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Integral Equations and Operator Theory

A Bitangential Interpolation Problem on the Closed Unit Ball for Multipliers of the Arveson Space

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Abstract. We solve a bitangential interpolation problem for contractive multipliers on the Arveson space with an arbitrary interpolating set in the closed unit ball $\mathbb{B}^d$ of $\mathbb{C}^d$. The solvability criterion is established in terms of positive kernels. The set of all solutions is parametrized by a Redheffer transform.

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1. Introduction

In this paper we study the bitangential interpolation problem for a class of contractive valued functions on the unit ball of $\mathbb{C}^d$. To introduce this class we first recall some definitions.

Let Ω be a domain in $\mathbb{C}^d$, let $\mathcal{E}$ be a separable Hilbert space and let $\mathcal{L}(\mathcal{E})$ stand for the set of all bounded linear operators on $\mathcal{E}$. A $\mathcal{L}(\mathcal{E})$–valued function $K(z,w)$ defined on $\Omega \times \Omega$ is called a positive kernel if

$$\sum_{j,\ell=1}^{n} c_j^* K(z^{(j)}, z^{(\ell)}) c_\ell \geq 0$$

for every choice of an integer n, of vectors $c_1, \dots, c_n \in \mathcal{E}$ and of points $z^{(1)}, \dots, z^{(n)} \in \Omega$. This property will be denoted by $K(z,w) \succeq 0$. In what follows we shall write $K_w(z)$ rather than $K(z,w)$ if the last function will be considered as a function of z with a fixed point $w \in \Omega$.

For example, the kernel

$$k_d(z,w) = \frac{1}{1 - \langle z,\, w \rangle_{\mathbb{C}^d}} \tag{1.1}$$

is positive on the unit ball $\mathbb{B}^d = \left\{z = (z_1, \dots, z_d) \in \mathbb{C}^d : \ |z_1|^2 + \dots + |z_d|^2 < 1\right\}$ of $\mathbb{C}^d$. Points in $\mathbb{C}^d$ will be denoted by $z = (z_1, \dots, z_d)$, where $z_j \in \mathbb{C}$. Throughout the paper

$$\langle z,\, w\rangle = \langle z,\, w\rangle_{\mathbb{C}^d} = \sum_{j=1}^{d} z_j \bar{w}_j \quad (z,\, w \in \mathbb{C}^d)$$

stands for the standard inner product in $\mathbb{C}^d$.

Let $\mathcal{H}(k_d)$ be the reproducing kernel Hilbert space with reproducing kernel k_d. This space exists and is unique by the result of Aronszajn [7]. For a Hilbert space $\mathcal{E}$ we consider the tensor product Hilbert space $\mathcal{H}(k_d) \otimes \mathcal{E}$, which can be viewed as the space of $\mathcal{E}$-valued functions in $\mathcal{H}(k_d)$. By $\mathcal{S}_d(\mathcal{E},\, \mathcal{E}_*)$ we denote the *Schur class of the unit ball*, which consists of all $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued analytic functions S on $\mathbb{B}^d$ such that the multiplication operator

$$\mathbf{M}_S(f(z)) = S(z)f(z),$$

maps contractively $\mathcal{H}(k_d) \otimes \mathcal{E}$ into $\mathcal{H}(k_d) \otimes \mathcal{E}_*$. The latter condition means that the following kernel is positive on $\mathbb{B}^d$:

$$K^S(z,w) = \frac{I_{\mathcal{E}_*} - S(z)S(w)^*}{1 - \langle z,\, w\rangle} \succeq 0 \qquad (z,w \in \mathbb{B}^d).$$

The following alternative characterizations of the class $\mathcal{S}_d(\mathcal{E},\, \mathcal{E}_*)$ in terms of isometric d–variable colligations is given in [16]. In what fallows, the symbol $\mathcal{L}(\mathcal{H}, \mathcal{G})$ stands for the set of all bounded operators acting from $\mathcal{H}$ into $\mathcal{G}$.

Theorem 1.1. *Let S be a $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$–valued function analytic in $\mathbb{B}^d$. The following are equivalent:*

1. *S belongs to $\mathcal{S}_d(\mathcal{E},\, \mathcal{E}_*)$.*
2. *There is an auxiliary Hilbert space $\mathcal{H}$ and an analytic $\mathcal{L}(\mathcal{H}, \mathcal{E}_*)$-valued function $H(z)$ on $\mathbb{B}^d$ so that*
$$\frac{I - S(z)S(w)^*}{1 - \langle z,\, w\rangle} = H(z)H(w)^*. \tag{1.2}$$
3. *There are analytic $\mathcal{L}(\mathcal{E}, \mathcal{H})$-valued functions $G_1, \dots, G_d$ on $\mathbb{B}_d$ such that*
$$I - S(z)^*S(w) = G(z)^*\left(I - Z(z)^*Z(w)\right)G(w), \tag{1.3}$$
where
$$G(z) = \begin{bmatrix} G_1(z) \\ \vdots \\ G_d(z) \end{bmatrix} \quad \textit{and} \quad Z(z) = \begin{bmatrix} z_1 I_{\mathcal{H}} & \dots & z_d I_{\mathcal{H}} \end{bmatrix}. \tag{1.4}$$
4. *There is a unitary operator*
$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \ \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \oplus_1^d \mathcal{H} \\ \mathcal{E}_* \end{bmatrix}$$

such that

$$S(z) = D + C\left(I_{\mathcal{H}} - Z(z)A\right)^{-1} Z(z)B. \tag{1.5}$$

For S of the form (1.5) *relations* (1.2) *and* (1.3) *hold with*

$$H(z) = C\left(I_{\mathcal{H}} - Z(z)A\right)^{-1} \quad \textit{and} \quad G(z) = \left(I_{\oplus_1^d \mathcal{H}} - AZ(z)\right)^{-1} B \tag{1.6}$$

and moreover,

$$\begin{aligned} S(z) - S(w) &= H(z)\left(Z(z) - Z(w)\right)G(w) \\ &= C\left(I_{\mathcal{H}} - Z(z)A\right)^{-1}\left(Z(z) - Z(w)\right)\left(I_{\oplus_1^d \mathcal{H}} - AZ(w)\right)^{-1} B \end{aligned} \tag{1.7}$$

The representation (1.5) is called *a unitary realization* of $S \in \mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$.

Let Ω_R, Ω_L and $\Omega_b \subset \Omega_L$ be three sets. The data set for the interpolation problem is as follows. We are given a one-to-one function

$$\tau = (\tau_1, \ldots, \tau_d) : \Omega_R \to \mathbb{B}^d,$$

and a one-to-one function

$$\sigma = (\sigma_1, \ldots, \sigma_d) : \Omega_L \to \overline{\mathbb{B}}^d$$

which maps $\Omega_L \backslash \Omega_b$ and Ω_b into the unit ball $\mathbb{B}^d$ and into the unit sphere $\mathbb{S}^d = \partial(\mathbb{B}^d)$, respectively. We are also given auxiliary Hilbert spaces $\mathcal{E}_L$ and $\mathcal{E}_R$ and functions

$$\begin{aligned} \mathbf{a} &: \Omega_L \to \mathcal{L}(\mathcal{E}_L, \mathcal{E}_*), \qquad & \mathbf{b} &: \Omega_R \to \mathcal{L}(\mathcal{E}_R, \mathcal{E}), \\ \mathbf{c} &: \Omega_L \to \mathcal{L}(\mathcal{E}_L, \mathcal{E}), \qquad & \mathbf{d} &: \Omega_R \to \mathcal{L}(\mathcal{E}_R, \mathcal{E}_*). \end{aligned}$$

Finally we are given d kernels

$$\Lambda_j(\xi, \mu) : (\Omega_L \backslash \Omega_b) \times \Omega_R \to \mathcal{L}(\mathcal{E}_R, \mathcal{E}_L) \quad (j = 1, \ldots, d),$$

d^2 kernels

$$\Phi_{j,\ell}(\xi, \mu) : \Omega_R \times \Omega_R \to \mathcal{L}(\mathcal{E}_R, \mathcal{E}_R) \quad (j, \ell = 1, \ldots, d)$$

and a function $\Psi(\xi)$ on Ω_b, whose values are positive bounded operators on $\mathcal{E}_L$:

$$\Psi(\xi) : \Omega_b \to \mathcal{L}(\mathcal{E}_L), \quad \Psi(\xi) \geq 0.$$

Problem 1.2. Find all functions $S \in \mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ such that S satisfies the following interpolation conditions:

$$\lim_{r \to 1} S(r\sigma(\xi))^* \mathbf{a}(\xi) = \mathbf{c}(\xi) \quad (\xi \in \Omega_L), \qquad S(\tau(\mu))\mathbf{b}(\mu) = \mathbf{d}(\mu) \quad (\mu \in \Omega_R), \tag{1.8}$$

where the limit is meant in the strong sense, and for some choice of associated functions $H(z)$ and $G_1(z), \ldots, G_d(z)$ in the representations (1.2), (1.3), it holds that

$$\mathbf{b}(\xi)^* G_j(\tau(\xi))^* G_\ell(\tau(\mu))\mathbf{b}(\mu) = \Phi_{j\ell}(\xi, \mu) \quad (\xi, \mu \in \Omega_R;\ j, \ell = 1, \ldots, d), \tag{1.9}$$

$$\mathbf{a}(\xi)^* H(\sigma(\xi)) G_j(\tau(\mu))\mathbf{b}(\mu) = \Lambda_j(\xi, \mu) \quad (\xi \in \Omega_L \backslash \Omega_b,\ \mu \in \Omega_R;\ j = 1, \ldots, d), \tag{1.10}$$

and finally,

$$\lim_{r\to 1}\mathbf{a}(\xi)^*\frac{I_{\mathcal{E}_*}-S(r\sigma(\xi))S(r\sigma(\xi))^*}{1-r^2}\mathbf{a}(\xi)\le\Psi(\xi)\quad(\xi\in\Omega_b),\tag{1.11}$$

where the limit in (1.11) is assumed to exist in the weak sense.

Note that for $\xi\in\Omega_b$, the first condition in (1.8) fixes the directional value of the radial boundary limit of the interpolant S at the point $\sigma(\xi)\in\mathbb{S}^d$, whereas for $\xi\in\Omega_L\backslash\Omega_b$, it reduces to an ordinary left sided condition

$$S(\sigma(\xi))^*\mathbf{a}(\xi)=\mathbf{c}(\xi)\quad(\xi\in\Omega_L\backslash\Omega_b).$$

Note also that we are given only the left sided interpolation boundary condition in (1.8); however, the "boundary part" of the problem is also bitangential: it follows by an analogue of Carathéodory–Julia theorem for functions $S\in\mathcal{S}_d(\mathcal{E},\,\mathcal{E}_*)$ (see Lemma 2.2 below) that conditions (1.8) and (1.11) imply that

$$\|\mathbf{a}(\xi)\|=\|\mathbf{c}(\xi)\|\quad(\xi\in\Omega_b)\tag{1.12}$$

and the right sided interpolation condition

$$\lim_{r\to 1}S(r\sigma(\xi))\mathbf{c}(\xi)=\mathbf{a}(\xi)$$

is satisfied for every $\xi\in\Omega_b$.

Note two opposite particular cases of Problem 1.2: the nonboundary bitangential problem (when Ω_b is the empty set) and the tangential boundary problem (when $\Omega_L\backslash\Omega_b$ is the empty set):

Problem 1.3. Given one-to-one functions $\sigma:\Omega_L\to\mathbb{B}^d$ and $\tau:\Omega_R\to\mathbb{B}^d$, find all functions $S\in\mathcal{S}_d(\mathcal{E},\,\mathcal{E}_*)$ such that S satisfies interpolation conditions

$$S(\sigma(\xi))^*\mathbf{a}(\xi)=\mathbf{c}(\xi)\quad(\xi\in\Omega_L),\qquad S(\tau(\mu))\mathbf{b}(\mu)=\mathbf{d}(\mu)\quad(\mu\in\Omega_R),$$

and, for some choice of associated functions $H(z)$ and $G_1(z),\dots,G_d(z)$ in the representations (1.2), (1.3), it holds that

$$\begin{aligned}\mathbf{a}(\xi)^*H(\sigma(\xi))G_j(\tau(\mu))\mathbf{b}(\mu)&=\Lambda_j(\xi,\mu)\quad(\xi\in\Omega_L,\ \mu\in\Omega_R;\ j=1,\dots,d),\\ \mathbf{b}(\xi)^*G_j(\tau(\xi))^*G_\ell(\tau(\mu))\mathbf{b}(\mu)&=\Phi_{j\ell}(\xi,\mu)\quad(\xi,\mu\in\Omega_R;\ j,\ell=1,\dots,d).\end{aligned}$$

Problem 1.4. Given a function $\sigma:\Omega_b\to\mathbb{S}^d$, find all functions $S\in\mathcal{S}_d(\mathcal{E},\,\mathcal{E}_*)$ such that

$$\lim_{r\to 1}S(r\sigma(\xi))^*\mathbf{a}(\xi)=\mathbf{c}(\xi),$$

and

$$\lim_{r\to 1}\mathbf{a}(\xi)^*\frac{I_{\mathcal{E}_*}-S(r\sigma(\xi))S(r\sigma(\xi))^*}{1-r^2}\mathbf{a}(\xi)\le\Psi(\xi).$$

The following remark shows that Problem 1.2 is in fact more general than one might expect at first sight.

Remark 1.5. The interpolation problem with n left interpolation conditions

$$S(\sigma^{(j)}(\xi))^* \mathbf{a}_j(\xi) = \mathbf{c}_j(\xi) \quad (j = 1, \dots, n)$$

where

$$\mathbf{a}_j : \Omega_j \to \mathcal{L}(\mathcal{E}_L, \mathcal{E}_*), \quad \mathbf{c}_j : \Omega_j \to \mathcal{L}(\mathcal{E}_L, \mathcal{E})$$

and $\Omega_1, \dots, \Omega_n$ are disjoint sets in $\mathbb{C}^d$, also can be included in the framework of Problem 1.2 upon setting

$$\Omega = \bigcup_{j=1}^{n} \Omega_j, \quad \mathbf{a}|_{\Omega_j} = \mathbf{a}_j \quad \text{and} \quad \mathbf{c}|_{\Omega_j} = \mathbf{c}_j.$$

Several right sided and two sided conditions can be considered in much the same way.

In this paper we obtain a definitive solution of Problem 1.2. The existence criterion (see Theorem 3.2) is in terms of the positivity of a certain operator-valued Pick kernel $\mathbb{P}(\xi, \mu)$ (defined on $(\Omega_L \cup \Omega_R) \times (\Omega_L \cup \Omega_R)$ completely in terms of the problem data) which must also satisfy a certain Stein identity (3.10). Following the method of [16] and [12] (see also [25] and [27] for a more abstract version of the method for one-variable problems), we show that solutions of the interpolation problem correspond to unitary colligation extensions of a partially defined isometric colligation constructed explicitly from the interpolation data. In addition we obtain a linear fractional parametrization for the set of all solutions (see Theorem 5.1) by a simple adaptation of the method of Arov and Grossman (see [8, 9]) for the univariate case. This canonical form of the solution is really the motivation behind the seemingly mysterious form of the interpolation conditions (1.9) and (1.10).

We mention that various special cases of Problem 1.2 have been considered before in the literature. The special case Problem 1.3 (with Ω_L and Ω_R taken to be finite sets) was considered in [16] and [12]; the formulation of the interpolation conditions via an operator argument actually makes the problem considered in [12] more general than Problem 1.3 in that interpolation conditions involving arbitrarily high order derivatives are incorporated as well. The existence criterion for Problem 1.2 was obtained by use of a lifting theorem for a noncommutative Cuntz-Toeplitz operator algebra setup in the work of [6, 22, 31]. The special case of Problem 1.3 (with finite Ω_b and finite dimensional $\mathcal{E}$ and $\mathcal{E}_*$) was solved (including with the linear fractional parametrization for the set of all solutions) in [5] via (1) a recursive multivariable adaptation of the Schur algorithm, and (2) an adaptation of reproducing kernel Hilbert space methods. The paper [17] solved the problem by a multivariable adaptation of Potapov's method [33]. The contribution of this paper is to extend the method of [16] to handle Problem 1.2 in full generality (with simultaneous interior and boundary interpolation conditions).

For the single-variable case ($d = 1$), boundary interpolation on the unit disk for scalar-valued functions appears already in the work of Nevanlinna [34] as well as in [3]. The paper [38] obtains necessary and sufficient conditions for the interpolation problem to be solved with the inequality condition (1.11) replaced by equality. In general it is known that choosing a strictly contractive Schur-class free parameter (which is always possible except in cases where the solution is unique) leads to a solution with equality holding in condition (1.11); although we do not prove this point here, this phenomenon also holds in our setting—as is illustrated by the example which we give in Section 7 below. It is also known that choosing the free parameter to be zero leads to an interpolant with interesting special properties (the so-called "central solution") and that choosing the free parameter to be a contractive constant leads to a rational solution with McMillan degree equal to at most the number of interpolation nodes (for the scalar case). Recent work of Byrnes, Lindquist and collaborators (see [20] for a recent survey) obtains a complete parametrization of such low degree interpolants—analogues of this result for the matrix-valued case and for our multivariable setting remain interesting open problems. We refer to the papers [11, 15, 18, 14] and the books [13, 23] for operator-theoretic treatments of boundary interpolation problems for the matrix-valued Schur-class for the $d = 1$ case.

