_{C,\mathbf{A}}(\lambda,\zeta)y, y'\rangle_{\mathcal{Y}} \tag{3.6}$$

from which we conclude that $K_S(\lambda,\zeta) = K_{C,\mathbf{A}}(\lambda,\zeta)$. It now follows from Theorem 2.4 that there is a choice of $B_j : \mathcal{U} \to \mathcal{H}(K_S)$ with $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathcal{H}(K_S) \oplus \mathcal{U} \to \mathcal{H}(K_S)^d \oplus \mathcal{Y}$ weakly coisometric so that $S(\lambda) = D + C(I - Z(\lambda)A)^{-1}Z(\lambda)B$. This completes the proof. □

Equality (3.1) means that the operator tuple $\mathbf{A} = (A_1,\dots,A_d)$ solves *the Gleason problem* [19] for $\mathcal{H}(K_S)$. Let us say that $\mathbf{A}$ *is a contractive solution of the Gleason problem* if in addition relation (3.2) holds for every $f \in \mathcal{H}(K_S)$ or, equivalently, if the pair $(C, \mathbf{A})$ is contractive where $C : \mathcal{H}(K_S) \to \mathcal{Y}$ is defined as in (3.3). Theorem 3.1 shows that any contractive solution $\mathbf{A} = (A_1,\dots,A_d)$ of the Gleason problem for $\mathcal{H}(K_S)$ gives rise to a weakly coisometric realization for $S \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$ (not unique, in general). Let us call any such weakly coisometric realization a *generalized functional-model* realization of $S(\lambda)$. A consequence of formula (3.4) is that *any generalized functional-model realization of S is observable.*

Note also that any contractive realization

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}(K_S) \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(K_S)^d \\ \mathcal{Y} \end{bmatrix} \tag{3.7}$$

with C given as in (3.3) and the state space tuple $(A_1,\dots,A_d)$ a contractive solution to the Gleason problem on $\mathcal{H}(S)$ is automatically weakly coisometric (i.e., a generalized functional-model realization), as follows from calculation (3.6) and Proposition 1.5.

For a generalized functional-model realization, we have the following explicit formulas for the characters appearing in Lemma 2.3.

Proposition 3.2. *Suppose that* $\mathbf{U}$ *of the form* (3.7) *is a generalized functional model realization for the Schur-class function* $S \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$ *and that the spaces* $\mathcal{D}$ *and* $\mathcal{R}_V$ *are defined as in* (1.13) *and* (2.6)*. Then the spaces* $\mathcal{D}$, $\mathcal{D}^\perp = \mathcal{H}(K_S)^d \ominus \mathcal{D}$, $\mathcal{R}_V$ *and* $\mathcal{R}_V^\perp = \mathcal{H}(K_S) \ominus \mathcal{R}_V$ *can be described in the following explicit functional forms*:

$$\mathcal{D} = \overline{\operatorname{span}}\{Z(\zeta)^*K_S(\cdot,\zeta)y:\ \zeta \in \mathbb{B}^d,\ y \in \mathcal{Y}\},$$
$$\mathcal{R}_V = \overline{\operatorname{span}}\left\{\begin{bmatrix} K_S(\cdot,\zeta)y \\ S(\zeta)^*y \end{bmatrix}:\ \zeta \in \mathbb{B}^d,\ y \in \mathcal{Y}\right\},$$
$$\mathcal{D}^\perp = \{h \in \mathcal{H}(K_S)^d:\ Z(\lambda)h(\lambda) \equiv 0\}, \tag{3.8}$$
$$\mathcal{R}_V^\perp = \left\{\begin{bmatrix} h \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(K_S) \\ \mathcal{U} \end{bmatrix}:\ h(\lambda) + S(\lambda)u \equiv 0\right\}. \tag{3.9}$$

