ACADEMIC PRESS

J. Math. Anal. Appl. 273 (2002) 328–348

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.academicpress.com

A tangential interpolation problem on the distinguished boundary of the polydisk for the Schur–Agler class

Joseph A. Ball [a,*] and Vladimir Bolotnikov [b]

[a] *Department of Mathematics, Virginia Polytechnic Institute, Blacksburg, VA 24061-0123, USA*
[b] *Department of Mathematics, The College of William and Mary, Williamsburg, VA 23187-8795, USA*

Received 26 February 2002

Submitted by Steven Krantz

Abstract

We study a tangential interpolation problem with an arbitrary set of interpolating points on the distinguished boundary of the unit polydisk for Schur–Agler class. The description of all solutions is parametrized in terms of a linear fractional transformation.

Keywords: Boundary interpolation; Schur–Agler functions; Linear fractional parametrization

1. Introduction

In this paper we consider a boundary interpolation problem for the class of contractive valued functions in the polydisk introduced by Agler in [1]. This class, which we denote by $\mathcal{S}_d(\mathcal{E},\mathcal{E}_*)$ and call the *Schur–Agler class of the polydisk*, consists of all $\mathcal{L}(\mathcal{E},\mathcal{E}_*)$-valued functions S analytic on the d-fold polydisk $\mathbb{D}^d$:

$$\mathbb{D}^d = \left\{z = (z_1,\ldots,z_d) \in \mathbb{C}^d\colon\ |z_k| < 1\right\}$$

* Corresponding author.
E-mail address: ball@math.vt.edu (J.A. Ball).

0022-247X/02/$ – see front matter
PII: S0022-247X(02)00226-3

and such that

$$\sup_{r<1} \left\| S(rT_1, \dots, rT_d) \right\| \leqslant 1$$

for any $r < 1$ and for any d-tuple of commuting contractions $(T_1, \dots, T_d)$. In the latter relation $S(rT_1, \dots, rT_d)$ can be defined by the Cauchy integral formula

$$S(rT_1, \dots, rT_n) = \frac{1}{(2\pi i)^d} \int_{r\mathbb{T}^d} S(z) \otimes (z_1 I - T_1)^{-1} \dots (z_d I - T_d)^{-1} \, dz_1 \dots dz_d.$$

Throughout the paper $\mathcal{E}$ and $\mathcal{E}_*$ are separable Hilbert spaces and $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ denotes the set of all bounded linear operators from $\mathcal{E}$ into $\mathcal{E}_*$. It was shown in [2] that S belongs to $\mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ if and only if there exist d analytic operator-valued functions $H_k(z)$ on $\mathbb{D}^d$ with values equal to operators from an auxiliary Hilbert space $\mathcal{H}_k$ into $\mathcal{E}_*$ so that

$$I_{\mathcal{E}_*} - S(z)S(w)^* = \sum_{k=1}^{d} (1 - z_k \bar{w}_k) H_k(z) H_k(w)^* \tag{1.1}$$

for every choice of points $z = (z_1, \dots, z_d)$ and $w = (w_1, \dots, w_d)$ in $\mathbb{D}^d$. Let

$$\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_d \tag{1.2}$$

and let P_k be orthogonal projections of $\mathcal{H}$ onto $\mathcal{H}_k$. Then the operator-valued functions

$$Z(z) = z_1 P_1 + \dots + z_d P_d \quad \text{and} \quad H(z) = H_1(z)P_1 + \dots + H_d(z)P_d \tag{1.3}$$

admit the block representations

$$Z(z) = \begin{bmatrix} z_1 I_{\mathcal{H}_1} & & 0 \\ & \ddots & \\ 0 & & z_d I_{\mathcal{H}_d} \end{bmatrix} \quad \text{and} \quad H(z) = \begin{bmatrix} H_1(z) & \dots & H_d(z) \end{bmatrix} \tag{1.4}$$

with respect to the decomposition (1.2) and allow us to rewrite (1.1) in a more compact form as

$$I_{\mathcal{E}_*} - S(z)S(w)^* = H(z)\big(I_{\mathcal{H}} - Z(z)Z(w)^*\big)H(w)^*. \tag{1.5}$$

The following alternative characterization of the class $\mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ in terms of unitary d-variable colligations is given in [1] and [14].

Theorem 1.1. *A $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued function S analytic in $\mathbb{D}^d$ belongs to $\mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ if and only if there is an auxiliary Hilbert space $\mathcal{H}$ and a unitary operator*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{E}_* \end{bmatrix}$$

and a d-fold orthogonal decomposition (1.2) *of $\mathcal{H}$ such that*

$$S(z) = D + C\bigl(I_{\mathcal{H}} - Z(z)A\bigr)^{-1} Z(z)B, \tag{1.6}$$

where $Z(z)$ is given in (1.3). *For S of the form* (1.6), *the representations* (1.1) *and* (1.5) *are valid for*

$$H_k(z) = C\bigl(I - Z(z)A\bigr)^{-1} P_k \quad \textit{and} \quad H(z) = C\bigl(I - Z(z)A\bigr)^{-1}, \tag{1.7}$$

respectively.

The representation (1.6) is called *a unitary realization* of $S \in \mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$.