The paper is organized as follows. Section 2 presents preliminaries on the tangential analogue of the Julia-Carathéodory theorem implicit in the formulation of the boundary interpolation conditions in Problem 1.2. Section 3 formulates the existence criterion and proves the necessity part of the existence theorem. Section 4 establishes the correspondence between solutions of the interpolation problem and characteristic functions of unitary colligations which extend a particular isometric colligation constructed explicitly from the interpolation data. Section 5 introduces the so-called universal unitary colligation and its characteristic function which gives the linear fractional map which parametrizes the set of all solutions of the interpolation problem in terms of a free Schur-class parameter. Section 6 presents various applications of the main results, namely (1) a version of the Leech's theorem for this setting, and (2) a tangential interpolation problem for contractive multipliers from the space of constants to the Arveson space. The final Section 7 illustrates the theory for a simple sample problem with two interpolation nodes.

2. Preliminaries

In this section we present some preliminary results which are probably of independent interest.

Lemma 2.1. *Let A be a contraction on a Hilbert space $\mathcal{H}$. Then the following limits exist in the weak sense*

$$R := \lim_{r\to 1}(1-r)\,(I_{\mathcal{H}} - rA)^{-1}, \tag{2.1}$$

$$Q := \lim_{r\to 1}\,(I_{\mathcal{H}} - rA)^{-1}\,(I_{\mathcal{H}} - A), \tag{2.2}$$

and

$$\lim_{r\to 1}(1-r)^2\,(I_{\mathcal{H}} - rA^*)^{-1}\,(I_{\mathcal{H}} - A^*A)\,(I_{\mathcal{H}} - rA)^{-1} = 0. \tag{2.3}$$

Moreover, R and Q are in fact orthogonal projections onto $\operatorname{Ker}(I_{\mathcal{H}} - A)$ *and* $\overline{\operatorname{Ran}(I_{\mathcal{H}} - A^*)}$*, respectively.*

Proof: Since $\| (I_{\mathcal{H}} - rA)^{-1} \| \leq \frac{1}{1-r}$, it follows that

$$\sup_{0 \leq r < 1} \|(1-r)(I_{\mathcal{H}} - rA)^{-1}\| \leq 1.$$

Therefore,

$$\lim_{r\to 1}(1-r)^2 (I_{\mathcal{H}} - rA)^{-1} = 0 \tag{2.4}$$

and

$$\begin{aligned}\lim_{r\to 1}(1-r)(I_{\mathcal{H}} - rA)^{-1}(I_{\mathcal{H}} - A) &= \lim_{r\to 1}(1-r)\Big(I_{\mathcal{H}} - (1-r)(I_{\mathcal{H}} - rA)^{-1}A\Big) \\ &= 0. \end{aligned}\tag{2.5}$$

On the other hand, the $\mathcal{L}(\mathcal{H})$–valued function

$$\Phi(z) = (I_{\mathcal{H}} - zA)^{-1}(I_{\mathcal{H}} + zA)$$

is analytic and has a nonnegative real part in the unit disk: indeed,

$$\frac{\Phi(z) + \Phi(z)^*}{2} = (I_{\mathcal{H}} - zA)^{-1}(I_{\mathcal{H}} - |z|^2 AA^*)(I_{\mathcal{H}} - \bar{z}A^*)^{-1} \geq 0. \tag{2.6}$$

Therefore, it admits the Riesz–Herglotz representation

$$\Phi(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\sigma(t)$$

for a nonnegative $\mathcal{L}(\mathcal{H})$–valued measure $d\sigma(t)$, such that $\displaystyle\int_0^{2\pi} d\sigma(t) = \Phi(0) = I_{\mathcal{H}}$. Moreover, it follows by the principle of dominated convergence, that

$$\lim_{r\to 1}(1-r)\Phi(r) = 2\sigma(\{0\}), \tag{2.7}$$

where $\sigma(\{0\})$ on the right hand side denotes the operator assigned to the point $t = 0$ by the measure σ and where the limit on the left hand side is meant in the weak sense (for the proof see [23, Lemma 8.1]). Rewrite the last equality as

$$\lim_{r\to 1}(1-r)(I_{\mathcal{H}} - rA)^{-1}(I_{\mathcal{H}} + rA) = 2\sigma(\{0\}) \tag{2.8}$$

and take the average of (2.5) and (2.8):

$$\lim_{r\to 1}(1-r)(I_{\mathcal{H}} - rA)^{-1}(I_{\mathcal{H}} - (1-r)A) = \sigma(\{0\}),$$

which on account of (2.4), is equivalent to

$$\lim_{r\to 1}(1-r)(I_{\mathcal{H}} - rA)^{-1} = \sigma(\{0\}).$$

Thus, the weak limit in (2.1) exists and defines a positive semidefinite operator $R = \sigma(\{0\})$. For every vector $x \in \mathcal{K} = \operatorname{Ker}(I - A)$ (i.e., such that $Ax = x$), it holds that $(1-r)(I_{\mathcal{H}} - rA)^{-1} x = x$ and therefore,

$$Rx = x, \quad x \in \mathcal{K}. \tag{2.9}$$

Furthermore, it follows from (2.1) that A and R commute and moreover,

$$\begin{aligned} AR = RA = \lim_{r\to 1}(1-r)\,(I_{\mathcal{H}}-rA)^{-1}\,A &= \lim_{r\to 1}\frac{1-r}{r}\left((I_{\mathcal{H}}-rA)^{-1}-I_{\mathcal{H}}\right) \\ &= \lim_{r\to 1}\frac{1-r}{r}\,(I_{\mathcal{H}}-rA)^{-1} = R. \end{aligned} \tag{2.10}$$

Since R is selfadjoint, we conclude that $R(I_{\mathcal{H}}-A^*)=0$ and thus,

$$Rx=0, \quad x\in\overline{\operatorname{Ran}(I_{\mathcal{H}}-A^*)}=\mathcal{K}^{\perp},$$

which together with (2.9) imply that R is the the orthogonal projection onto $\mathcal{K}$.

Let Υ stand for the limit on the left hand side in (2.3). To show that $\Upsilon=0$ we start with the following evident equality

$$\begin{aligned} \Upsilon &= \lim_{r\to 1}(1-r)^2\,(I_{\mathcal{H}}-rA^*)^{-1}\,(I_{\mathcal{H}}-r^2A^*A)\,(I_{\mathcal{H}}-rA)^{-1} \\ &\quad -\lim_{r\to 1}(1-r)^2(1-r^2)\,(I_{\mathcal{H}}-rA^*)^{-1}\,A^*A\,(I_{\mathcal{H}}-rA)^{-1}. \end{aligned} \tag{2.11}$$

By (2.7),

$$\lim_{r\to 1}(1-r)^2(\Phi(r)+\Phi(r)^*)=0,$$

which in view of (2.6), can be written as

$$\lim_{r\to 1}(1-r)^2\,(I_{\mathcal{H}}-rA)^{-1}\,(I_{\mathcal{H}}-r^2AA^*)\,(I_{\mathcal{H}}-rA^*)^{-1}=0$$

and thus, the first term on the right hand side of (2.11) is equal to zero. Using the estimate $\|(I_{\mathcal{H}}-rA)^{-1}\|\le(1-r)^{-1}$ we see that the second term is equal to zero as well. Thus, $\Upsilon=0$.

Finally, by (2.10),

$$\begin{aligned} \lim_{r\to 1}\,(I_{\mathcal{H}}-rA)^{-1}\,(I_{\mathcal{H}}-A) &= I_{\mathcal{H}}-\lim_{r\to 1}(1-r)\,(I_{\mathcal{H}}-rA)^{-1}\,A \\ &= I_{\mathcal{H}}-RA=I_{\mathcal{H}}-R. \end{aligned}$$

Thus, $Q=I_{\mathcal{H}}-R$ and (2.2) follows from (2.1). $\square$

Remark 2.2. Note that in fact, the limits in (2.1)–(2.3) exist in the strong sense. Although we do not use this fact in our consideration, below we outline the sketch of the proof, omitting details.

Let $d\Sigma(t)$ be the spectral measure of a unitary dilation $U\in\mathcal{L}(\widehat{\mathcal{H}})$ of A $(t\in T)$. It follows (from the proof of Lemma 2.1) that

$$(1-r)(I-rU)^{-1}=\int_{\mathbb{T}}\frac{1-r}{1-rt}d\Sigma(t)\to\Sigma(\{1\}) \quad (r\to 1) \tag{2.12}$$

weakly (as $r \to 1$), where $\Sigma(\{1\})$ is the projection onto the space of fixed vectors of U. Fix a vector $g \in \widehat{\mathcal{H}}$. Since $\{\Sigma(t)\}$ is a family of orthogonal projections,

$$\begin{aligned}
\left\| \int_{\mathbb{T}} \frac{1-r}{1-rt} d\Sigma(t) g \right\|_{\widehat{\mathcal{H}}}^2 &= \left\langle \int_{\mathbb{T}} \frac{1-r}{1-rt} d\Sigma(t) g, \int_{\mathbb{T}} \frac{1-r}{1-r\tau} d\Sigma(\tau) g \right\rangle_{\widehat{\mathcal{H}}} \\
&= \left\langle \int_{\mathbb{T}} \frac{(1-r)^2}{|1-rt|^2} d\Sigma(t) g, g \right\rangle_{\widehat{\mathcal{H}}} \\
&= \int_{\mathbb{T}} \frac{(1-r)^2}{|1-rt|^2} \langle d\Sigma(t) g, g \rangle_{\widehat{\mathcal{H}}}.
\end{aligned}$$

By the standard arguments applied to the scalar measure $\langle d\Sigma(t)g, g\rangle$ we get

$$\lim_{r\to 1} \|(1-r)(I-rU)^{-1} g\|_{\widehat{\mathcal{H}}}^2 = \langle \Sigma(\{1\}) g, g\rangle_{\widehat{\mathcal{H}}} = \|\Sigma(\{1\}) g\|_{\widehat{\mathcal{H}}}^2,$$

which together with weak convergence in (2.12) implies the strong convergence. Since U is a unitary dilation of A, it follows that $(I - rA)^{-1} = P(I - rU)^{-1}|_{\mathcal{H}}$, where P is the projection of $\widehat{\mathcal{H}}$ onto $\mathcal{H}$, and thus, the limit R in (2.1) exists in the strong sense. Now the strong convergence in (2.2) and (2.3) is clear.

The next lemma contains a tangential multivariable analogue of the Julia–Carathéodory theorem. The classical (scalar single-variable) case is due G. Julia [24] and [21] (see [37] for historic comments and also for an elegant proof involving Hilbert space arguments), matrix single–variable generalizations were obtained in [13, Chapter 21], [23, Section 8] and [29].

Lemma 2.3. *Let $S \in \mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$, $\beta \in \mathbb{S}^d$, $\mathbf{x} \in \mathcal{E}_*$ and let H be a $\mathcal{L}(\mathcal{H}, \mathcal{E}_*)$–valued function from the representation* (1.2). *Then:*
I. The following three statements are equivalent:

1. *S is subject to* $\mathbf{L} := \sup\limits_{0 \le r < 1} \mathbf{x}^* \dfrac{I_{\mathcal{E}_*} - S(r\beta)S(r\beta)^*}{1-r^2} \mathbf{x} < \infty.$
2. *The radial limit* $L := \lim\limits_{r\to 1} \mathbf{x}^* \dfrac{I_{\mathcal{E}^*} - S(r\beta)S(r\beta)^*}{1-r^2} \mathbf{x}$ *exists.*
3. *The radial limit*
$$\lim_{r\to 1} S(r\beta)^* \mathbf{x} = \mathbf{y} \tag{2.13}$$
exists in the strong sense and serves to define the vector $\mathbf{y} \in \mathcal{E}$. Furthermore,
$$\lim_{r\to 1} S(r\beta)\mathbf{y} = \mathbf{x}, \qquad \|\mathbf{y}\| = \|\mathbf{x}\|, \tag{2.14}$$
(the limit is understood in the strong sense) and the radial limit
$$\widetilde{L} = \lim_{r\to 1} \frac{\mathbf{y}^*\mathbf{y} - \mathbf{x}^* S(r\beta)\mathbf{y}}{1-r} \tag{2.15}$$
exists.

II. Any two of the three equalities in (2.13) *and* (2.14) *imply the third.*

III. If any of the three statements in part I holds true, then the radial limit

$$T_0 = \lim_{r\to 1} H(r\beta)^*\mathbf{x} \tag{2.16}$$

exists in the strong sense and

$$T_0^*T_0 = L = \widetilde{L} \le \mathbf{L}. \tag{2.17}$$

Proof: First we prove all the statements for the single-variable case ($d = 1$). The proofs of the two first statements and of the two last relations in (2.17) (in other words, all the assertions that are not related to T_0) for the single-variable case are contained in [23, Lemma 8.3, Lemma 8.4 and Theorem 8.5]. In fact, the proofs in [23] are given for the case when the spaces $\mathcal{E}$ and $\mathcal{E}_*$ are finite dimensional; however, the same arguments go through for the infinite dimensional case, if the limits in (2.13) and (2.14) are understood in the weak sense. Furthermore, estimates

$$\|S(r\beta)^*\mathbf{x} - \mathbf{y}\|^2 \le 2(1-r)\mathbf{L} \quad \text{and} \quad \|S(r\beta)\mathbf{y} - \mathbf{x}\|^2 \le 2|\Re\langle S(r\beta)^*\mathbf{x} - \mathbf{y},\, \mathbf{y}\rangle|$$

(established in the proof of [23, Lemma 8.3]) imply that the limits in (2.13) and (2.14) exist in the strong sense.

Due to the second equality in (2.14) and since $|\beta| = 1$, the rule

$$V:\ \begin{bmatrix} \sqrt{\mathbf{L}} \\ \mathbf{y} \end{bmatrix} \to \begin{bmatrix} \beta\sqrt{\mathbf{L}} \\ \mathbf{x} \end{bmatrix}$$

defines a rank one isometry from

$$\operatorname{span}\left\{\begin{bmatrix} \sqrt{\mathbf{L}} \\ \mathbf{y} \end{bmatrix}\right\} \subset \begin{bmatrix} \mathbb{C} \\ \mathcal{E} \end{bmatrix} \quad \text{onto} \quad \operatorname{span}\left\{\begin{bmatrix} \beta\sqrt{\mathbf{L}} \\ \mathbf{x} \end{bmatrix}\right\} \subset \begin{bmatrix} \mathbb{C} \\ \mathcal{E}_* \end{bmatrix}.$$

It was shown in [27] (see also [26] and [19]) that every Schur function S satisfying interpolation boundary conditions (2.13) and

$$\lim_{r\to 1} \mathbf{x}^* \frac{I_{\mathcal{E}_*} - S(r\beta)S(r\beta)^*}{1-r^2}\mathbf{x} \le \mathbf{L},$$

admits a unitary realization

$$S(z) = D + zC(I_{\mathcal{H}} - zA)^{-1}B \tag{2.18}$$

with an operator $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{E}_* \end{bmatrix}$, which is a unitary extension of the isometry V, that is,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \sqrt{\mathbf{L}}\,\mathbf{e} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \beta\sqrt{\mathbf{L}}\,\mathbf{e} \\ \mathbf{x} \end{bmatrix}, \tag{2.19}$$

where $\mathbb{C}$ is identified with span$\{\mathbf{e}\}$ for an appropriate choice of a unit vector $\mathbf{e} \in \mathcal{H}$. Since $\mathbf{U}$ is unitary,

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} \beta\sqrt{\mathbf{L}}\,\mathbf{e} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \sqrt{\mathbf{L}}\,\mathbf{e} \\ \mathbf{y} \end{bmatrix},$$

which reduces to

$$C^*\mathbf{x} = \sqrt{\mathbf{L}}\,(I_{\mathcal{H}} - \beta A^*)\,\mathbf{e} \quad \text{and} \quad D^*\mathbf{x} = \mathbf{y} - \beta\sqrt{\mathbf{L}}B^*\mathbf{e}. \tag{2.20}$$

Making use of (1.6) (for $d = 1$ and $z = r\beta$ and of the first relation in (2.20) we get

$$H(r\beta)^*\mathbf{x} = \left(I - r\bar{\beta}A^*\right)^{-1} C^*\mathbf{x} = \left(I - r\bar{\beta}A^*\right)^{-1} \left(I - \bar{\beta}A^*\right) \sqrt{\mathbf{L}}\,\mathbf{e}.$$