Proof. Substituting (3.5) into (1.13) and (2.6) gives the two first of the four representations above. Given the formula for $\mathcal{D}$, the formula for $\mathcal{D}^\perp$ follows via a standard calculation in reproducing kernel Hilbert spaces using the reproducing kernel property: indeed $f \in \mathcal{H}(K_S)^d \ominus \mathcal{D}$ means that

$$0 = \langle f, Z(\zeta)^*K_S(\cdot,\zeta)y\rangle_{\mathcal{H}(K_S)^d} = \langle Z(\zeta)f, K_S(\cdot,\zeta)\rangle_{\mathcal{H}(K_S)} = \langle Z(\zeta)f(\zeta), y\rangle_{\mathcal{Y}}$$

holds for every $\zeta \in \mathbb{B}^d$ and $y \in \mathcal{Y}$ forcing $Z(\lambda)f(\lambda) \equiv 0$. The formula (3.9) follows similarly. □

Remark 3.3. Note that in case $d = 1$, Theorem 3.1 collapses to Theorem 1.2. Indeed, in this case equality (3.1) reads

$$f(\lambda) - f(0) = \lambda(Af)(\lambda)$$

and completely defines the operator A as in formula (1.5). By (3.8), $\mathcal{D}^\perp = \{0\}$ and hence $\mathcal{D} = \mathcal{H}(K_S)$. Therefore any weakly coisometric realization is automatically coisometric. On account of (3.5), formula (2.5) for $C_V : \mathcal{D} = \mathcal{H}(K_S) \to \mathcal{U}$ takes the form

$$C_V\colon\ \bar{\zeta} K_S(\cdot, \boldsymbol{\zeta}) y \mapsto \big(S(\zeta)^* - S(0)^*\big) y \quad \text{for } \zeta \in \mathbb{D}\ y \in \mathcal{Y}$$

and the formula for its adjoint $C_V^* : \mathcal{U} \to \mathcal{H}(K_S)$,

$$C_V^* : u \to \frac{S(\zeta) - S(0)}{\zeta} u, \tag{3.10}$$

follows from equalities

$$\begin{aligned} \big\langle \big(C_V^* u\big)(\zeta), y\big\rangle_{\mathcal{Y}} &= \big\langle C_V^* u, K_S(\cdot, \zeta) y\big\rangle_{\mathcal{H}(K_S)} = \big\langle u, C_V K_S(\cdot, \zeta) y\big\rangle_{\mathcal{U}} \\ &= \left\langle u, \frac{S(\zeta)^* - S(0)^*}{\bar{\zeta}} y\right\rangle_{\mathcal{U}} = \left\langle \frac{S(\zeta) - S(0)}{\zeta} u, y\right\rangle_{\mathcal{Y}}. \end{aligned}$$

Since $\mathcal{D}^\perp = \{0\}$, formula (2.16) gives that the only B such that $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is coisometric is $B = C_V^*$. By (3.10), this B is the same as in (1.5).

We next present the result concerning the universality of generalized functional-model realizations among weakly coisometric realizations. We say that two colligations

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{X} \oplus \mathcal{U} \to \mathcal{X}^d \oplus \mathcal{Y} \quad \text{and} \quad \widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} : \widetilde{\mathcal{X}} \oplus \mathcal{U} \to \widetilde{\mathcal{X}}^d \oplus \mathcal{Y}$$

are *unitarily equivalent* if there is a unitary operator $U : \mathcal{X} \to \widetilde{\mathcal{X}}$ such that

$$\begin{bmatrix} \bigoplus_{k=1}^d U & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}.$$

Theorem 3.4. *Any observable weakly coisometric realization of a Schur function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is unitarily equivalent to some generalized functional-model realization of S.*

Proof. Let $S(\boldsymbol{\lambda}) = D + \widetilde{C}(I_{\mathcal{X}} - Z(\boldsymbol{\lambda})\widetilde{A})^{-1} Z(\boldsymbol{\lambda})\widetilde{B}$ be an observable weakly coisometric realization of S with the state space $\widetilde{\mathcal{X}}$. Then $\mathcal{H}(K_S) = \mathcal{H}(K_{\widetilde{C},\widetilde{\mathbf{A}}})$ by Proposition 1.5. The observability operator $\mathcal{O}_{\widetilde{C},\widetilde{\mathbf{A}}} : x \to \widetilde{C}(I_{\mathcal{X}} - Z(\boldsymbol{\lambda})\widetilde{A})^{-1} x$ associated with the contractive observable pair $(\widetilde{C}, \widetilde{\mathbf{A}})$ is isometric as an operator from $\widetilde{\mathcal{X}}$ into $\mathcal{H}(K_{\widetilde{C},\widetilde{\mathbf{A}}}) = \mathcal{H}(K_S)$ by part (2) in Theorem 2.1. Let us define the operator tuple $\mathbf{A} = (A_1, \ldots, A_d)$ on the functional-model state space $\mathcal{X} := \mathcal{H}(K_S) = \operatorname{Ran} \mathcal{O}_{\widetilde{C},\widetilde{\mathbf{A}}}$ by

$$A_j \mathcal{O}_{\widetilde{C},\widetilde{\mathbf{A}}} = \mathcal{O}_{\widetilde{C},\widetilde{\mathbf{A}}} \widetilde{A}_j x \quad \text{for } j = 1, \ldots, d. \tag{3.11}$$

Then for the generic element $f(\boldsymbol{\lambda}) = \widetilde{C}(I_{\mathcal{X}} - Z(\boldsymbol{\lambda})\widetilde{A})^{-1} x$ of $\mathcal{H}(K_S)$ we have

$$\begin{aligned} f(\boldsymbol{\lambda}) - f(0) &= \widetilde{C}\big(I - Z(\boldsymbol{\lambda})\widetilde{A}\big)^{-1}x - \widetilde{C}x \\ &= \widetilde{C}\big(I - Z(\boldsymbol{\lambda})\widetilde{A}\big)^{-1}Z(\boldsymbol{\lambda})\widetilde{A}x \\ &= \sum_{j=1}^{d} \lambda_j \widetilde{C}\big(I - Z(\boldsymbol{\lambda})\widetilde{A}\big)^{-1}\widetilde{A}_j x \\ &= \sum_{j=1}^{d} \lambda_j \cdot (\mathcal{O}_{\widetilde{C},\widetilde{\mathbf{A}}}\widetilde{A}_j x)(\boldsymbol{\lambda}) \\ &= \sum_{j=1}^{d} \lambda_j \cdot (A_j \mathcal{O}_{\widetilde{C},\widetilde{\mathbf{A}}} x)(\boldsymbol{\lambda}) = \sum_{j=1}^{d} \lambda_j \cdot (A_j f)(\boldsymbol{\lambda}) \end{aligned}$$

which means that the operators $A_1, \dots, A_d$ solve the Gleason problem on $\mathcal{H}(K_S)$. For the same generic element $f(\boldsymbol{\lambda})$ of $\mathcal{H}(K_S)$ and for the operator $C : \mathcal{H}(K_S) \to \mathcal{Y}$ defined as in (3.3) we also have

$$C\mathcal{O}_{\widetilde{C},\widetilde{\mathbf{A}}}x = Cf = f(0) = \widetilde{C}x$$

and, since the vector $x \in \mathcal{X}$ is arbitrary, it follows that

$$C\mathcal{O}_{\widetilde{C},\widetilde{\mathbf{A}}} = \widetilde{C}. \tag{3.12}$$

Now we let

$$B_j := \mathcal{O}_{\widetilde{C},\widetilde{\mathbf{A}}}\widetilde{B}_j \quad \text{for } j = 1, \dots, d. \tag{3.13}$$