Taking limits in the last identity as r tends to one and applying Lemma 2.1 (with A replaced by $\bar{\beta}A^*$), we get

$$T_0 := \lim_{r\to 1} H(r\beta)^*\mathbf{x} = \mathbf{P}_{\overline{\mathrm{Ran}(I_{\mathcal{H}}-\beta A)}}\sqrt{\mathbf{L}}\,\mathbf{e}, \tag{2.21}$$

where the limit understood in the weak sense. Furthermore, by (2.18) and (2.20),

$$\begin{aligned} S(r\beta)^*\mathbf{x} &= D^*\mathbf{x} + r\bar{\beta}B^*\left(I - r\bar{\beta}A^*\right)^{-1} C^*\mathbf{x} \\ &= \mathbf{y} - \bar{\beta}B^*\sqrt{\mathbf{L}}\,\mathbf{e} + r\bar{\beta}B^*\left(I - r\bar{\beta}A^*\right)^{-1}\left(I - \bar{\beta}A^*\right)\sqrt{\mathbf{L}}\,\mathbf{e} \\ &= \mathbf{y} - (1-r)\bar{\beta}B^*\left(I - r\bar{\beta}A^*\right)^{-1}\sqrt{\mathbf{L}}\,\mathbf{e}. \end{aligned}$$

Therefore,

$$\frac{\mathbf{y}^*\mathbf{y} - \mathbf{y}^*S(r\beta)^*\mathbf{x}}{1-r} = \bar{\beta}\mathbf{y}^*B^*\left(I - r\bar{\beta}A^*\right)^{-1}\sqrt{\mathbf{L}}\,\mathbf{e}. \tag{2.22}$$

It follows from (2.19) that

$$A\sqrt{\mathbf{L}}\,\mathbf{e} + B\mathbf{y} = \bar{\beta}\sqrt{\mathbf{L}}\,\mathbf{e}$$

and therefore,

$$\mathbf{y}^*B^* = \sqrt{\mathbf{L}}\,\mathbf{e}^*(\beta I_{\mathcal{H}} - A^*).$$

Substituting the latter equality into (2.22) and taking into account that $\beta \in \mathbb{T}$, we get

$$\frac{\mathbf{y}^*\mathbf{y} - \mathbf{y}^*S(r\beta)^*\mathbf{x}}{1-r} = \mathbf{L}\,\mathbf{e}^*(I - \bar{\beta}A^*)\left(I - r\bar{\beta}A^*\right)^{-1}\,\mathbf{e}.$$

Taking limits in the last identity as r tends to one and applying Lemma 2.1 (with A replaced by $\bar{\beta}A^*$), we get, on account of (2.15) and (2.21), that

$$\widetilde{L} = \mathbf{L}\,\mathbf{e}^*\mathbf{P}_{\overline{\mathrm{Ran}(I_{\mathcal{H}}-\beta A)}}\mathbf{e} = T_0^*T_0.$$

Since $L = \widetilde{L}$, we get also $L = T_0^*T_0$. It follows from (1.2) that

$$\mathbf{x}^*\,\frac{I_{\mathcal{E}_*} - S(r\beta)S(r\beta)^*}{1-r^2}\mathbf{x} = \mathbf{x}^*H(r\beta)H(r\beta)^*\mathbf{x}$$

and thus,

$$L := \lim_{r\to 1}\|H(r\beta)^*\mathbf{x}\|^2 = \|T_0\|^2.$$

The last equality together with the weak convergence of $H(r\beta)^*\xi$ to T_0 as $r \to 1$ implies (by the standard fact from functional analysis) that convergence in (2.21) is in fact, strong. This completes the proof of lemma in the case $d = 1$.

For the case $d \geq 2$, let us introduce the slice-functions

$$S_\beta(\zeta) := S(\zeta\beta) \quad \text{and} \quad H_\beta(\zeta) := H(\zeta\beta) \qquad (\zeta \in \mathbb{D}),$$

the first of which clearly belongs to the classical Schur class $\mathcal{S}_1(\mathcal{E}, \mathcal{E}_*)$. Since $\beta \in \mathbb{S}^d$, it follows that $Z(\zeta\beta)Z(\omega\beta)^* = \zeta\bar{\omega}I_{\mathcal{H}}$ for every pair of $\zeta, \omega \in \mathbb{C}$ and thus, by (1.2),

$$I_{\mathcal{E}_*} - S_\beta(\zeta)S_\beta(\omega)^* = (1 - \zeta\bar{\omega})H_\beta(\zeta)H_\beta(\omega)^*.$$

Applying one-variable results to slice-functions S_β and H_β and then returning to the original functions S and H, we come to all of the desired assertions. □

Note that multivariable analogues of Julia–Carathéodory theorem in the setting of holomorphic maps from $\mathbb{B}^{d_1}$ into $\mathbb{B}^{d_2}$ (and more generally, for holomorphic maps between strongly pseudoconvex domains in unit balls) can be found in [36, Section 8.5] and [1].

3. The solvability criterion

In this section we establish the solvability criterion of Problem 1.2. First we note that if S meets the first interpolation condition (1.8) and the limit in (1.11) exists, then condition (1.12) holds true and therefore, it is a necessary condition for Problem 1.2 to have a solution.

Lemma 3.1. *Let $S \in \mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ satisfy* (1.11) *and the first interpolation condition in* (1.8). *Then the $\mathcal{L}(\mathcal{E}_L)$-valued kernel*

$$\Lambda(\xi,\mu) = \begin{cases} \Psi(\xi) & \xi = \mu \in \Omega_b \\ \dfrac{\mathbf{a}(\xi)^*\mathbf{a}(\mu) - \mathbf{c}(\xi)^*\mathbf{c}(\mu)}{1 - \langle \sigma(\xi),\, \sigma(\mu)\rangle} & \textit{otherwise} \end{cases} \tag{3.1}$$

is positive on Ω_L and satisfies

$$\Lambda(\xi,\mu) - \sum_{j=1}^{d} \sigma_j(\xi)\overline{\sigma_j(\mu)}\Lambda(\xi,\mu) = \mathbf{a}(\xi)^*\mathbf{a}(\mu) - \mathbf{c}(\xi)^*\mathbf{c}(\mu) \quad (\xi,\mu \in \Omega_L). \tag{3.2}$$

Proof: Since S belongs to $\mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$, the identity (1.2) holds for some $\mathcal{L}(\mathcal{H}, \mathcal{E}_*)$-valued function H which is analytic on $\mathbb{B}^d$. Let $T_0(\xi)$ stand for the following strong limit

$$T_0(\xi) := \lim_{r \to 1} H(r\sigma(\xi))^*\mathbf{a}(\xi) \quad (\xi \in \Omega_L), \tag{3.3}$$

which exists at every point $\xi \in \Omega_b$ by Lemma 2.3 and at every $\xi \in \Omega_L \backslash \Omega_b$ by analyticity of H. Setting $z = r\sigma(\xi)$ and $w = r\sigma(\mu)$ in (1.2) and multiplying the resulting equality by $\mathbf{a}(\xi)^*$ on the left and by $\mathbf{a}(\mu)$ on the right, we get

$$\mathbf{a}(\xi)^*\frac{I_{\mathcal{E}_*} - S(r\sigma(\xi))S(r\sigma(\mu))^*}{1 - r^2\langle\sigma(\xi),\, \sigma(\mu)\rangle}\mathbf{a}(\mu) = \mathbf{a}(\xi)^*H(r\sigma(\xi))H(r\sigma(\mu))^*\mathbf{a}(\mu).$$

Assuming that $\xi \neq \mu$ or $\xi \notin \Omega_b$ or $\mu \notin \Omega_b$ we let $r \to 1$ in the last identity. In view of (1.8) and (3.3), we get

$$\frac{\mathbf{a}(\xi)^*\mathbf{a}(\mu) - \mathbf{c}(\xi)^*\mathbf{c}(\mu)}{1 - \langle \sigma(\xi), \, \sigma(\mu) \rangle} = T_0(\xi)^* T_0(\mu). \tag{3.4}$$

The case when $\xi = \mu \in \Omega_b$, is handled by (1.11). Comparing (3.4) and (1.11) with (3.1) we conclude that

$$\Lambda(\xi, \mu) = T_0(\xi)^* T_0(\mu) + \gamma(\xi)^* \gamma(\mu) \quad (\xi, \mu \in \Omega_L), \tag{3.5}$$

where $\gamma(\xi)$ is a function on Ω whose values are bounded operators from $\mathcal{E}_L$ to an auxiliary Hilbert space $\mathcal{C}$ which satisfies

$$\gamma(\xi)^* \gamma(\mu) = \begin{cases} \delta(\xi) \geq 0 & \text{if } \xi = \mu \in \Omega_b \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

In particular, $\gamma(\xi) = 0$ for every $\xi \in \Omega_L \backslash \Omega_b$.

It follows from (3.5) that the kernel Λ is positive on Ω_L. Identity (3.2) holds true for $\xi = \mu \in \Omega_b$ due to (1.12) (in this case expressions on the left and right hand sides of (3.2) both equal zero). For all other choices of ξ and μ in Ω_L, (3.2) follows from (3.4) and (3.5). □

Theorem 3.2. *Let $\Lambda(\xi, \mu)$ be given by* (3.1) *and let*

$$E_1 = \begin{bmatrix} I_{\mathcal{E}_R} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 \\ I_{\mathcal{E}_R} \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad E_d = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{\mathcal{E}_R} \end{bmatrix}. \tag{3.7}$$

Then Problem 1.2 has a solution if and only if the kernel $\mathbb{P} : \ (\Omega_L \cup \Omega_R) \times (\Omega_L \cup \Omega_R) \to \mathcal{L}(\mathcal{E}_L \oplus (\oplus_1^d \mathcal{E}_R))$ *given by*

$$\mathbb{P}(\xi, \mu) = \begin{cases} \Lambda(\xi, \mu) & \text{if } \ \xi, \mu \in \Omega_L, \\ \begin{bmatrix} \Lambda_1(\xi, \mu) & \ldots & \Lambda_d(\xi, \mu) \end{bmatrix} & \text{if } \ \xi \in \Omega_L, \ \mu \in \Omega_R, \\ \begin{bmatrix} \Lambda_1(\mu, \xi) & \ldots & \Lambda_d(\mu, \xi) \end{bmatrix}^* & \text{if } \ \xi \in \Omega_R, \ \mu \in \Omega_L, \\ \begin{bmatrix} \Phi_{11}(\xi, \mu) & \ldots & \Phi_{1d}(\xi, \mu) \\ \vdots & & \vdots \\ \Phi_{d1}(\xi, \mu) & \ldots & \Phi_{dd}(\xi, \mu) \end{bmatrix} & \text{if } \ \xi, \mu \in \Omega_R. \end{cases} \tag{3.8}$$

is positive on $\Omega_L \cup \Omega_R$,

$$\mathbb{P}(\xi, \mu) \succeq 0 \qquad (\xi, \, \mu \in \Omega_L \cup \Omega_R), \tag{3.9}$$

and satisfies the Stein identity

$$N(\xi)^* \mathbb{P}(\xi, \mu) N(\mu) - \sum_{j=1}^{d} N_j(\xi)^* \mathbb{P}(\xi, \mu) N_j(\mu) = X(\xi)^* X(\mu) - Y(\xi)^* Y(\mu), \tag{3.10}$$

for every $\xi,\mu\in\Omega_L\cup\Omega_R$*, where*

$$N(\xi)=\begin{cases} I_{\mathcal{E}_L} & \textit{if } \xi\in\Omega_L,\\ \begin{bmatrix}\tau_1(\xi)I_{\mathcal{E}_R}\\ \vdots\\ \tau_d(\xi)I_{\mathcal{E}_R}\end{bmatrix} & \textit{if } \xi\in\Omega_R,\end{cases}\qquad N_j(\xi)=\begin{cases}\overline{\sigma_j(\xi)}I_{\mathcal{E}_L} & \textit{if } \xi\in\Omega_L,\\ E_j & \textit{if } \xi\in\Omega_R\end{cases}\tag{3.11}$$

and

$$X(\xi)=\begin{cases}\mathbf{a}(\xi) & \textit{if } \xi\in\Omega_L,\\ \mathbf{d}(\xi) & \textit{if } \xi\in\Omega_R,\end{cases}\qquad Y(\xi)=\begin{cases}\mathbf{c}(\xi) & \textit{if } \xi\in\Omega_L,\\ \mathbf{b}(\xi) & \textit{if } \xi\in\Omega_R.\end{cases}\tag{3.12}$$

Proof: Here we check the necessity of conditions (3.9), (3.10). The proof of the sufficiency part is postponed until Section 4 where it will be obtained as a consequence of stronger results. Let S be a solution of Problem 1.2, that is, let relations (1.2), (1.3) and (1.7)–(1.11) be in force. By Lemma 2.3, the strong limit in (3.3) exists for every $\xi\in\Omega_L$ and defines the function T_0 on Ω_L. Then from (1.10) we have

$$\Lambda_j(\xi,\mu)=T_0(\xi)^*G_j(\tau(\mu))\mathbf{b}(\mu)\quad (\xi\in\Omega_L,\ \mu\in\Omega_R;\ j=1,\dots,d).\tag{3.13}$$

Let for short

$$T_j(\xi)=G_j(\tau(\xi))\mathbf{b}(\xi)\quad (\xi\in\Omega_R;\ j=1,\dots,d)\tag{3.14}$$

and

$$T(\xi)=\begin{cases} T_0(\xi) & \text{if } \xi\in\Omega_L,\\ \begin{bmatrix}T_1(\xi) & \dots & T_d(\xi)\end{bmatrix} & \text{if } \xi\in\Omega_R.\end{cases}\tag{3.15}$$

It follows from (3.1), (3.5), (3.13) and interpolation conditions (1.9) that

$$\mathbb{P}(\xi,\mu)=T(\xi)^*T(\mu)+\gamma(\xi)^*\gamma(\mu),\tag{3.16}$$

where $\gamma(\mu)$ is defined as in (3.6) for $\mu\in\Omega_L$ and is defined to be equal to 0 for $\mu\in\Omega_R$. Note that (3.9) now follows immediately from (3.16).

From the definitions we see that the generalized Stein equation (3.10) breaks out into three distinct cases, depending on whether $\xi,\mu\in\Omega_L$, $\xi\in\Omega_L$ with $\mu\in\Omega_R$, or $\xi,\mu\in\Omega_R$. The first case is equivalent to (3.1) and holds by definition of Λ. The other two are:

$$\sum_{j=1}^d(\sigma_j(\xi)-\tau_j(\mu))\Lambda_j(\xi,\mu)=\mathbf{c}(\xi)^*\mathbf{b}(\mu)-\mathbf{a}(\xi)^*\mathbf{d}(\mu)\quad (\xi\in\Omega_L,\ \mu\in\Omega_R)\tag{3.17}$$

and

$$\sum_{j=1}^d\Phi_{jj}(\xi,\mu)-\sum_{j=1}^d\sum_{\ell=1}^d\overline{\tau_j(\xi)}\Phi_{j\ell}(\xi,\mu)\tau_\ell(\mu)=\mathbf{b}(\xi)^*\mathbf{b}(\mu)-\mathbf{d}(\xi)^*\mathbf{d}(\mu)\quad (\xi,\mu\in\Omega_R).\tag{3.18}$$

The verification of the two last equalities is similar to the proof of Lemma 3.1. We multiply (1.7) by $\mathbf{a}(\xi)^*$ on the left and by $\mathbf{b}(\mu)$ on the right and set $z = r\sigma(\xi)$ and $w = \tau(\mu)$ (where $\xi \in \Omega_L$ and $\mu \in \Omega_R$) in the resulting identity:

$$\begin{aligned}
&\mathbf{a}(\xi)^* \left(S(r\sigma(\xi)) - S(\tau(\mu))\right) \mathbf{b}(\mu) \\
&= \mathbf{a}(\xi)^* H(r\sigma(\xi)) \left(Z(r\sigma(\xi)) - Z(\tau(\mu))\right) G(\tau(\mu))\mathbf{b}(\mu) \\
&= \mathbf{a}(\xi)^* H(r\sigma(\xi)) \sum_{j=1}^{d} (r\sigma_j(\xi) - \tau_j(\mu)) G_j(\tau(\mu))\mathbf{b}(\mu).
\end{aligned}$$

Making use of interpolation conditions (1.8), we get

$$\lim_{r\to 1} \mathbf{a}(\xi)^* \left(S(r\sigma(\xi)) - S(\tau(\mu))\right) \mathbf{b}(\mu) = \mathbf{c}(\xi)^*\mathbf{b}(\mu) - \mathbf{a}(\xi)^*\mathbf{d}(\mu),$$

whereas conditions (3.13) leads to

$$\lim_{r\to 1} \sum_{j=1}^{d} (r\sigma_j(\xi) - \tau_j(\mu))\mathbf{a}(\xi)^* H(r\sigma(\xi)) G_j(\tau(\mu)))\mathbf{b}(\mu) = \sum_{j=1}^{d} (\sigma_j(\xi) - \tau_j(\mu))\Lambda_j(\xi,\mu).$$