It is readily seen that B_j maps $\mathcal{U}$ into $\mathcal{H}(K_S)$ and it follows from (3.11)–(3.13) that the realization $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is unitarily equivalent to the original realization $\widetilde{\mathbf{U}} = \left[\begin{smallmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & D \end{smallmatrix}\right]$ via the unitary operator $\mathcal{O}_{\widetilde{C},\widetilde{\mathbf{A}}} : \mathcal{X} \to \mathcal{H}(K_S)$:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \mathcal{O}_{\widetilde{C},\widetilde{\mathbf{A}}} & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} \bigoplus_{k=1}^{d} \mathcal{O}_{\widetilde{C},\widetilde{\mathbf{A}}} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix}\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & D \end{bmatrix}.$$

Therefore this realization $\mathbf{U}$ is also weakly coisometric. Also it is a generalized functional-model realization since the state space $\mathcal{X}$ is the functional-model state space $\mathcal{H}(K_S)$, the output operator C is given by evaluation at 0, and the state-space operators $A_1, \dots, A_d$ on $\mathcal{X} = \mathcal{H}(K_S)$ solve the Gleason problem in $\mathcal{H}(K_S)$. □

As we have already seen, a Schur class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ can admit more than one (not unitarily equivalent) weakly coisometric realizations of the form (1.10) with the same $A_1, \dots, A_d$ and C. Theorem 3.1 indicates another source for nonuniqueness: the kernel K_S can be represented in the form $K_{C,\mathbf{A}}$ in more than one way, or equivalently, the Gleason problem for the space $\mathcal{H}(K_S)$ may have contractive solutions that are not unitarily equivalent. A description of all contractive solutions of the Gleason problem lies beyond the scope of this paper and will be presented elsewhere. Here we present an example showing that the nonuniqueness of the representing pair $(C, \mathbf{A})$ indeed may occur.

Example 3.5. Let us introduce the matrices

$$C = \begin{bmatrix} \frac{1}{2} & 0 & 0 \end{bmatrix}, \qquad A_{0,1} = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \qquad A_{0,2} = \begin{bmatrix} 0 & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.14}$$

$$B_{0,1}=\begin{bmatrix} 0 & \frac{\sqrt{15}}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2\sqrt{3}} & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \end{bmatrix}, \tag{3.15}$$

$$B_{0,2}=\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\sqrt{15}}{4} & 0 & 0 \\ -\frac{1}{2\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.16}$$

$$D=\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.17}$$

so that the 7×10 matrix

$$\mathbf{U}_0=\begin{bmatrix} A_{0,1} & B_{0,1} \\ A_{0,2} & B_{0,2} \\ C & D \end{bmatrix}$$

is coisometric. Then the characteristic function

$$S(\boldsymbol{\lambda})=D+C(I-\lambda_1 A_{0,1}-\lambda_2 A_{0,2})^{-1}(\lambda_1 B_{0,1}+\lambda_2 B_{0,2}) \tag{3.18}$$

of the colligation $\mathbf{U}_0$ belongs to the Schur class $\mathcal{S}_2(\mathbb{C}^7, \mathbb{C})$. It is readily seen that

$$C(I-\lambda_1 A_{0,1}-\lambda_2 A_{0,2})^{-1}=\frac{1}{2}\begin{bmatrix} \frac{4}{4-\lambda_1\lambda_2} & \frac{\lambda_1}{4-\lambda_1\lambda_2} & \frac{\lambda_2}{4-\lambda_1\lambda_2} \end{bmatrix} \tag{3.19}$$

which being substituted along with (3.16)–(3.17) into (3.18) gives the explicit formula

$$S(\boldsymbol{\lambda})=\frac{1}{2(4-\lambda_1\lambda_2)}\begin{bmatrix} \frac{12-4\lambda_1\lambda_2}{\sqrt{3}} & \sqrt{15}\lambda_1 & \lambda_1^2 & \frac{\lambda_1\lambda_2}{\sqrt{6}} & \sqrt{15}\lambda_2 & \frac{\lambda_1\lambda_2}{\sqrt{2}} & \lambda_2^2 \end{bmatrix}. \tag{3.20}$$