The three last equalities imply (3.17). Furthermore, for the case ξ and μ in Ω_R, the identity

$$\begin{aligned}
&\mathbf{b}(\xi)^*\mathbf{b}(\mu) - \mathbf{b}(\xi)^* S(\tau(\xi))^* S(\tau(\mu))\mathbf{b}(\mu) \\
&= \mathbf{b}(\xi)^* G(\tau(\xi))^* \left(I - Z(\tau(\xi))^* Z(\tau(\mu))\right) G(\tau(\mu))\mathbf{b}(\mu)
\end{aligned}$$

follows from (1.3). By the second condition in (1.8),

$$\mathbf{b}(\xi)^*\mathbf{b}(\mu) - \mathbf{b}(\xi)^* S(\tau(\xi))^* S(\tau(\mu))\mathbf{b}(\mu) = \mathbf{b}(\xi)^*\mathbf{b}(\mu) - \mathbf{d}(\xi)^*\mathbf{d}(\mu),$$

and by (1.9),

$$\begin{aligned}
&\mathbf{b}(\xi)^* G(\tau(\xi))^* \left(I - Z(\tau(\xi))^* Z(\tau(\mu))\right) G(\tau(\mu))\mathbf{b}(\mu) \\
&= \sum_{j=1}^{d} \mathbf{b}(\xi)^* G_j(\tau(\xi))^* G_j(\tau(\mu))\mathbf{b}(\mu) \\
&\quad - \mathbf{b}(\xi)^* \left(\sum_{j=1}^{d} \overline{\tau_j(\xi)} G_j(\tau(\xi))^* \right) \left(\sum_{j=1}^{d} \tau_j(\mu) G_j(\tau(\mu)) \right) \mathbf{b}(\mu) \\
&= \sum_{j=1}^{d} \Phi_{jj}(\xi,\mu) - \sum_{j=1}^{d}\sum_{\ell=1}^{d} \overline{\tau_j(\xi)} \Phi_{j\ell}(\xi,\mu)\tau_\ell(\mu)
\end{aligned}$$

and the three last equalities imply (3.18). □

It follows from the last theorem that necessary and sufficient conditions for Problems 1.3 and 1.4 to have a solution are that the kernel $\mathbb{P}$ and the kernel Λ are positive on $\Omega_L \cup \Omega_R$ and Ω_L, respectively.

4. Solutions to the interpolation problem and unitary extensions

We recall that a d–variable colligation is defined as a quadruple

$$\mathcal{Q} = \{\mathcal{H},\ \mathcal{F},\ \mathcal{G},\ \mathbf{U}\} \tag{4.1}$$

consisting of three Hilbert spaces $\mathcal{H}$ (*the state space*), $\mathcal{F}$ (*the input space*) and $\mathcal{G}$ (*the output space*), together with a connecting operator

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \quad \begin{bmatrix} \mathcal{H} \\ \mathcal{F} \end{bmatrix} \rightarrow \begin{bmatrix} \oplus_1^d \mathcal{H} \\ \mathcal{G} \end{bmatrix}.$$

The colligation is said to be *unitary* if the connecting operator $\mathbf{U}$ is unitary. A colligation

$$\widetilde{\mathcal{Q}} = \{\widetilde{\mathcal{H}},\ \mathcal{F},\ \mathcal{G},\ \widetilde{\mathbf{U}}\}$$

is said to be *unitarily equivalent* to the colligation $\mathcal{Q}$ if there is a unitary operator $M :\ \mathcal{H} \rightarrow \widetilde{\mathcal{H}}$ such that

$$\begin{bmatrix} \mathbf{M} & 0 \\ 0 & I_{\mathcal{G}} \end{bmatrix} \mathbf{U} = \widetilde{\mathbf{U}} \begin{bmatrix} M & 0 \\ 0 & I_{\mathcal{F}} \end{bmatrix},$$

where

$$\mathbf{M} = \begin{bmatrix} M & & \\ & \ddots & \\ & & M \end{bmatrix} :\ \oplus_1^d \mathcal{H} \rightarrow \oplus_1^d \mathcal{H}. \tag{4.2}$$

The *characteristic function* of the colligation $\mathcal{Q}$ is defined as

$$S_{\mathcal{Q}}(z) = D + C\,(I_{\mathcal{H}} - Z(z)A)^{-1}\, Z(z)B, \tag{4.3}$$

where $Z(z)$ is defined as in (1.4). Thus, Theorem 1.1 asserts that a $\mathcal{L}(\mathcal{E},\, \mathcal{E}_*)$–valued function S analytic in $\mathbb{B}^d$ belongs to the class $\mathcal{S}_d(\mathcal{E},\, \mathcal{E}_*)$ if and only if it is the characteristic function of some d–variable unitary colligation $\mathcal{Q}$ of the form (4.1).

Remark 4.1. Unitarily equivalent colligations have the same characteristic function.

In this section we associate a certain unitary colligation to Problem 1.2. It turns out that the characteristic function of this colligation is the transfer function of the Redheffer transform describing the set of all solutions of Problem 1.2. Assuming that necessary conditions (3.9) and (3.10) for Problem 1.2 to have a solution are in force, let $\mathbb{P}$, N, N_j, X and Y be the functions defined in (3.8), (3.11) and (3.12).

Let $\mathcal{H}_L$ and $\mathcal{H}_R$ be linear spaces of $\mathcal{E}_L$-valued functions defined on Ω_L and $\mathcal{E}_R$-valued functions defined on Ω_R, respectively, and taking nonzero values at at most finitely many points. Let us set for short

$$\mathcal{H}_0 = \mathcal{H}_L \oplus (\oplus_1^d \mathcal{H}_R), \quad \mathcal{H}_1 = \mathcal{H}_L \oplus \mathcal{H}_R. \tag{4.4}$$

Thus, elements of $\mathcal{H}_0$ can be viewed as functions f of finite support on $\Omega_L \cup \Omega_R$ such that $f(\xi) \in \mathcal{E}_L$ for $\xi \in \Omega_L$ and $f(\xi) \in \oplus_1^d \mathcal{H}$ for $\xi \in \Omega_R$, and similarly for $\mathcal{H}_1$. Let $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{E}_*)$ and $Y \in \mathcal{L}(\mathcal{H}_1, \mathcal{E})$ be operators defined by

$$Xf = \sum_{\xi} X(\xi)f(\xi), \qquad Yf = \sum_{\xi} Y(\xi)f(\xi) \tag{4.5}$$

and let $D(h,g)$ be the quadratic form on $\mathcal{H}_0 \times \mathcal{H}_0$ defined as

$$D(h,g) = \sum_{\xi_i,\xi_j} \langle \mathbb{P}(\xi_i,\xi_j)h(\xi_j),\ g(\xi_i)\rangle_{\mathcal{E}_0}. \tag{4.6}$$

Then it follows from (3.10) that

$$D(Nh,\ Ng) - \sum_{j=1}^{d} D(N_j h,\ N_j g) = \langle Xh,\ Xg\rangle_{\mathcal{E}_*} - \langle Yh,\ Yg\rangle_{\mathcal{E}}. \tag{4.7}$$

We say that $h_1 \sim h_2$ if and only if $D(h_1 - h_2,\ y) = 0$ for all $y \in \mathcal{H}_0$ and denote $[h]$ the equivalence class of h with respect to the above equivalence. The linear space of equivalence classes endowed with the inner product

$$\langle [h],\ [y]\rangle = D(h,\ y) \tag{4.8}$$

is a prehilbert space, whose completion we denote by $\widehat{\mathcal{H}}$. Rewriting (4.7) as

$$\langle [Nf],\ [Nf]\rangle_{\widehat{\mathcal{H}}} + \langle Yf,\ Yf\rangle_{\mathcal{E}} = \sum_{j=1}^{d} \langle [N_j f],\ [N_j f]\rangle_{\widehat{\mathcal{H}}} + \langle Xf,\ Xf\rangle_{\mathcal{E}_*}$$

we conclude that the linear map

$$\mathbf{V}:\ \begin{bmatrix} [Nf] \\ Yf \end{bmatrix} \to \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_d f] \\ Xf \end{bmatrix} \tag{4.9}$$

is an isometry from

$$\mathcal{D}_{\mathbf{V}} = \operatorname{Clos}\left\{ \begin{bmatrix} [Nf] \\ Yf \end{bmatrix},\ f \in \mathcal{H}_1 \right\} \subset \begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E} \end{bmatrix} \tag{4.10}$$

onto

$$\mathcal{R}_{\mathbf{V}} = \operatorname{Clos}\left\{ \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_d f] \\ Xf \end{bmatrix},\ f \in \mathcal{H}_1 \right\} \subset \begin{bmatrix} \oplus_1^d \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix}. \tag{4.11}$$

Theorems 4.2 and 4.3 below establish a correspondence between solutions S to Problem 1.2 and unitary extensions of the partially defined isometry $\mathbf{V}$ given in (4.6).

Theorem 4.2. *Any solution S of Problem* 1.2 *is the characteristic function of a unitary colligation*

$$\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} : \begin{bmatrix} \widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \oplus_1^d(\widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}}) \\ \mathcal{E}_* \end{bmatrix}, \tag{4.12}$$

which is an extension of the isometry $\mathbf{V}$ defined from the data of Problem 1.2 *as in* (4.9).

Proof: Let S be a solution to Problem 1.2. In particular, S belongs to $\mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ and by Theorem 1.1, it is the characteristic function of some unitary colligation $\mathcal{Q}$ of the form (4.1). In other words, S admits a unitary realization (1.5) with the state space $\mathcal{H}$ and equalities (1.2), (1.3) hold for functions H and G defined via (1.6). The functions H and G are analytic and take respectively $\mathcal{L}(\mathcal{H}, \mathcal{E}_*)$ and $\mathcal{L}(\mathcal{E}, \oplus_1^d\mathcal{H})$–valued in $\mathbb{B}^d$. We shall use representations

$$S(z) = D + H(z)Z(z)B = D + CZ(z)G(z), \tag{4.13}$$

of S (where H and G are defined in (1.6)), each of which is equivalent to (1.5).

The interpolation conditions (1.8)–(1.11) satisfied by S by assumption lead to certain restrictions on the connecting operator $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Substituting (4.13) into (1.8) we get (strongly)

$$\begin{aligned} \lim_{r\to 1} \left(D^* + B^*Z(r\sigma(\xi))^*H(r\sigma(\xi))^*\right)\mathbf{a}(\xi) &= \mathbf{c}(\xi) \quad (\xi \in \Omega_L), \\ \left(D + CZ(\tau(\xi))G(\tau(\xi))\right)\mathbf{b}(\xi) &= \mathbf{d}(\xi) \quad (\xi \in \Omega_R). \end{aligned} \tag{4.14}$$

Making use of functions $T_0, \ldots, T_d$ introduced in (3.3) and (3.14), one can rewrite the two last relations as

$$D^*\mathbf{a}(\xi) + B^*Z(\sigma(\xi))^*T_0(\xi) = \mathbf{c}(\xi) \quad (\xi \in \Omega_L) \tag{4.15}$$

and

$$D\mathbf{b}(\xi) + C\sum_{j=1}^{d} \tau_j(\xi)T_j(\xi) = \mathbf{d}(\xi) \quad (\xi \in \Omega_R). \tag{4.16}$$

It also follows from (1.6) that

$$C + H(z)Z(z)A = H(z), \qquad B + AZ(z)G(z) = G(z)$$

and therefore, that

$$C^*\mathbf{a}(\xi) + \lim_{r\to 1} A^*Z(r\sigma(\xi))^*H(r\sigma(\xi))^*\mathbf{a}(\xi) = \lim_{r\to 1} H(r\sigma(\xi))^*\mathbf{a}(\xi) \quad (\xi \in \Omega_L)$$

and

$$B\mathbf{b}(\xi) + AZ(\tau(\xi))G(\tau(\xi))\mathbf{b}(\xi) = G(\tau(\xi))\mathbf{b}(\xi) \quad (\xi \in \Omega_R).$$

The two last equalities can be written in terms of functions $T_0, \ldots, T_d$ as

$$C^*\mathbf{a}(\xi) + A^*Z(\sigma(\xi))^*T_0(\xi) = T_0(\xi) \quad (\xi \in \Omega_L) \tag{4.17}$$

and

$$B\mathbf{b}(\xi)+A\sum_{j=1}^{d}\tau_j(\xi)T_j(\xi)=\begin{bmatrix} T_1(\xi)\\ \vdots \\ T_d(\xi)\end{bmatrix}\quad (\xi\in\Omega_R). \tag{4.18}$$

The equalities (4.15) and (4.17) can be written in matrix form as

$$\begin{bmatrix} A^* & C^* \\ B^* & D^*\end{bmatrix}\begin{bmatrix} Z(\sigma(\xi))^*T_0(\xi) \\ \mathbf{a}(\xi)\end{bmatrix}=\begin{bmatrix} T_0(\xi)\\ \mathbf{c}(\xi)\end{bmatrix}\quad (\xi\in\Omega_L), \tag{4.19}$$

whereas the equalities (4.16) and (4.18) are equivalent to

$$\begin{bmatrix} A & B \\ C & D\end{bmatrix}\begin{bmatrix} \sum_{j=1}^{d}\tau_j(\xi)T_j(\xi) \\ \mathbf{b}(\xi)\end{bmatrix}=\begin{bmatrix} T_1(\xi)\\ \vdots \\ T_d(\xi) \\ \mathbf{d}(\xi)\end{bmatrix}\quad (\xi\in\Omega_R).$$

Since the operator $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ is unitary, we conclude from (4.19) that

$$\begin{bmatrix} A & B \\ C & D\end{bmatrix}\begin{bmatrix} T_0(\xi) \\ \mathbf{c}(\xi)\end{bmatrix}=\begin{bmatrix} \sigma_1(\xi)^*T_0(\xi)\\ \vdots \\ \sigma_d(\xi)^*T_0(\xi) \\ \mathbf{a}(\xi)\end{bmatrix}\quad (\xi\in\Omega_L).$$

The two last relations can be combined into the single equation

$$\begin{bmatrix} A & B \\ C & D\end{bmatrix}\begin{bmatrix} T(\xi)N(\xi) \\ Y(\xi)\end{bmatrix}f(\xi)=\begin{bmatrix} T(\xi)N_1(\xi)\\ \vdots \\ T(\xi)N_d(\xi) \\ X(\xi)\end{bmatrix}f(\xi)\quad (\xi\in\Omega_L\cup\Omega_R), \tag{4.20}$$

for $f\in\mathcal{H}_1$, where T is given in (3.15). The interpolation conditions (1.9)–(1.11) guarantee (see the proof of Theorem 3.2) the representation (3.16) of the kernel $\mathbb{P}(\xi,\mu)$ with a $\mathcal{L}(\mathcal{E}_L,\mathcal{C})$–valued function $\gamma(\xi)$ on Ω_L satisfying (3.6). Setting

$$\widehat{T}_0(\xi)=\begin{bmatrix} T_0(\xi)\\ \gamma(\xi)\end{bmatrix}\quad (\xi\in\Omega_L)\quad\text{and}\quad \widehat{T}_j(\xi)=\begin{bmatrix} T_j(\xi)\\ 0\end{bmatrix}\quad (\xi\in\Omega_R;\ j=1,\dots,d)$$

and

$$\widehat{T}(\xi)=\begin{cases} \widehat{T}_0(\xi) & \text{if } \xi\in\Omega_L,\\ \begin{bmatrix}\widehat{T}_1(\xi) & \dots & \widehat{T}_d(\xi)\end{bmatrix} & \text{if } \xi\in\Omega_R,\end{cases}$$

we get from (3.16) the following factorization:

$$\mathbb{P}(\xi,\mu)=\widehat{T}(\xi)^*\widehat{T}(\mu). \tag{4.21}$$

Let $\ell^2_{\mathbb{C}}(\Omega_L)$ be the Hilbert space of $\mathbb{C}$–valued functions g on Ω_L such that $g(\xi)^*g(\mu) = 0$ for $\xi \neq \mu$, with the norm $\|g\|^2_{\ell^2_{\mathbb{C}}(\Omega_L)} = \sum_{\xi\in\Omega_L} \|g(\xi)\|^2_{\mathbb{C}}$. Let $\mathcal{H}_L$ denote the set of all $\mathcal{E}_L$–valued functions on Ω_L taking nonzero values at at most finitely many points. Then the set

$$\mathcal{G} = \{h(\xi) = \gamma(\xi)y(\xi) \ \text{ for some } y \in \mathcal{H}_L\}$$

is a subspace of $\ell^2_{\mathbb{C}}(\Omega_L)$. Furthermore, the operator

$$\mathcal{A} = \begin{bmatrix} A_{11} \\ \vdots \\ A_{d1} \end{bmatrix} : \ \gamma(\xi)y(\xi) \to \begin{bmatrix} \sigma_1(\xi)^* \\ \vdots \\ \sigma_d(\xi)^* \end{bmatrix} \gamma(\xi)y(\xi) = Z(\sigma(\xi))^*\gamma(\xi)y(\xi) \quad (y \in \mathcal{H}_L) \tag{4.22}$$

is an isometry from $\mathcal{G}$ into $\oplus_1^d\mathcal{G}$. Let

$$\begin{bmatrix} A_{12} \\ \vdots \\ A_{d2} \end{bmatrix} : \ \widehat{\mathcal{G}} \to \oplus_1^d\widehat{\mathcal{G}}$$

be a unitary extension of $\mathcal{A}$, where $\widehat{\mathcal{G}}$ is a Hilbert space containing $\mathcal{G}$. Thus,

$$A_{j2}(\gamma(\xi)y(\xi)) = \sigma_j(\xi)^*\gamma(\xi)y(\xi) \quad (y \in \mathcal{H}_L;\ j = 1, \dots, d). \tag{4.23}$$

Let

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}$$

be the block decompositions of operators A and B from the unitary realization (1.5) of S. Let

$$\widehat{A} = \begin{bmatrix} \widehat{A}_1 \\ \vdots \\ \widehat{A}_d \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} \widehat{B}_1 \\ \vdots \\ \widehat{B}_d \end{bmatrix} \quad \text{and} \quad \widehat{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \tag{4.24}$$

where

$$\widehat{A}_j = \begin{bmatrix} A_j & 0 \\ 0 & A_{j2} \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \widehat{\mathcal{G}} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \widehat{\mathcal{G}} \end{bmatrix} \quad \text{and} \quad \widehat{B}_j = \begin{bmatrix} B_j \\ 0 \end{bmatrix} : \mathcal{E} \to \begin{bmatrix} \mathcal{H} \\ \widehat{\mathcal{G}} \end{bmatrix}. \tag{4.25}$$

It is readily seen that the operator

$$\widehat{\mathbf{U}} = \begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & D \end{bmatrix} : \quad \begin{bmatrix} \mathcal{H} \oplus \widehat{\mathcal{G}} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \oplus_1^d (\mathcal{H} \oplus \widehat{\mathcal{G}}) \\ \mathcal{E}_* \end{bmatrix}$$

is a unitary extension of $\mathbf{U}$ and that the unitary colligation

$$\widehat{\mathcal{Q}} = \{\mathcal{H} \oplus \widehat{\mathcal{G}},\ \mathcal{E},\ \mathcal{E}_*,\ \widehat{\mathbf{U}}\}$$

has the same characteristic function as $\mathcal{Q}$, that is, $S(z)$. We show that

$$\begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & D \end{bmatrix} \begin{bmatrix} \widehat{T}(\xi)N(\xi) \\ Y(\xi) \end{bmatrix} f(\xi) = \begin{bmatrix} \widehat{T}(\xi)N_1(\xi) \\ \vdots \\ \widehat{T}(\xi)N_d(\xi) \\ X(\xi) \end{bmatrix} f(\xi), \tag{4.26}$$

or, equivalently, that

$$\begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & D \end{bmatrix} \begin{bmatrix} \widehat{T}_0(\xi) \\ \mathbf{c}(\xi) \end{bmatrix} f(\xi) = \begin{bmatrix} \sigma_1(\xi)^* \widehat{T}_0(\xi) \\ \vdots \\ \sigma_d(\xi)^* \widehat{T}_0(\xi) \\ \mathbf{a}(\xi) \end{bmatrix} f(\xi) \quad (\xi \in \Omega_L),$$

$$\begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & D \end{bmatrix} \begin{bmatrix} \sum_{j=1}^d \tau_j(\xi)\widehat{T}_j(\xi) \\ \mathbf{b}(\xi) \end{bmatrix} f(\xi) = \begin{bmatrix} \widehat{T}_1(\xi) \\ \vdots \\ \widehat{T}_d(\xi) \\ \mathbf{d}(\xi) \end{bmatrix} f(\xi) \quad (\xi \in \Omega_R) \tag{4.27}$$

for every $f \in \mathcal{H}_1$. Indeed, due to (4.20) and (4.22)–(4.25), for $\xi \in \Omega_L$ we have

$$\begin{aligned} \widehat{A}_j \widehat{T}_0(\xi) f(\xi) + \widehat{B}_j \mathbf{c}(\xi) f(\xi) &= \begin{bmatrix} (A_j T_0(\xi) + B_j \mathbf{c}(\xi)) f(\xi) \\ A_{j2}\gamma(\xi) f(\xi) \end{bmatrix} \\ &= \begin{bmatrix} \sigma_j(\xi)^* T_0(\xi) \\ \sigma_j(\xi)^* \gamma(\xi) \end{bmatrix} f(\xi) \\ &= \sigma_j(\xi)^* \widehat{T}_0(\xi) f(\xi), \end{aligned}$$

while for $\xi \in \Omega_R$ we have

$$\begin{aligned} \widehat{A}_j \left(\sum_{j=1}^d \tau_j(\xi) \widehat{T}_j(\xi) f(\xi) \right) + \widehat{B}_j \mathbf{b}(\xi) f(\xi) &= \begin{bmatrix} A_j(\sum_{j=1}^d \tau_j(\xi) T_j(\xi)) + B_j \mathbf{b}(\xi) \\ 0 \end{bmatrix} f(\xi) \\ &= \begin{bmatrix} T_j(\xi) \\ 0 \end{bmatrix} f(\xi) \\ &= \widehat{T}_j(\xi) f(\xi), \\ \widehat{C}\widehat{T}_j(\xi) f(\xi) &= C T_j(\xi) f(\xi) \quad (j = 1, \dots, d) \end{aligned}$$

and (4.27) easily follows from the three last relations.

Let $\mathbf{T}: \ \mathcal{H}_0 \to \mathcal{H} \oplus \mathcal{G}$ be the operator given by

$$(\mathbf{T}h)(\xi) = \sum_{\xi} \left(T(\xi)h(\xi) + \gamma(\xi)h(\xi) \right). \tag{4.28}$$

Upon making subsequent use of (4.8), (4.6), (3.16) and (4.28), we have

$$\begin{aligned}
\langle [h], \, [y] \rangle_{\widehat{\mathcal{H}}} &= D(h, \, y) \\
&= \sum_{\xi_i, \xi_j} \langle \mathbb{P}(\xi_i, \xi_j) h(\xi_j), \ y(\xi_i) \rangle_{\mathcal{E}_0} \\
&= \sum_{\xi_i, \xi_j} \langle \widehat{T}(\xi_j) h(\xi_j), \ \widehat{T}(\xi_i) y(\xi_i) \rangle_{\mathcal{H} \oplus \mathbb{C}} \\
&= \sum_{\xi_i, \xi_j} \left(\langle T(\xi_j) h(\xi_j), \ T(\xi_i) y(\xi_i) \rangle_{\mathcal{H}} + \langle \gamma(\xi_j) h_0(\xi_j), \ \gamma(\xi_i) y_0(\xi_i) \rangle_{\mathbb{C}} \right) \\
&= \left\langle \sum_{\xi_j} T(\xi_j) h(\xi_j), \ \sum_{\xi_i} T(\xi_i) y(\xi_i) \right\rangle_{\mathcal{H}} + \langle \gamma(\cdot) h(\cdot), \ \gamma(\cdot) y_0(\cdot) \rangle_{\mathcal{G}} \\
&= \langle \mathbf{T}h, \ \mathbf{T}y \rangle_{\mathcal{H} \oplus \mathcal{G}} .
\end{aligned}$$

Therefore, the linear transformation $U: \ \operatorname{Ran} \mathbf{T} \to \widehat{\mathcal{H}}$ defined by the rule

$$U \ : \ \ \mathbf{T}f \to [f] \qquad (f \in \mathcal{H}_0) \tag{4.29}$$

can be extended to the unitary map (which still is denoted by U) from $\overline{\operatorname{Ran} \mathbf{T}}$ onto $\widehat{\mathcal{H}}$. Setting

$$\mathcal{N} := (\mathcal{H} \oplus \mathcal{G}) \ominus \overline{\operatorname{Ran} \mathbf{T}} \quad \text{and} \quad \widetilde{\mathcal{H}} := \widehat{\mathcal{H}} \oplus \mathcal{N},$$

let us define the unitary map $M \ : \ \ \mathcal{H} \oplus \mathcal{G} \to \widetilde{\mathcal{H}}$ by the rule

$$Mg = \begin{cases} Ug & \text{if} \quad g \in \overline{\operatorname{Ran} \mathbf{T}} \\ g & \text{if} \quad g \in \mathcal{N}. \end{cases} \tag{4.30}$$

and let $\mathbf{M}$ be the block diagonal operator defined as in (4.2). Introducing the operators

$$\widetilde{A} = \mathbf{M} \widehat{A} M^*, \qquad \widetilde{B} = \mathbf{M} \widehat{B}, \qquad \widetilde{C} = \widehat{C} M^*, \qquad \widetilde{D} = D \tag{4.31}$$

we construct the colligation

$$\widetilde{\mathcal{Q}} = \left\{ \widetilde{\mathcal{H}}, \ \mathcal{E}, \ \mathcal{E}_*, \ \left[\begin{array}{cc} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{array} \right] \right\}.$$

By definition, $\widetilde{\mathcal{Q}}$ is unitarily equivalent to the initial colligation $\mathcal{Q}$ defined in (4.1). By Remark 4.1, $\widetilde{\mathcal{Q}}$ has the same characteristic function as $\mathcal{Q}$, that is, $S(z)$. It remains to check that the connecting operator of $\widetilde{\mathcal{Q}}$ is an extension of $\mathbf{V}$, that is

$$\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} [Nf] \\ Yf \end{bmatrix} \rightarrow \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_d f] \\ Xf \end{bmatrix}, \qquad f \in \mathcal{H}_1. \tag{4.32}$$

To this end, note that by (4.28)–(4.30) and block partitionings (3.11) and (4.2) of N, N_j and $\mathbf{M}$,

$$\begin{aligned} M^*[Nf] = \mathbf{T}(Nf) &= \sum_{\xi} T(\xi)N(\xi)f(\xi) + \gamma(\xi)f(\xi) \\ &= \sum_{\xi\in\Omega_L} \begin{bmatrix} T_0(\xi) \\ \gamma(\xi) \end{bmatrix} f(\xi) + \sum_{\xi\in\Omega_R} \begin{bmatrix} \sum_{j=1}^d \tau_j(\xi)T_j(\xi) \\ 0 \end{bmatrix} f(\xi) \\ &= \sum_{\xi\in\Omega_L} \widehat{T}_0(\xi)f(\xi) + \sum_{\xi\in\Omega_R} \left(\sum_{j=1}^d \tau_j(\xi)\widehat{T}_j(\xi) \right) f(\xi) \\ &= \sum_{\xi} \widehat{T}(\xi)N(\xi)f(\xi) \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} M\left(\sum_{\xi} \widehat{T}(\xi)N_j(\xi) \right) &= M\left(\sum_{\xi\in\Omega_L} \overline{\sigma_j(\xi)}\widehat{T}_0(\xi)f(\xi) + \sum_{\xi\in\Omega_R} \widehat{T}_j(\xi)f(\xi) \right) \\ &= M\mathbf{T}(N_j f) = [N_j f] \end{aligned}$$

for every function $f \in \mathcal{H}_1$ and $j = 1, \ldots, d$. Taking into account the diagonal structure (3.8), (4.2) of E_j and $\mathbf{M}$, we now get from the last equality that

$$\mathbf{M}\left(\sum_{\xi} \begin{bmatrix} \widehat{T}(\xi)N_1(\xi) \\ \vdots \\ \widehat{T}(\xi)N_d(\xi) \end{bmatrix} f(\xi) \right) = \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_d f] \end{bmatrix}. \tag{4.34}$$

Thus, upon making the subsequent use of (4.31), (4.33), (4.26) and (4.34), we get

$$\begin{aligned}
\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} [Nf] \\ Yf \end{bmatrix} &= \begin{bmatrix} \mathbf{M} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & D \end{bmatrix} \begin{bmatrix} M^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} [Nf] \\ Yf \end{bmatrix} f \\
&= \begin{bmatrix} \mathbf{M} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & D \end{bmatrix} \left(\sum_{\xi} \begin{bmatrix} \widehat{T}(\xi)N(\xi) \\ Y(\xi) \end{bmatrix} f(\xi) \right) \\
&= \begin{bmatrix} \mathbf{M} & 0 \\ 0 & I \end{bmatrix} \left(\sum_{\xi} \begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & D \end{bmatrix} \begin{bmatrix} \widehat{T}(\xi)N(\xi) \\ Y(\xi) \end{bmatrix} f(\xi) \right) \\
&= \begin{bmatrix} \mathbf{M} & 0 \\ 0 & I \end{bmatrix} \left(\sum_{\xi} \begin{bmatrix} \widehat{T}(\xi)N_1(\xi) \\ \vdots \\ \widehat{T}(\xi)N_d(\xi) \\ X(\xi) \end{bmatrix} f(\xi) \right) \\
&= \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_d f] \\ Xf \end{bmatrix},
\end{aligned} \tag{4.35}$$

which proves (4.32) and completes the proof of the lemma. □

Theorem 4.3. *Let $\widetilde{\mathbf{U}}$ of the form* (4.12) *be a unitary extension of the isometry* $\mathbf{V}$ *given in* (4.9). *Then the characteristic function S of the colligation $\widetilde{\mathcal{Q}} = \{\widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}}, \mathcal{E}, \mathcal{E}_*, \widetilde{\mathbf{U}}\}$,*

$$S(z) = \widetilde{D} + \widetilde{C}\left(I_{\widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}}} - Z(z)\widetilde{A}\right)^{-1} Z(z)\widetilde{B},$$

is a solution to Problem 1.2.

Proof: We fix a factorization

$$\mathbb{P}(\xi, \mu) = T(\xi)^* T(\mu), \tag{4.36}$$

where $T(\xi)$ is partitioned into blocks as in (3.15) and define the operator $\mathbf{T}$: $\mathcal{H}_0 \to \mathcal{H}$ by

$$\mathbf{T}h = \sum_{\xi} T(\xi)h(\xi). \tag{4.37}$$

Due to (4.36), the linear transformation $U:\ \operatorname{Ran}\mathbf{T} \to \widehat{\mathcal{H}}$ defined in (4.29) can be extended to the unitary map (which still is denoted by U) from $\overline{\operatorname{Ran}\mathbf{T}}$ onto $\widehat{\mathcal{H}}$. Setting

$$\mathcal{N} := (\mathcal{H}) \ominus \overline{\operatorname{Ran}\mathbf{T}} \quad \text{and} \quad \widetilde{\mathcal{H}} := \overline{\operatorname{Ran}\mathbf{T}} \oplus \mathcal{N},$$

we define the unitary map $M: \mathcal{H} \to \widetilde{\mathcal{H}}$ as in (4.30) and let $\mathbf{M}$ be the block diagonal operator defined as in (4.2). Then relations

$$M^*[Nf] = \sum_{\xi} T(\xi)N(\xi) \quad \text{and} \quad \mathbf{M}\left(\sum_{\xi} \begin{bmatrix} T(\xi)N_1(\xi) \\ \vdots \\ T(\xi)N_d(\xi) \end{bmatrix} f(\xi)\right) = \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_d f] \end{bmatrix}$$

hold by construction. Therefore, the operator

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathbf{M}^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}$$

satisfies (4.20) (or equivalently, (4.15)–(4.18)), which can be easily seen from (4.35). By Remark 4.1, the colligations $\mathbf{U}$ and $\widetilde{\mathbf{U}}$ have the same characteristic functions and thus, S can be taken in the form (1.5). Then the functions H and G from representations (1.2) and (1.3) can be taken as in (1.6). We shall use the representations (4.13) of $S(z)$ which are equivalent to (1.5).