By (3.19), identity

$$C(I-\lambda_1 A_{0,1}-\lambda_2 A_{0,2})^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\frac{1}{2}\cdot\frac{4x_1+x_2\lambda_1+x_3\lambda_2}{4-\lambda_1\lambda_2}\equiv 0$$

implies $x_1=x_2=x_3=0$ and therefore the pair $(C, \mathbf{A}_0)$ is observable. Thus, representation (3.18) is a coisometric (and therefore, also weakly coisometric) observable realization of the function $S \in \mathcal{S}_2(\mathbb{C}^7, \mathbb{C})$ given by (3.20). Then we also have

$$\begin{aligned} K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) &= C(I-\lambda_1 A_{0,1}-\lambda_2 A_{0,2})^{-1}\big(I-\bar{\zeta}_1 A_{0,1}^*-\bar{\zeta}_2 A_{0,2}^*\big)^{-1}C^* \\ &= K_{C,\mathbf{A}_0}(\boldsymbol{\lambda}, \boldsymbol{\zeta}). \end{aligned} \tag{3.21}$$

Now let us consider the matrices

$$A_{\gamma,1}=\begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \\ \frac{1}{2}+\gamma & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_{\gamma,2}=\begin{bmatrix} 0 & 0 & \frac{1}{4} \\ \frac{1}{2}-\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.22}$$

where $\gamma \in \mathbb{C}$ is a parameter, and note that

$$C(I-\lambda_1 A_{\gamma,1}-\lambda_2 A_{\gamma,2})^{-1}=\frac{1}{2}\begin{bmatrix} \frac{4}{4-\lambda_1\lambda_2} & \frac{\lambda_1}{4-\lambda_1\lambda_2} & \frac{\lambda_2}{4-\lambda_1\lambda_2} \end{bmatrix}$$

for every γ. In particular, the pair $(C, \mathbf{A}_\gamma)$ is observable for every γ. The latter equality together with (3.21) gives

$$K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta})=K_{C,\mathbf{A}_\gamma}(\boldsymbol{\lambda}, \boldsymbol{\zeta}). \tag{3.23}$$

Now pick any γ so that $|\gamma| < \frac{1}{2\sqrt{2}}$. As it is easily seen, the latter inequality is equivalent to the pair $(C, \mathbf{A}_\gamma)$ being contractive. Thus, we have a Schur-class function S and a contractive pair $(C, \mathbf{A}_\gamma)$ such that equality (3.23) holds. Then by Theorem 2.4, there exist operators $B_{\gamma,1}$ and $B_{\gamma,2}$ so that the operator

$$\mathbf{U}_\gamma = \begin{bmatrix} A_{\gamma,1} & B_{\gamma,1} \\ A_{\gamma,2} & B_{\gamma,2} \\ C & D \end{bmatrix}$$

is weakly coisometric and S can be realized as

$$S(\boldsymbol{\lambda}) = D + C(I - \lambda_1 A_{\gamma,1} - \lambda_2 A_{\gamma,2})^{-1}(\lambda_1 B_{\gamma,1} + \lambda_2 B_{\gamma,2}).$$

It remains to note that the pairs $(C, \mathbf{A}_\gamma)$ and $(C, \mathbf{A}_{\gamma'})$ are not unitarily equivalent (which is shown by another elementary calculation) unless $\gamma = \gamma'$.

4. Overlapping spaces

The subspaces $\mathcal{D}^\perp$ and $\mathcal{R}_V^\perp$ as described in (3.8), (3.9) are particular examples of a general notion of *overlapping spaces* appearing in the theory of reproducing kernel Hilbert spaces as developed by de Branges and Rovnyak [16,17]. In general, suppose that $\mathbb{M} = \mathbb{M}(\boldsymbol{\lambda}, \boldsymbol{\zeta})$ is a positive kernel on $\Omega \times \Omega$ with values in $\mathcal{L}(\mathcal{X})$ (for some Hilbert space $\mathcal{X}$) inducing a reproducing kernel Hilbert space $\mathcal{H}(\mathbb{M})$ of $\mathcal{X}$-valued functions via the Aronszajn construction, and suppose that F is a function on Ω with values equal to operators from $\mathcal{X}$ to another Hilbert space $\mathcal{X}'$. (In our application, of course, we will take $\Omega = \mathbb{B}^d$.) Then

$$\mathbb{M}_F(\boldsymbol{\lambda}, \boldsymbol{\zeta}) := F(\boldsymbol{\lambda})\mathbb{M}(\boldsymbol{\lambda}, \boldsymbol{\zeta})F(\boldsymbol{\zeta})^* \tag{4.1}$$

is also a positive kernel on $\Omega \times \Omega$ with values in $\mathcal{L}(\mathcal{X}')$ inducing a reproducing kernel Hilbert space $\mathcal{H}(\mathbb{M}_F)$ of $\mathcal{X}'$-valued functions on Ω. The sets of finite linear combinations of kernel functions

$$\mathcal{S}_\mathbb{M} := \left\{ \sum_{k=1}^N \mathbb{M}(\cdot, \boldsymbol{\zeta}_k)x_k\colon\ \boldsymbol{\zeta}_k \in \Omega,\ x_k \in \mathcal{X},\ N = 1, 2, 3, \ldots \right\},$$

$$\mathcal{S}_{\mathbb{M}_F} := \left\{ \sum_{k=1}^N \mathbb{M}_F(\cdot, \boldsymbol{\zeta}_k)x_k\colon\ \boldsymbol{\zeta}_k \in \Omega,\ x_k \in \mathcal{X},\ N = 1, 2, 3, \ldots \right\}$$

form dense sets in $\mathcal{H}(\mathbb{M})$ and $\mathcal{H}(\mathbb{M}_F)$, respectively. Moreover, the computation

$$\begin{aligned}
\left\| \sum_{k=1}^N \mathbb{M}_F(\cdot, \boldsymbol{\zeta}_k)x'_k \right\|^2 &= \sum_{k,\ell=1}^N \big\langle \mathbb{M}_F(\cdot, \boldsymbol{\zeta}_k)x'_k, \mathbb{M}_F(\cdot, \boldsymbol{\zeta}_\ell)x'_\ell \big\rangle_{\mathcal{H}(\mathbb{M}_F)} \\
&= \sum_{k,\ell=1}^N \big\langle \mathbb{M}_F(\boldsymbol{\zeta}_\ell, \boldsymbol{\zeta}_k)x'_k, x'_\ell \big\rangle_{\mathcal{X}'} \\
&= \sum_{k,\ell=1}^N \big\langle \mathbb{M}(\boldsymbol{\zeta}_\ell, \boldsymbol{\zeta}_k)F(\boldsymbol{\zeta}_k)^*x'_k, F(\boldsymbol{\zeta}_\ell)^*x'_\ell \big\rangle_{\mathcal{X}} \\
&= \sum_{k,\ell=1}^N \big\langle \mathbb{M}(\cdot, \boldsymbol{\zeta}_k)F(\boldsymbol{\zeta}_k)^*x'_k, \mathbb{M}(\cdot, \boldsymbol{\zeta}_\ell)F(\boldsymbol{\zeta}_\ell)^*x'_\ell \big\rangle_{\mathcal{H}(\mathbb{M})}
\end{aligned}$$

$$= \left\| \sum_{k=1}^{N} \mathbb{M}(\cdot, \boldsymbol{\zeta}_k) F(\boldsymbol{\zeta}_k)^* x'_k \right\|^2_{\mathcal{H}(\mathbb{M})}$$

shows that the map

$$\Psi : \mathbb{M}_F(\cdot, \boldsymbol{\zeta}) x' \mapsto \mathbb{M}(\cdot, \boldsymbol{\zeta}) F(\boldsymbol{\zeta})^* x'$$