Since $\tau(\xi) \in \mathbb{B}^d$ for every $\xi \in \Omega_R$, the operator $I - Z(\tau(\xi))A$ is boundedly invertible for every $\xi \in \Omega_R$. Rewriting (4.18) as

$$B\mathbf{b}(\xi) + AZ(\tau_j(\xi)) \begin{bmatrix} T_1(\xi) \\ \vdots \\ T_d(\xi) \end{bmatrix} = \begin{bmatrix} T_1(\xi) \\ \vdots \\ T_d(\xi) \end{bmatrix}$$

and making use of (1.6), we get

$$\begin{bmatrix} T_1(\xi) \\ \vdots \\ T_d(\xi) \end{bmatrix} = (I - AZ(\tau(\xi)))^{-1}B\mathbf{b}(\xi) = \begin{bmatrix} G_1(\tau(\xi)) \\ \vdots \\ G_d(\tau(\xi)) \end{bmatrix} \mathbf{b}(\xi), \tag{4.38}$$

which being substituted into (4.14), leads to

$$D\mathbf{b}(\xi) + CZ(\tau(\xi))(I - AZ(\tau(\xi)))^{-1}B\mathbf{b}(\xi) = \mathbf{d}(\xi) \quad (\xi \in \Omega_R),$$

which in turn, coincides with the second condition in (1.8). Moreover, by (3.8), (4.36) and (4.38)

$$\begin{aligned} \Phi_{j\ell}(\xi, \mu) &= T_j(\xi)^* T_\ell(\mu) \\ &= \mathbf{b}(\xi)^* G_j(\tau(\xi))^* G_\ell(\tau(\mu))\mathbf{b}(\mu) \quad (\xi, \mu \in \Omega_R;\ j, \ell = 1, \ldots, d), \end{aligned}$$

which coincides with (1.9).

For $\xi \in \Omega_L$, by (2.3) (with A replaced by $A^*Z(\sigma(\xi))^*$, it follows that

$$\lim_{r\to 1}(1-r)^2 \left(I - rZ(\sigma(\xi))A\right)^{-1} \left(I - Z(\sigma(\xi))AA^*Z(\sigma(\xi))^*\right) \left(I - rA^*Z(\sigma(\xi))^*\right)^{-1} = 0,$$

which is equivalent, since $AA^* + BB^* = I$ and $Z(\sigma(\xi))Z(\sigma(\xi))^* = I$, to

$$\lim_{r\to 1}(1-r)^2 \left(I - rZ(\sigma(\xi))A\right)^{-1} Z(\sigma(\xi))BB^*Z(\sigma(\xi))^* \left(I - rA^*Z(\sigma(\xi))^*\right)^{-1} = 0.$$

Therefore,

$$\lim_{r\to 1}(1-r)B^*Z(\sigma(\xi))^*\left(I-rA^*Z(\sigma(\xi))^*\right)^{-1}x=0 \quad (\text{for every } x\in\mathcal{H}). \tag{4.39}$$

Using expressions for $D^*\mathbf{a}(\xi)$ and $C^*\mathbf{a}(\xi)$ derived from (4.15) and (4.17), respectively, we get

$$\begin{aligned} S(r\sigma(\xi))^*\mathbf{a}(\xi) &= D^*\mathbf{a}(\xi)+rB^*Z(\sigma(\xi))^*\left(I-rA^*Z(\sigma(\xi))^*\right)^{-1}C^*\mathbf{a}(\xi)\\ &= \mathbf{c}(\xi)-B^*Z(\sigma(\xi))^*T_0(\xi)\\ &\quad +rB^*Z(\sigma(\xi))^*\left(I-rA^*Z(\sigma(\xi))^*\right)^{-1}\left(I-A^*Z(\sigma(\xi))^*\right)T_0(\xi)\\ &= \mathbf{c}(\xi)-(1-r)B^*Z(\sigma(\xi))^*\left(I-rA^*Z(\sigma(\xi))^*\right)^{-1}T_0(\xi). \end{aligned} \tag{4.40}$$

Taking limits in the last identity as r tends to one and taking into account (4.39), we come to the first interpolation condition in (1.8).

Making use of (1.6) and of the expression for $C^*\mathbf{a}(\xi)$ derived from (4.17) we get

$$\begin{aligned} H(r\sigma(\xi))^*\mathbf{a}(\xi) &= \left(I-A^*Z(r\sigma(\xi))^*\right)^{-1}C^*\mathbf{a}(\xi)\\ &= \left(I-rA^*Z(\sigma(\xi))^*\right)^{-1}\left(I-A^*Z(\sigma(\xi))^*\right)T_0(\xi). \end{aligned} \tag{4.41}$$

Taking limits in the last identity as r tends to one and applying Lemma 2.1 (with A replaced by $A^*Z(\sigma(\xi))^*$, we get

$$\widetilde{T}_0(\xi):=\lim_{r\to 1}H(r\sigma(\xi))^*\mathbf{a}(\xi)=\mathbf{P}_{\overline{\operatorname{Ran}(I_{\mathcal{H}}-Z(\sigma(\xi))A)}}T_0(\xi) \quad (\xi\in\Omega_L), \tag{4.42}$$

where the convergence is understood (so far) in the weak sense. By (4.40),

$$\frac{\mathbf{c}(\xi)^*\mathbf{c}(\xi)-\mathbf{c}(\xi)^*S(r\sigma(\xi))^*\mathbf{a}(\xi)}{1-r}=\mathbf{c}(\xi)^*B^*Z(\sigma(\xi))^*\left(I-rA^*Z(\sigma(\xi))^*\right)^{-1}T_0(\xi). \tag{4.43}$$

It follows from (4.20) that

$$AT_0(\xi)+B\mathbf{c}(\xi)=Z(\sigma(\xi))^*T_0(\xi)$$

and therefore,

$$\mathbf{c}(\xi)^*B^*=T_0(\xi)^*(Z(\sigma(\xi)-A^*).$$

Substituting the latter equality into (4.43) and taking into account that

$$Z(\sigma(\xi))Z(\sigma(\xi))^*=I,$$

we get

$$\begin{aligned} &\frac{\mathbf{c}(\xi)^*\mathbf{c}(\xi)-\mathbf{c}(\xi)^*S(r\sigma(\xi))^*\mathbf{a}(\xi)}{1-r}\\ &=T_0(\xi)^*(I-A^*Z(\sigma(\xi)^*)\left(I-rA^*Z(\sigma(\xi))^*\right)^{-1}T_0(\xi). \end{aligned}$$

Taking (weak) limits in the last identity as r tends to one and applying Lemma 2.1 (with A replaced by $A^*Z(\sigma(\xi))^*$, we get

$$\lim_{r\to 1}\frac{\mathbf{c}(\xi)^*\mathbf{c}(\xi)-\mathbf{c}(\xi)^*S(r\sigma(\xi))^*\mathbf{a}(\xi)}{1-r}=T_0(\xi)^*\mathbf{P}_{\overline{\operatorname{Ran}(I_{\mathcal{H}}-Z(\sigma(\xi))A)}}T_0(\xi) \quad (\xi\in\Omega_L).$$

Existence of the last limit together with (1.12) and the first equality in (1.8) imply, by Lemma 2.3, the existence of the weak limit

$$\lim_{r\to 1} \mathbf{a}(\xi)^* \frac{I_{\mathcal{E}_*} - S(r\sigma(\xi))S(r\sigma(\xi))^*}{1-r^2}\mathbf{a}(\xi),$$

the strong convergence in (4.42), and the equality

$$\lim_{r\to 1} \mathbf{a}(\xi)^* \frac{I_{\mathcal{E}_*} - S(r\sigma(\xi))S(r\sigma(\xi))^*}{1-r^2}\mathbf{a}(\xi) = \widetilde{T}_0(\xi)^*\widetilde{T}_0(\xi) \quad (\xi \in \Omega_L).$$

By (4.36), (4.42), (3.8) and (3.1), it follows that for every $\xi \in \Omega_0$,

$$\begin{aligned}
\lim_{r\to 1} \mathbf{a}(\xi)^* \frac{I_{\mathcal{E}_*} - S(r\sigma(\xi))S(r\sigma(\xi))^*}{1-r^2}\mathbf{a}(\xi) &= \widetilde{T}_0(\xi)^*\widetilde{T}_0(\xi) \\
&= T_0(\xi)^*\mathbf{P}_{\overline{\mathrm{Ran}(I_{\mathcal{H}} - Z(\sigma(\xi))A)}}T_0(\xi) \\
&\leq T_0(\xi)^*T_0(\xi) = \Lambda(\xi,\xi) = \Psi(\xi),
\end{aligned}$$

which proves (1.11). Finally, if $\xi \in \Omega\backslash\Omega_0$, then $Z(\sigma(\xi))A$ is strictly contractive and it follows from (4.42) that $\widetilde{T}_0 = H(\sigma(\xi))^*\mathbf{a}(\xi) = T_0(\xi)$. By (4.36), (3.8) and (4.38), we have for $\xi \in \Omega_L\backslash\Omega_b$ and $\mu \in \Omega_R$,

$$\mathbf{a}(\xi)^*H(\sigma(\xi))G_j(\tau(\mu))\mathbf{b}(\mu) = T_0(\xi)^*T_j(\mu) = \Lambda_j(\xi,\mu)$$

which proves (1.10) and completes the proof of theorem. □

5. The universal unitary colligation associated with the interpolation problem

A general result of Arov and Grossman (see [8], [9]) describes how to parametrize the set of all unitary extensions of a given partially defined isometry $\mathbf{V}$. Their result has been extended to the multivariable case in [16] and can be applied to the present setting.

Let $\mathbf{V}: \mathcal{D}_{\mathbf{V}} \to \mathcal{R}_{\mathbf{V}}$ be the isometry given in (4.9) with $\mathcal{D}_{\mathbf{V}}$ and $\mathcal{R}_{\mathbf{V}}$ given in (4.10) and (4.11). Introducing the defect spaces

$$\Delta = \begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E} \end{bmatrix} \ominus \mathcal{D}_{\mathbf{V}} \quad \text{and} \quad \Delta_* = \begin{bmatrix} \oplus_1^d \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix} \ominus \mathcal{R}_{\mathbf{V}}$$

and let $\widetilde{\Delta}$ be another copy of Δ and $\widetilde{\Delta}_*$ be another copy of Δ_* with unitary identification maps

$$i: \Delta \to \widetilde{\Delta} \quad \text{and} \quad i_*: \Delta_* \to \widetilde{\Delta}_*.$$

Define a unitary operator $\mathbf{U}_0$ from $\mathcal{D}_{\mathbf{V}} \oplus \Delta \oplus \widetilde{\Delta}_*$ onto $\mathcal{R}_{\mathbf{V}} \oplus \Delta_* \oplus \widetilde{\Delta}$ by the rule

$$\mathbf{U}_0 x = \begin{cases} \mathbf{V}x, & \text{if } x \in \mathcal{D}_{\mathbf{V}} \\ i(x) & \text{if } x \in \Delta, \\ i_*^{-1}(x) & \text{if } x \in \widetilde{\Delta}_*. \end{cases} \tag{5.1}$$

Identifying $\begin{bmatrix} \mathcal{D}_{\mathbf{V}} \\ \Delta \end{bmatrix}$ with $\begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E} \end{bmatrix}$ and $\begin{bmatrix} \mathcal{R}_{\mathbf{V}} \\ \Delta_* \end{bmatrix}$ with $\begin{bmatrix} \oplus_1^d \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix}$, we decompose $\mathbf{U}_0$ defined by (5.1) according to

$$\mathbf{U}_0 = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & 0 \end{bmatrix} : \quad \begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E} \\ \widetilde{\Delta}_* \end{bmatrix} \to \begin{bmatrix} \oplus_1^d \widehat{\mathcal{H}} \\ \mathcal{E}_* \\ \widetilde{\Delta} \end{bmatrix}.$$

The "33" block in this decomposition is zero, since (by definition (5.1)), for every $x \in \widetilde{\Delta}_*$, the vector $\mathbf{U}_0 x$ belongs to Δ, which is a subspace of $\begin{bmatrix} \oplus_1^d \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix}$ and therefore, is orthogonal to $\widetilde{\Delta}$ (in other words $\mathbf{P}_{\widetilde{\Delta}} \mathbf{U}_0|_{\widetilde{\Delta}_*} = 0$, where $\mathbf{P}_{\widetilde{\Delta}}$ stands for the orthogonal projection of $\mathcal{R}_{\mathbf{V}} \oplus \Delta_* \oplus \widetilde{\Delta}$ onto $\widetilde{\Delta}$).

The unitary operator $\mathbf{U}_0$ is the connecting operator of the unitary colligation

$$\mathcal{Q}_0 = \left\{ \widehat{\mathcal{H}}, \begin{bmatrix} \mathcal{E} \\ \widetilde{\Delta}_* \end{bmatrix}, \begin{bmatrix} \mathcal{E}_* \\ \widetilde{\Delta} \end{bmatrix}, \mathbf{U}_0 \right\}, \tag{5.2}$$

which is called *the universal unitary colligation* associated with the interpolation problem. According to (4.3), the characteristic function of this colligation is given by

$$\begin{aligned} \Sigma(z) &= \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} \\ &= \begin{bmatrix} U_{22} & U_{23} \\ U_{32} & 0 \end{bmatrix} + \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} \left(I_n - Z(z)U_{11}\right)^{-1} Z(z) \begin{bmatrix} U_{12} & U_{13} \end{bmatrix} \end{aligned} \tag{5.3}$$

and belongs to the class $\mathcal{S}_d(\mathcal{E} \oplus \widetilde{\Delta}_*, \ \mathcal{E}_* \oplus \widetilde{\Delta})$, by Theorem 1.1.

Theorem 5.1. *Let* $\mathbf{V}$ *be the isometry defined in* (4.9)*, let* Σ *be the function constructed as above and let* S *be a* $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$*-valued function. Then the following are equivalent:*

1. *S is a characteristic function of a colligation*

$$\Omega = \{\mathbb{C}^n \oplus \mathcal{H}, \ \mathcal{E}, \ \mathcal{E}_*, \begin{bmatrix} A & B \\ C & D \end{bmatrix}\}$$

   with the connecting operator being a unitary extension of $\mathbf{V}$.
2. *S is of the form*

$$S(z) = \Sigma_{11}(z) + \Sigma_{12}(z)\mathcal{T}(z) \left(I_{\mathcal{E}_* \oplus \widetilde{\Delta}} - \Sigma_{22}(z)\mathcal{T}(z) \right)^{-1} \Sigma_{21}(z) \tag{5.4}$$

   where $\mathcal{T}$ is a function from the class $\mathcal{S}_d(\mathcal{E} \oplus \widetilde{\Delta}_*, \ \mathcal{E}_* \oplus \widetilde{\Delta})$.

This result (which has been proved in [16] for a more general setting) together with Theorems 4.2 and 4.3 leads to a description of all solutions of Problem 1.2.

As a corollary we obtain the sufficiency part of Theorem 3.2: under assumptions (3.9) and (3.10) the set of all functions parametrized by formula (5.4) is nonempty.

Remark 5.2. Formula (5.4) with appropriately chosen coefficients Σ_{ij} parametrizes the set of all solutions of Problem 1.3 and 1.4.

6. Applications

6.1. Leech's Theorem

In conclusion we present some more corollaries. The functions $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are assumed to be defined on $\mathbb{B}^d$.

Theorem 6.1. *There is a function $S \in \mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ such that*

$$\mathbf{A}(z)S(z) \equiv \mathbf{C}(z) \tag{6.1}$$

if and only if the kernel

$$\Lambda(z, w) = \frac{\mathbf{A}(z)\mathbf{A}(w)^* - \mathbf{C}(z)\mathbf{C}(w)^*}{1 - \langle z, w\rangle} \tag{6.2}$$

is positive on $\mathbb{B}^d$.

For the proof it is enough to apply Theorem 3.2 for

$$\Omega_L = \mathbb{B}^d, \quad \Omega_R = \emptyset \quad \sigma(z) = \tau(z) \equiv z$$

and $\mathbf{b} = \mathbf{d} = 0$, $\mathbf{a}(z) = \mathbf{A}(z)^*$, $\mathbf{c}(z) = \mathbf{C}(z)^*$, $\Lambda_j = 0$, $\Phi_{\ell,j} = 0$.

Several remarks are in order. First we note that in the last theorem the functions $\mathbf{A}$ and $\mathbf{B}$ are not assumed to be analytic. Under the assumption that these functions are analytic and matrix–valued, the result appears in [4], where it is obtained as a direct consequence of the fact that the kernel $\frac{1}{1-z\bar{w}}$ is a complete Nevanlinna–Pick kernel (for this independently interesting topic we refer to [2], [16], [30], [32]). Note also that in the one variable formulation, Theorem 6.1 was obtained in [28]. Under the assumption that $\mathbf{A}$ and $\mathbf{B}$ are analytic, the one-variable result is known as Leech's theorem and becomes an easy but elegant consequence of the commutant lifting theorem [35, p.107].

The preceding analysis allows us to move further and to solve the following

Problem 6.2. Given functions $\mathbf{A}$ and $\mathbf{C}$ on $\mathbb{B}^d$ such that the kernel (6.2) is positive on $\mathbb{B}^d$, find all functions $S \in \mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ giving factorization (6.1).

Indeed, the above setting includes Problem 6.2 in the general scheme of Problem 1.2 and thus, the set of all solutions is parametrized in terms of a Redheffer transform as in Theorem 5.1.

Note also that [28] presents a two-sided one-variable version of the Theorem 6.1, which is not so nice for the multivariable case. Nevertheless, below we give a right-sided and a two-sided multivariable versions of Theorem 6.1 (see Theorems 6.3 and 6.5 below), which are also consequences of Theorem 3.2.