(for $\boldsymbol{\zeta} \in \Omega$ and $x' \in \mathcal{X}'$) extends by linearity and continuity to define an isometry, still called Ψ, from $\mathcal{H}(\mathbb{M}_F)$ into $\mathcal{H}(\mathbb{M})$. Another computation, where $f \in \mathcal{H}(\mathbb{M})$, $\boldsymbol{\zeta} \in \Omega$ and $x' \in \mathcal{X}'$,

$$\begin{aligned} \langle \Psi^* f, \mathbb{M}_F(\cdot, \boldsymbol{\zeta}) x' \rangle &= \langle f, \Psi \mathbb{M}_F(\cdot, \boldsymbol{\zeta}) x' \rangle_{\mathcal{H}(\mathbb{M})} \\ &= \langle f, \mathbb{M}(\cdot, \boldsymbol{\zeta}) F(\boldsymbol{\zeta})^* x' \rangle_{\mathcal{H}(\mathbb{M})} \\ &= \langle f(\boldsymbol{\zeta}), F(\boldsymbol{\zeta})^* x' \rangle_{\mathcal{X}} \\ &= \langle F(\boldsymbol{\zeta}) f(\boldsymbol{\zeta}), x' \rangle_{\mathcal{X}'} \end{aligned}$$

shows that the adjoint of Ψ is the multiplication operator

$$\Psi^* = M_F : f(\boldsymbol{\lambda}) \mapsto F(\boldsymbol{\lambda}) f(\boldsymbol{\lambda}).$$

Since we saw above that Ψ is an isometry, we conclude that M_F is a coisometry from $\mathcal{H}(\mathbb{M})$ onto $\mathcal{H}(\mathbb{M}_F)$ and that $\mathcal{H}(\mathbb{M}_F)$ can be characterized as

$$\mathcal{H}(\mathbb{M}_F) = \{ F \cdot f : \ f \in \mathcal{H}(\mathbb{M}) \}$$

with norm given by

$$\begin{aligned} \|F \cdot f\|_{\mathcal{H}(\mathbb{M}_F)} &= \inf\{ \|f'\|_{\mathcal{H}(\mathbb{M})} : \ F(\boldsymbol{\lambda}) f'(\boldsymbol{\lambda}) = F(\boldsymbol{\lambda}) f(\boldsymbol{\lambda}) \text{ for all } \boldsymbol{\lambda} \in \Omega \} \\ &= \|Qf\|_{\mathcal{H}(\mathbb{M})} \quad \text{where } Q = P_{(\operatorname{Ker} M_F)^\perp}. \end{aligned} \tag{4.2}$$

The associated *overlapping space* $\mathcal{L}(F, \mathbb{M})$ is defined to be

$$\mathcal{L}(F, \mathbb{M}) = \operatorname{Ker} M_F \subset \mathcal{H}(\mathbb{M})$$

with norm inherited from $\mathcal{H}(\mathbb{M})$. We then have the unitary identification map

$$\Gamma := \begin{bmatrix} M_F \\ P_{\operatorname{Ker} M_F} \end{bmatrix} : \mathcal{H}(\mathbb{M}) \to \begin{bmatrix} \mathcal{H}(\mathbb{M}_F) \\ \mathcal{L}(F, \mathbb{M}) \end{bmatrix}.$$

When there are canonical operators on $\mathcal{H}(\mathbb{M})$, it is often of interest to work out the induced canonical operators on $\mathcal{H}(\mathbb{M}_F) \oplus \mathcal{L}(F, \mathbb{M})$. We discuss two particular instances here related to Proposition 3.2; in these examples, $\Omega = \mathbb{B}^d$.