Theorem 6.3. *There is a function $S \in \mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ such that*

$$S(z)\mathbf{B}(z) \equiv \mathbf{D}(z) \tag{6.3}$$

if and only if there exists a positive kernel

$$\Phi(z, w) = [\Phi_{j\ell}(z, w)]_{j,\ell=1}^d \tag{6.4}$$

on $\mathbb{B}^d$, such that

$$\sum_{j,\ell=1}^d z_j \bar{w}_\ell \Phi_{j\ell}(z, w) - \sum_{j=1}^d \Phi_{jj}(z, w) = \mathbf{D}(z)\mathbf{D}(w)^* - \mathbf{B}(z)\mathbf{B}(w)^*. \tag{6.5}$$

Note that Theorem 5.1 parametrizes the set of all solutions of the following

Problem 6.4. Given functions $\mathbf{B}$ and $\mathbf{D}$ on $\mathbb{B}^d$ and given a positive kernel $\Phi(z, w)$ of the form (6.4) subject to (6.5), find all functions $\in \mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ giving factorization (6.3).

Theorem 6.5. *There is a function $S \in \mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ subject to* (6.1) *and* (6.3) *if and only if there exist a positive kernel $\Phi(z, w)$ of the form* (6.4) *subject to* (6.5) *and kernels $\Lambda_1, \ldots, \Lambda_d$ subject to*

$$\sum_{j=1}^d (z_j - \bar{w}_j)\Lambda_j(z, w) = \mathbf{C}(z)\mathbf{B}(w)^* - \mathbf{A}(z)\mathbf{D}(w)^*,$$

such that the kernel

$$\mathbb{P}(z, w) = \begin{bmatrix} \Lambda(z, w) & \Lambda_1(z, w) & \cdots & \Lambda_d(z, w) \\ \Lambda_1(z, w)^* & \Phi_{11}(z, w) & \cdots & \Phi_{1d}(z, w) \\ \vdots & \vdots & & \vdots \\ \Lambda_d(z, w)^* & \Phi_{d1}(z, w) & \cdots & \Phi_{dd}(z, w) \end{bmatrix}$$

is positive on $\mathbb{B}^d$ (Λ is the kernel given by (6.2)*).*

6.2. Tangential interpolation on the Arveson space

Let k_d be the kernel given by (1.1), let $\mathcal{H}(k_d)$ be the corresponding reproducing kernel Hilbert space. In this section we apply the preceding analysis to the operator-valued $\mathcal{H}(k_d)$-functions. To be more precise, let $\mathcal{E}$ and $\mathcal{E}_*$ be two Hilbert spaces and let $\mathcal{H}(k_d, \mathcal{E}, \mathcal{E}_*)$ denote the space of $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued functions $F(z)$ such that the function $z \to \langle F(z)x,\ y\rangle_{\mathcal{E}_*}$ belongs to $\mathcal{H}(k_d)(= \mathcal{H}(k_d, \mathbb{C}, \mathbb{C}))$ for every choice of $x \in \mathcal{E}$ and $y \in \mathcal{E}_*$. The space $\mathcal{H}(k_d, \mathcal{E}, \mathcal{E}_*)$ can (and will) be identified with the tensor product Hilbert space $\mathcal{H}(k_d) \otimes \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$. For multiindicies $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ we shall use the standard notations

$$n_1 + n_2 + \ldots + n_d = |\mathbf{n}|, \qquad n_1!n_2!\ldots n_d! = \mathbf{n}!, \qquad z_1^{n_1} z_2^{n_2} \ldots z_d^{n_d} = z^{\mathbf{n}}.$$

It can be shown (see, e.g., [10, Lemma 3.8]) that in the metric of $\mathcal{H}(k_d)$,

$$\langle z^{\mathbf{n}},\, z^{\mathbf{m}}\rangle_{\mathcal{H}(k_d)} = \begin{cases} \dfrac{\mathbf{n}!}{|\mathbf{n}|!} & \text{if} \quad \mathbf{n}=\mathbf{m} \\ 0 & \quad\text{otherwise} \end{cases} \tag{6.6}$$

which enables us to characterize $\mathcal{H}(k_d,\mathcal{E},\mathcal{E}_*)$ as

$$\mathcal{H}(k_d,\mathcal{E},\mathcal{E}_*) = \left\{ F(z) = \sum_{\mathbf{n}\in\mathbb{N}^d} F_{\mathbf{n}} z^{\mathbf{n}} :\; F_{\mathbf{n}} \in \mathcal{L}(\mathcal{E},\,\mathcal{E}_*) \text{ and } \sum_{\mathbf{n}\in\mathbb{N}^d} \frac{\mathbf{n}!}{|\mathbf{n}|!} F_{\mathbf{n}}^* F_{\mathbf{n}} \in \mathcal{L}(\mathcal{E}) \right\}.$$

The next step is to introduce the operator–valued sesquilinear form

$$[X,\, Y]_{\mathcal{H}(k_d)} = \sum_{\mathbf{n}\in\mathbb{N}^d} \frac{\mathbf{n}!}{|\mathbf{n}|!} Y_{\mathbf{n}}^* X_{\mathbf{n}}, \tag{6.7}$$

which makes sense and is $\mathcal{L}(\mathcal{E}_1,\,\mathcal{E}_2)$-valued for every choice of

$$Y(z) = \sum_{\mathbf{n}\in\mathbb{N}^d} Y_{\mathbf{n}} z^{\mathbf{n}} \in \mathcal{H}(k_d,\mathcal{E}_1,\mathcal{E}) \quad \text{and} \quad X(z) = \sum_{\mathbf{n}\in\mathbb{N}^d} X_{\mathbf{n}} z^{\mathbf{n}} \in \mathcal{H}(k_d,\mathcal{E}_2,\mathcal{E}).$$

Similarly to classical $\mathbf{H}^2$ functions of the unit disk, the functions $F \in \mathcal{H}(k_d,\mathcal{E},\mathcal{E}_*)$ can be characterized as functions for which the multiplication operator $\mathbf{M}_F : \mathcal{E} \to \mathcal{H}(k_d,\mathbb{C},\mathcal{E}_*)$ defined by the rule

$$\mathbf{M}_F\, x = F(z)x \quad (x \in \mathcal{E}) \tag{6.8}$$

is bounded. Let us denote by $\mathcal{B}_d(\mathcal{E},\,\mathcal{E}_*)$ the set of all contractive multipliers between $\mathcal{E}$ and $\mathcal{H}(k_d,\mathbb{C},\mathcal{E}_*)$, i.e., the set of all functions $F \in \mathcal{H}(k_d,\mathcal{E},\mathcal{E}_*)$ for which the corresponding multiplication operator $\mathbf{M}_F$ is a contraction. This set is characterized in terms of the form (6.7) as

$$\mathcal{B}_d(\mathcal{E},\,\mathcal{E}_*) = \left\{ F \in \mathcal{H}(k_d,\mathcal{E},\mathcal{E}_*) :\;\; [F,\, F]_{\mathcal{H}(k_d)} \le I_{\mathcal{E}_*} \right\} \tag{6.9}$$

We consider the following tangential interpolation problem in the class $\mathcal{B}_d(\mathcal{E},\,\mathcal{E}_*)$:

Problem 6.6. Given a set Ω and a function $\sigma = (\sigma_1,\dots,\sigma_d) :\; \Omega \to \mathbb{B}^d$, given functions $\mathbf{h}$ and $\mathbf{g}$, which are respectively, $\mathcal{L}(\mathcal{E}_L,\mathcal{E}_*)$ and $\mathcal{L}(\mathcal{E}_L,\mathcal{E})$-valued, find all functions $F \in \mathcal{B}_d(\mathcal{E},\,\mathcal{E}_*)$ such that

$$F(\sigma(\xi))^*\mathbf{h}(\xi) = \mathbf{g}(\xi). \tag{6.10}$$

The next two lemmas will allow us to reduce Problem 6.6 to a tangential problem for Schur functions.

Lemma 6.7. *The following are equivalent:*

1. *F belongs to $\mathcal{B}_d(\mathcal{E},\,\mathcal{E}_*)$.*
2. *The kernel K_F defined below is positive on $\mathbb{B}^d$:*

$$K_F(z,w) = \frac{I_{\mathcal{E}_*}}{1-\langle z,\, w\rangle} - F(z)F(w)^* \succeq 0 \quad (z,\, w \in \mathbb{B}^d). \tag{6.11}$$

3. *F admits a representation*

$$F(z) = S_0(z)\left(I_{\mathcal{E}} - Z(z)S_1(z)\right)^{-1} \tag{6.12}$$

for some Schur function $S \in \mathcal{S}_d(\mathcal{E},\, \mathcal{E}_* \oplus (\oplus_1^d \mathcal{E}))$:

$$S(z) = \begin{bmatrix} S_0(z) \\ S_1(z) \end{bmatrix} : \ \mathcal{E} \to \begin{bmatrix} \mathcal{E}^* \\ \oplus_1^d \mathcal{E} \end{bmatrix}, \quad Z(z) = \begin{bmatrix} z_1 I_{\mathcal{E}} & \dots & z_d I_{\mathcal{E}} \end{bmatrix}. \tag{6.13}$$

Proof: The equivalence (**1** $\Leftrightarrow$ **2**) follows from a more general fact that F is a contractive multiplier between two reproducing kernel Hilbert spaces $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ of functions analytic on a set Ω if and only if the kernel

$$K_2(z,w) - F(z)K_1(z,w)F(w)^*$$

is positive on Ω. To show that (**2** $\Leftrightarrow$ **3**), we represent K_F as

$$K_F(z,w) = \frac{I_{\mathcal{E}} - F(z)F(w)^* + F(z)Z(z)Z(w)^*F(w)^*}{1 - \langle z,\, w\rangle},$$

or, equivalently, as

$$K_F(z,w) = \frac{\mathbf{A}(z)\mathbf{A}(w)^* - \mathbf{B}(z)\mathbf{B}(w)^*}{1 - \langle z,\, w\rangle},$$

where

$$\mathbf{A}(z) = \begin{bmatrix} I_{\mathcal{E}} & F(z)Z(z) \end{bmatrix} \quad \text{and} \quad \mathbf{B}(z) = F(z).$$

By Theorem 6.1, K_F is positive on $\mathbb{B}^d$ if and only if there is a Schur function S as in (6.13), so that

$$F(z) = \mathbf{B}(z) = \mathbf{A}(z)S(z) = \begin{bmatrix} I_{\mathcal{E}} & F(z)Z(z) \end{bmatrix} \begin{bmatrix} S_0(z) \\ S_1(z) \end{bmatrix} = S_0(z) + F(z)Z(z)S_1(z). \tag{6.14}$$

It remains to note that for $z \in \mathbb{B}^d$,

$$\|Z(z)S_1(z)\| < \|S_1(z)\| \le 1$$

and therefore, the operator $I_{\mathcal{E}} - Z(z)S_1(z)$ is boundedly invertible at every point $z \in \mathbb{B}^d$. Therefore, the representation (6.12) is equivalent to (6.14). $\square$

Lemma 6.8. *Let F be in $\mathcal{B}_d(\mathcal{E},\, \mathcal{E}_*)$ and admit a representation* (6.12) *for some Schur function S of the form* (6.13). *Then F satisfies the interpolation condition* (6.10) *if and only if S is subject to*

$$S(\sigma(\xi))^* \begin{bmatrix} \mathbf{h}(\xi) \\ Z(\sigma(\xi))^*\mathbf{g}(\xi) \end{bmatrix} = \mathbf{g}(\xi). \tag{6.15}$$

Proof: Let F be of the form (6.13). Then (6.10) takes the form

$$(I_{\mathcal{E}} - S_1(\sigma(\xi))^*Z(\sigma(\xi))^*)^{-1}S_0(\sigma(\xi))^*\mathbf{h}(\xi) = \mathbf{g}(\xi),$$

which is equivalent to

$$S_0(\sigma(\xi))^*\mathbf{h}(\xi) = (I_{\mathcal{E}} - S_1(\sigma(\xi))^*Z(\sigma(\xi))^*)\mathbf{g}(\xi),$$

that is, to

$$\begin{bmatrix} S_0(\sigma(\xi))^* & S_1(\sigma(\xi))^* \end{bmatrix} \begin{bmatrix} \mathbf{h}(\xi) \\ Z(\sigma(\xi))^*\mathbf{g}(\xi) \end{bmatrix} = \mathbf{g}(\xi).$$

The last equality coincides with (6.15), in view of (6.13). □

The next theorem parametrizes the set of all solutions of Problem 6.6 in terms of a Redheffer transform. In contrast to Problem 1.2, the transfer function of this transformation is not a Schur function anymore, but nevertheless, it has some special properties.

Theorem 6.9. *The set of all solutions F of Problem* 6.6 *is parametrized by the linear fractional transformation*

$$F(z) = \Theta_{11}(z) + \Theta_{12}(z)\left(I - \mathcal{T}(z)\Theta_{22}(z)\right)^{-1}\mathcal{T}(z)\Theta_{21}(z) \tag{6.16}$$

with the transfer function

$$\Theta(z) = \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix} : \begin{bmatrix} \mathcal{E} \\ \mathcal{E}_* \oplus (\oplus_1^d \mathcal{E}) \oplus \widetilde{\Delta} \end{bmatrix} \to \begin{bmatrix} \mathcal{E}_* \\ \mathcal{E} \oplus \widetilde{\Delta}_* \end{bmatrix} \tag{6.17}$$

and the parameter $\mathcal{T}$ from the class $\mathcal{S}_d(\mathcal{E}\oplus\widetilde{\Delta}_*,\ \mathcal{E}_*\oplus(\oplus_1^d\mathcal{E})\oplus\widetilde{\Delta})$ for some auxiliary Hilbert spaces $\widetilde{\Delta}$ and $\widetilde{\Delta}_*$. Moreover,

$$\begin{bmatrix} \Theta_{11}(z) \\ \Theta_{21}(z) \end{bmatrix} \in \mathcal{B}_d(\mathcal{E}, \begin{bmatrix} \mathcal{E}_* \\ \mathcal{E} \oplus \widetilde{\Delta}_* \end{bmatrix}) \quad \text{and} \quad \begin{bmatrix} \Theta_{12}(z) \\ \Theta_{22}(z) \end{bmatrix} \in \mathcal{S}_d(\mathcal{E}_* \oplus (\oplus_1^d \mathcal{E}) \oplus \widetilde{\Delta}, \begin{bmatrix} \mathcal{E}_* \\ \mathcal{E} \oplus \widetilde{\Delta}_* \end{bmatrix}). \tag{6.18}$$

Proof: By Theorem 5.1 the set of all Schur functions S of the form (6.13) satisfying the interpolation condition (6.15) is parametrized by the linear fractional transformation

$$S(z) = \begin{bmatrix} S_0(z) \\ S_1(z) \end{bmatrix} = \begin{bmatrix} \Sigma_{11}^0(z) \\ \Sigma_{11}^1(z) \end{bmatrix} + \begin{bmatrix} \Sigma_{12}^0(z) \\ \Sigma_{12}^1(z) \end{bmatrix} \left(I - \mathcal{T}(z)\Sigma_{22}(z)\right)^{-1}\mathcal{T}(z)\Sigma_{21}(z) \tag{6.19}$$

with the transfer function

$$\Sigma(z) = \begin{bmatrix} \Sigma_{11}^0(z) & \Sigma_{12}^0(z) \\ \Sigma_{11}^1(z) & \Sigma_{12}^1(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} : \begin{bmatrix} \mathcal{E} \\ \mathcal{E}_* \oplus (\oplus_1^d \mathcal{E}) \oplus \widetilde{\Delta} \end{bmatrix} \to \begin{bmatrix} \mathcal{E}_* \\ \oplus_1^d \mathcal{E} \\ \mathcal{E} \oplus \widetilde{\Delta}_* \end{bmatrix} \tag{6.20}$$

in the Schur class $\mathcal{S}_d(\mathcal{E}\oplus\mathcal{E}_*\oplus(\oplus_1^d\mathcal{E})\oplus\widetilde{\Delta},\ \mathcal{E}_*\oplus(\oplus_1^d\mathcal{E})\oplus\mathcal{E}\oplus\widetilde{\Delta}_*)$ and the parameter $\mathcal{T}$ from the class $\mathcal{S}_d(\mathcal{E}\oplus\widetilde{\Delta}_*,\ \mathcal{E}_*\oplus(\oplus_1^d\mathcal{E})\oplus\widetilde{\Delta})$, where $\widetilde{\Delta}$ and $\widetilde{\Delta}_*$ are auxiliary Hilbert spaces isomorphic to the defect spaces of the isometry $\mathbf{V}$ associated to the interpolation problem (6.15).

Parametrization (6.19) splits into

$$S_\ell(z) = \Sigma_{11}^\ell(z) + \Sigma_{12}^\ell(z)\left(I - \mathcal{T}(z)\Sigma_{22}(z)\right)^{-1}\mathcal{T}(z)\Sigma_{21}(z) \quad (\ell = 0, 1)$$

which, being substituted into representation (6.12), leads to

$$F = \left[\Sigma_{11}^0 + \Sigma_{12}^0 \left(I - \mathcal{T}\Sigma_{22}\right)^{-1} \mathcal{T}\Sigma_{21}\right] \left[I - Z(\Sigma_{11}^1 + \Sigma_{12}^1 \left(I - \mathcal{T}\Sigma_{22}\right)^{-1} \mathcal{T}\Sigma_{21})\right]^{-1}. \tag{6.21}$$

By Lemma 6.8, the last formula describes all the solutions FF of Problem 6.6. It remains to show that (6.21) can be rewritten in the form (6.16).

Since the block Σ_{11}^1 is a Schur function, the operator $I - Z(z)\Sigma_{11}^1(z)$ is boundedly invertible at every point $z \in \mathbb{B}^d$, which allows us to introduce functions

$$\begin{bmatrix}\Theta_{11}(z)\\ \Theta_{21}(z)\end{bmatrix} = \begin{bmatrix}\Sigma_{11}^0(z)\\ \Sigma_{21}(z)\end{bmatrix} \left(I - Z(z)\Sigma_{11}^1(z)\right)^{-1}, \tag{6.22}$$

$$\begin{bmatrix}\Theta_{12}(z)\\ \Theta_{22}(z)\end{bmatrix} = \begin{bmatrix}\Sigma_{12}^0(z)\\ \Sigma_{22}(z)\end{bmatrix} + \begin{bmatrix}\Sigma_{11}^0(z)\\ \Sigma_{21}(z)\end{bmatrix} \left(I - Z(z)\Sigma_{11}^1(z)\right)^{-1} Z(z)\Sigma_{12}^1(z). \tag{6.23}$$

It is easily verified by a straightforward computation that

$$\begin{aligned}&\left[I - Z\Sigma_{11}^1 - Z\Sigma_{12}^1 \left(I - \mathcal{T}\Sigma_{22}\right)^{-1} \mathcal{T}\Sigma_{21}\right]^{-1}\\ &= (I - Z\Sigma_{11}^1)^{-1} \left(I + Z\Sigma_{12}^1 \left(I - \mathcal{T}\Theta_{22}\right)^{-1} \mathcal{T}\Theta_{21}\right)\end{aligned} \tag{6.24}$$

and that

$$\begin{aligned}&\left(I - \mathcal{T}\Sigma_{22}\right)^{-1} \mathcal{T}\Sigma_{21}(I - Z\Sigma_{11}^1)^{-1} Z\Sigma_{12}^1 \left(I - \mathcal{T}\Theta_{22}\right)^{-1}\\ &= \left(I - \mathcal{T}\Theta_{22}\right)^{-1} - \left(I - \mathcal{T}\Sigma_{22}\right)^{-1}.\end{aligned} \tag{6.25}$$

Substituting (6.24) into (6.21), making use of (6.25) and taking into account (6.22) and (6.23), we get

$$\begin{aligned} F &= \left[\Sigma_{11}^0 + \Sigma_{12}^0 \left(I - \mathcal{T}\Sigma_{22}\right)^{-1} \mathcal{T}\Sigma_{21}\right] (I - Z\Sigma_{11}^1)^{-1}\\ &\quad + \Sigma_{11}^0 (I - Z\Sigma_{11}^1)^{-1} Z\Sigma_{12}^1 \left(I - \mathcal{T}\Theta_{22}\right)^{-1} \mathcal{T}\Theta_{21}\\ &\quad + \Sigma_{12}^0 \left[\left(I - \mathcal{T}\Theta_{22}\right)^{-1} - \left(I - \mathcal{T}\Sigma_{22}\right)^{-1}\right] \mathcal{T}\Theta_{21}\\ &= \Sigma_{11}^0 (I - Z\Sigma_{11}^1)^{-1} + \Sigma_{11}^0 (I - Z\Sigma_{11}^1)^{-1} Z\Sigma_{12}^1 \left(I - \mathcal{T}\Theta_{22}\right)^{-1} \mathcal{T}\Theta_{21}\\ &\quad + \Sigma_{12}^0 \left(I - \mathcal{T}\Theta_{22}\right)^{-1} \mathcal{T}\Theta_{21}\\ &= \Theta_{11} + \Theta_{12} \left(I - \mathcal{T}\Theta_{22}\right)^{-1} \mathcal{T}\Theta_{21}, \end{aligned}$$

which coincides with (6.16). Furthermore, it follows from the block decomposition (6.20) of Σ that the function $\begin{bmatrix}\Sigma_{11}^0(z)\\ \Sigma_{21}(z)\\ \Sigma_{11}^1(z)\end{bmatrix}$ is a Schur function. Applying the assertion (**3** $\Rightarrow$ **1**) in Lemma 6.7 to the functions $S_0 = \begin{bmatrix}\Sigma_{11}^0\\ \Sigma_{21}\end{bmatrix}$ and $S_1 = \Sigma_{11}^1$ we conclude then that the function on the right hand side of (6.21) belongs to $\mathcal{B}_d(\mathcal{E}, \begin{bmatrix}\mathcal{E}_*\\ \mathcal{E} \oplus \widetilde{\Delta}_*\end{bmatrix})$. Finally, the function on the right hand side of (6.22) can be considered as the

Redheffer transform of the Schur function $Z(z)$. It belongs to the Schur class, since the transfer function of this transform,

$$\begin{bmatrix} \Sigma_{12}^0(z) & \Sigma_{11}^0(z) \\ \Sigma_{22}(z) & \Sigma_{21}(z) \\ \Sigma_{12}^1(z) & \Sigma_{11}^1(z) \end{bmatrix}$$

belongs to the Schur class, which follows immediately from (6.20). □

7. Appendix: a numerical example

In conclusion we illustrate the general construction done in Sections 4 and 5 by a simple numerical example. Let us consider the following boundary interpolation problem: *Given to points* $\beta_1 = (1, 0)$ *and* $\beta_2 = (0, 1)$ *on the unit sphere, find all scalar functions* $S \in \mathcal{S}_2(\mathbb{C}, \mathbb{C})$ *such that*

$$\lim_{r\to 1} S(r\beta_1) = 1, \quad \lim_{r\to 1} S(r\beta_2) = -1 \tag{7.1}$$

and

$$\lim_{r\to 1} \frac{1 - |S(r\beta_1)|^2}{1 - r^2} \le 1, \quad \lim_{r\to 1} \frac{1 - |S(r\beta_1)|^2}{1 - r^2} \le 5. \tag{7.2}$$

Thus $d = 2$, $\Omega_b = \{\beta_1, \beta_2\}$, $\mathbf{a}(\beta_1) = \mathbf{a}(\beta_2) = 1$, $\mathbf{c}(\beta_1) = 1$, $\mathbf{c}(\beta_2) = -1$, $\Psi(\beta_1) = 1$ and $\Psi(\beta_2) = 5$. Furthermore,

$$\mathbb{P} = \begin{bmatrix} \Psi(\beta_1) & \dfrac{\mathbf{a}(\beta_1)\overline{\mathbf{a}(\beta_2)} - \mathbf{c}(\beta_1)\overline{\mathbf{c}(\beta_2)}}{1 - \langle \beta_1, \beta_2 \rangle} \\ \dfrac{\mathbf{a}(\beta_2)\overline{\mathbf{a}(\beta_1)} - \mathbf{c}(\beta_2)\overline{\mathbf{c}(\beta_1)}}{1 - \langle \beta_2, \beta_1 \rangle} & \Psi(\beta_2) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

In the present setting, the isometry $\mathbf{V}$ defined in (4.9) takes the form

$$\mathbf{V} : \begin{bmatrix} \mathbb{P}^{\frac{1}{2}} f \\ Y f \end{bmatrix} \to \begin{bmatrix} \mathbb{P}^{\frac{1}{2}} N_1 f \\ \mathbb{P}^{\frac{1}{2}} N_2 f \\ X f \end{bmatrix}, \quad f \in \mathbb{C}^2.$$

Note that

$$\mathbb{P}^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbb{P}^{\frac{1}{2}} N_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbb{P}^{\frac{1}{2}} N_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}.$$

Thus the matrix of the operator $\mathbf{V}$ with respect to standard bases in $\mathbb{C}^3$ and $\mathbb{C}^5$ (which still is denoted by $\mathbf{V}$) satisfies

$$\mathbf{V} \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 3 \\ \sqrt{2} & \sqrt{2} \end{bmatrix}.$$

A routine calculation shows that

$$\mathbf{V} = \frac{1}{22}\begin{bmatrix} 5 & 3 & 7\sqrt{2} \\ 5 & 3 & 7\sqrt{2} \\ 1 & 5 & -3\sqrt{2} \\ 3 & 15 & -9\sqrt{2} \\ 6\sqrt{2} & 8\sqrt{2} & 8\sqrt{2} \end{bmatrix}.$$

The next step is construct the universal unitary colligation $\mathbf{U}_0$ defined in (5.1). In the present context $\mathbf{U}_0$ is the 6×6 unitary matrix whose upper left 5×3 submatrix coincides with $\mathbf{V}$. Omitting calculations, we present $\mathbf{U}_0$ explicitly:

$$\mathbf{U}_0 = \begin{bmatrix} \frac{5}{22} & \frac{3}{22} & \frac{7\sqrt{2}}{22} & -\frac{1}{\sqrt{2}} & 0 & -\frac{\sqrt{5}}{\sqrt{22}} \\ \frac{5}{22} & \frac{3}{22} & \frac{7\sqrt{2}}{22} & \frac{1}{\sqrt{2}} & 0 & -\frac{\sqrt{5}}{\sqrt{22}} \\ \frac{1}{22} & \frac{5}{22} & -\frac{3\sqrt{2}}{22} & 0 & -\frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{110}} \\ \frac{3}{22} & \frac{15}{22} & -\frac{9\sqrt{2}}{22} & 0 & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{110}} \\ \frac{3\sqrt{2}}{11} & \frac{4\sqrt{2}}{11} & \frac{4}{11} & 0 & 0 & \frac{\sqrt{5}}{\sqrt{11}} \\ -\frac{2\sqrt{2}}{\sqrt{11}} & \frac{\sqrt{2}}{\sqrt{11}} & \frac{1}{\sqrt{11}} & 0 & 0 & 0 \end{bmatrix}$$

By (5.3), the characteristic function Σ of the unitary colligation $\mathcal{Q}_0 = \left\{\mathbb{C}^2,\ \mathbb{C}^2,\ \mathbb{C}^2,\ \mathbf{U}_0\right\}$ takes the form

$$\begin{aligned} \Sigma(z) &= \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{11} & 0 & 0 & \frac{\sqrt{5}}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} & 0 & 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} \frac{3\sqrt{2}}{11} & \frac{4\sqrt{2}}{11} \\ -\frac{2\sqrt{2}}{\sqrt{11}} & \frac{\sqrt{2}}{\sqrt{11}} \end{bmatrix} \left(I_2 - z_1 \begin{bmatrix} \frac{5}{22} & \frac{3}{22} \\ \frac{5}{22} & \frac{3}{22} \end{bmatrix} - z_2 \begin{bmatrix} \frac{1}{22} & \frac{5}{22} \\ \frac{3}{22} & \frac{15}{22} \end{bmatrix} \right)^{-1} \\ &\quad \times \left(z_1 \begin{bmatrix} \frac{7\sqrt{2}}{22} & -\frac{1}{\sqrt{2}} & 0 & -\frac{\sqrt{5}}{\sqrt{22}} \\ \frac{7\sqrt{2}}{22} & \frac{1}{\sqrt{2}} & 0 & -\frac{\sqrt{5}}{\sqrt{22}} \end{bmatrix} + z_2 \begin{bmatrix} -\frac{3\sqrt{2}}{22} & 0 & -\frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{110}} \\ -\frac{9\sqrt{2}}{22} & 0 & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{110}} \end{bmatrix} \right) \end{aligned}$$

Setting for short

$$d(z_1, z_2) = z_1 z_2 - 4z_1 - 8z_2 + 11$$

and taking into account that

$$\left(I_2 - z_1 \begin{bmatrix} \frac{5}{22} & \frac{3}{22} \\ \frac{5}{22} & \frac{3}{22} \end{bmatrix} - z_2 \begin{bmatrix} \frac{1}{22} & \frac{5}{22} \\ \frac{3}{22} & \frac{15}{22} \end{bmatrix} \right)^{-1} = \frac{\begin{bmatrix} 22 - 3z_1 - 15z_2 & 3z_1 + 5z_2 \\ 5z_1 + 3z_2 & 22 - 5z_1 - z_2 \end{bmatrix}}{2d(z_1, z_2)},$$

we get

$$\begin{aligned}\Sigma_{11}(z) &= \frac{4}{11}+\frac{\begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 22-3z_1-15z_2 & 3z_1+5z_2 \\ 5z_1+3z_2 & 22-5z_1-z_2 \end{bmatrix}\begin{bmatrix} 7z_1-3z_2 \\ 7z_1-9z_2 \end{bmatrix}}{242d(z_1,z_2)} \\ &= \frac{3z_1-7z_2+4}{z_1z_2-4z_1-8z_2+11} \end{aligned} \tag{7.3}$$

and quite similarly,

$$\begin{aligned}\Sigma_{12}(z) &= \begin{bmatrix} \frac{z_1(1-z_1+2z_2)}{d(z_1,z_2)} & \frac{z_2(5z_2-2z_1-5)}{\sqrt{5}\,d(z_1,z_2)} & \frac{\sqrt{11}(5-5z_1-5z_2+z_1z_2)}{\sqrt{5}\,d(z_1,z_2)} \end{bmatrix}, \\ \Sigma_{21}(z) &= \frac{\sqrt{11}(1-z_1)(1-z_2)}{d(z_1,z_2)}, \\ \Sigma_{22}(z) &= \begin{bmatrix} \frac{\sqrt{11}z_1(3-z_1-2z_2)}{d(z_1,z_2)} & \frac{\sqrt{11}z_2(7-2z_1-5z_2)}{\sqrt{5}\,d(z_1,z_2)} & \frac{5z_1-z_2-4z_1z_2}{\sqrt{5}\,d(z_1,z_2)} \end{bmatrix}. \end{aligned}$$

By Theorem 5.1, all functions $S \in \mathcal{S}_2(\mathbb{C},\mathbb{C})$ satisfying interpolation conditions (7.1), (7.2) are parametrized by formula (5.4) with the parameter $\mathcal{T}$ varying over the class $\mathcal{S}_2(\mathbb{C},\mathbb{C}^3)$. Choosing $\mathcal{T}$ to be a constant vector in $\mathbb{C}^3$ of the norm $\|\mathcal{T}\| \le 1$, we get via (5.4) a family of rational solutions of the problem of McMillan degree not greater than four. The choice $\mathcal{T}=0$ leads to the function $S(z)=\Sigma_{11}(z)$. This function is analytic at β_1 and β_2 and it is readily seen that

$$\Sigma_{11}(\beta_1)=\Sigma_{11}((1,0))=1 \quad \text{and} \quad \Sigma_{11}(\beta_2)=\Sigma_{11}((0,1))=-1.$$

Furthermore,

$$\lim_{r\to 1}\frac{1-|\Sigma_{11}(r\beta_1)|^2}{1-r^2}=\lim_{r\to 1}\frac{1-\left(\frac{3r+4}{11-4r}\right)^2}{1-r^2}=1$$

and

$$\lim_{r\to 1}\frac{1-|\Sigma_{11}(r\beta_2)|^2}{1-r^2}=\lim_{r\to 1}\frac{1-\left(\frac{4-7r}{11-8r}\right)^2}{1-r^2}=5.$$

Another choice of

$$\mathcal{T}=\begin{bmatrix} \frac{2\sqrt{11}}{7} \\ 0 \\ \frac{\sqrt{5}}{7} \end{bmatrix}$$

leads via (5.4) to

$$S(z)=\frac{3z_1-7z_2+4}{z_1z_2-4z_1-8z_2+11}+\frac{(1-z_1)(1-z_2)}{z_1z_2-4z_1-8z_2+11}\cdot\frac{5+5z_1z_2-2z_1^2-3z_1-5z_2}{5z_1z_2-9z_1-5z_2+2z_1^2+7}.$$

A simple computation shows that

$$S(r\beta_1)=\frac{2r+3}{7-2r}, \quad S(r\beta_2)=\frac{3-5r}{7-5r}.$$

Thus S satisfies conditions (7.1) and

$$\lim_{r\to 1}\frac{1-|S(r\beta_1)|^2}{1-r^2}=\frac{4}{5}\le 1,\quad \lim_{r\to 1}\frac{1-|S(r\beta_2)|^2}{1-r^2}=5.$$

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