Example 4.1. Take $\mathbb{M}(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) \otimes I_{\mathbb{C}^d}$ and $F(\boldsymbol{\lambda}) = Z(\boldsymbol{\lambda})$. Then $\mathcal{H}(\mathbb{M}) = \mathcal{H}(K_S)^d$ and the associated kernel $\mathbb{M}_F(\boldsymbol{\lambda}, \boldsymbol{\zeta})$ is given by

$$(K_S \otimes I_{\mathbb{C}^d})_Z = \lambda_1 K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) \overline{\zeta_1} + \cdots + \lambda_d K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) \overline{\zeta_d}$$

with associated overlapping space $\mathcal{L}(F, \mathbb{M})$ given by

$$\mathcal{L}(Z, K_S \otimes I_{\mathbb{C}^d}) = \{ f \in \mathcal{H}(K_S)^d : \ Z(\boldsymbol{\lambda}) f(\boldsymbol{\lambda}) \equiv 0 \}.$$

Then $\mathcal{L}(Z, K_S \otimes I_{\mathbb{C}^d})$ is exactly the subspace $\mathcal{D}^\perp$ in (3.9). Thus $M_Z : f(\boldsymbol{\lambda}) \mapsto Z(\boldsymbol{\lambda}) f(\boldsymbol{\lambda})$ is unitary from $\mathcal{D}$ onto $\mathcal{H}((K_S \otimes I_{\mathbb{C}^d})_Z)$.

Example 4.2. Take $\mathbb{M}(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = \left[\begin{smallmatrix} K_S(\boldsymbol{\lambda},\boldsymbol{\zeta}) & 0 \\ 0 & I_{\mathcal{U}} \end{smallmatrix}\right]$ and $F(\boldsymbol{\lambda}) = [\, I_{\mathcal{Y}} \quad S(\boldsymbol{\lambda}) \,]$. Then the associated kernel $\mathbb{M}_F(\boldsymbol{\lambda}, \boldsymbol{\zeta})$ is

$$(K_S \oplus I_{\mathcal{U}})_{[\, I \;\; S\,]}(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) + S(\boldsymbol{\lambda})S(\boldsymbol{\zeta})^*$$

while the associated overlapping space $\mathcal{L}(F, \mathbb{M})$ is given by

$$\mathcal{L}\big([\, I_{\mathcal{Y}} \quad S\,], K_S \oplus I_{\mathcal{U}}\big) = \left\{ \begin{bmatrix} h \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(K_S) \\ \mathcal{U} \end{bmatrix} : \; h(\boldsymbol{\lambda}) + S(\boldsymbol{\lambda})u \equiv 0 \right\}$$

and is exactly equal to the space $\mathcal{R}_V^{\perp}$ in (3.9). Note that the space $\mathcal{U}^0$ defined in (2.21) is related to $\mathcal{L}([\, I_{\mathcal{Y}} \quad S\,], K_S \oplus I_{\mathcal{U}})$ according to

$$\begin{bmatrix} 0 \\ \mathcal{U}_S^0 \end{bmatrix} = \mathcal{L}\big([\, I_{\mathcal{Y}} \quad S\,], K_S \oplus I_{\mathcal{U}}\big) \cap \begin{bmatrix} 0 \\ \mathcal{U} \end{bmatrix}.$$

Overlapping spaces are usually considered only for the case where F and $\mathbb{M}$ have the special form

$$F(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = \big[\, F_1(\boldsymbol{\lambda}) \quad F_2(\boldsymbol{\lambda}) \,\big], \qquad \mathbb{M}(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = \begin{bmatrix} \mathbb{M}_1(\boldsymbol{\lambda}, \boldsymbol{\zeta}) & 0 \\ 0 & \mathbb{M}_2(\boldsymbol{\lambda}, \boldsymbol{\zeta}) \end{bmatrix}$$

(see [16,17]), but the case of any finite number (or even a continuum) of such positive kernels $\mathbb{M}_s(\boldsymbol{\lambda}, \boldsymbol{\zeta})$ has come up in some applications (see [12]).

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