In this paper we study an interpolation problem on $\mathbb{T}^d$, the distinguished boundary of the $\mathbb{D}^d$. Let Ω be a set. The data set for the interpolation problem is as follows. We are given a one-to-one function

$$\sigma = (\sigma_1, \dots, \sigma_d) : \Omega \to \mathbb{T}^d,$$

along with an auxiliary Hilbert space $\mathcal{M}_L$. We are also given functions **a** and **c** on Ω, which are, respectively, $\mathcal{L}(\mathcal{M}_L, \mathcal{E}_*)$, and $\mathcal{L}(\mathcal{M}_L, \mathcal{E})$-valued. Finally, we are given a function $\Psi(\xi)$ on Ω whose values are bounded positive operators on $\mathcal{L}(\mathcal{M}_L)$.

Problem 1.2. *Find all functions $S \in \mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ such that*

$$\lim_{r\to 1} S\bigl(r\sigma(\xi)\bigr)^* \mathbf{a}(\xi) = \mathbf{c}(\xi) \quad (\xi \in \Omega) \tag{1.8}$$

and

$$\lim_{r\to 1} \mathbf{a}(\xi)^* \frac{I_{\mathcal{E}_*} - S(r\sigma(\xi))S(r\sigma(\xi))^*}{1 - r^2} \mathbf{a}(\xi) \leqslant \Psi(\xi) \quad (\xi \in \Omega), \tag{1.9}$$

where the limits in (1.8) *and* (1.9) *are understood in the strong and in the weak sense, respectively.*

Condition (1.8) is called the *left-sided* interpolation condition for S. It follows by a multivariable matrix analogue of the classical Julia–Carathéodory theorem (see Lemma 2.1 below) that if the limit in (1.8) exists and equals $\mathbf{c}(\xi)$, then the necessary condition for the limits in (1.9) to exist and to be finite is

$$\|\mathbf{a}(\xi)\|_{\mathcal{E}} = \|\mathbf{c}(\xi)\|_{\mathcal{E}_*} \quad (\xi \in \Omega). \tag{1.10}$$

It follows again by (the third assertion of) Lemma 2.1, that S satisfies also the right-sided interpolation condition

$$\lim_{r\to 1} S\bigl(r\sigma(\xi)\bigr)\mathbf{c}(\xi) = \mathbf{a}(\xi) \quad (\xi \in \Omega).$$

Thus, Problem 1.2 is in fact a two-sided interpolation problem and conditions (1.10) are necessary for this problem to have a solution.

The breakthrough result on interpolation for the Schur–Agler class was the result for scalar-valued functions with finitely many interpolation nodes in the interior of the polydisk obtained in the preprint [2]. Extensions of this result to matrix-valued functions and to tangential and bitangential problems were obtained in [3,14]. The approach of these latter papers was to identify solutions of the interpolation problem as characteristic functions of unitary colligations which are unitary extensions of a certain partially defined isometric colligation constructed explicitly from the interpolation data. A method of Arov–Grossman (see [6,7]) then leads to an elegant linear-fractional parametrization for the set of all solutions in terms of a free Schur-class parameter. A more abstract formulation of this approach (called the abstract interpolation problem (AIP)) has been developed for a variety of single-variable interpolation problems (see [22,24]); indeed, the paper [23] obtains new results on the (operator-valued) Hamburger moment problem by first performing a change of variable to convert the problem to a boundary interpolation problem and then using the AIP method to analyze this boundary interpolation problem. The existence criterion for a variety of bitangential interpolation problems for the Schur–Agler class with interior interpolation nodes can now also be deduced from the general commutant lifting theorem over the polydisk obtained in [13].

The purpose of this paper is to identify the extension of the AIP approach required to solve the boundary interpolation problem on the distinguished boundary of the polydisk for the Schur–Agler class (Problem 1.2). For the multivariable case ($d > 1$), we are aware only of the paper [28] on boundary interpolation for the Schur–Agler class; this latter paper, however, treats interpolation on disks embedded in the nondistinguished boundary of the polydisk rather than the boundary interpolation on the distinguished boundary related to Carathéodory–Julia theory, as is treated here. The existence criterion (see Theorem 2.2) is in terms of what is called an LMI (linear matrix inequality) in the engineering literature, rather than the positivity of a single Pick matrix as in the univariate case; we refer to [17,20] for a thorough discussion of LMIs and their manifold applications in engineering, and to [21] for a discussion of LMIs in the specific context of (interior) polydisk interpolation. In this context (as already exhibited for the interior interpolation problem studied in [14]), it is only particular subclasses of solutions associated with some additional interpolation constraints which have a single linear-fractional parametrization in terms of a free Schur–Agler-class parameter (see Corollary 4.2). Each such set of auxiliary interpolation conditions corresponds to a particular choice $\mathbb{P}_1, \ldots, \mathbb{P}_d$ of solution of the LMI in the existence criterion; one then must sweep through all linear fractional maps corresponding to each such $\mathbb{P}_1, \ldots, \mathbb{P}_d$ as well as through all free Schur–Agler-class parameters to arrive at the set of all solutions of Problem 1.2.

For the single-variable case ($d = 1$), boundary interpolation on the unit disk for scalar-valued functions appears already in the work of Nevanlinna [27] as well as in [4]. There have been a number of operator-theoretic treatments for boundary interpolation problems for the matrix-valued Schur class (see [9,10, 12,15,16,18,25,26]) and the problem is treated in the books [11] and [19]. We mention that the paper [29] obtains necessary and sufficient conditions for the inequality condition (1.9) to be solved with equality (for the single-variable scalar-valued case with finitely many interpolation nodes).

The paper is organized as follows. After the present Introduction, Section 2 presents necessary and sufficient conditions for Problem 1.2 to have a solution, Section 3 establishes a correspondence between the set of all solutions and unitary extensions of partially defined isometries, Section 4 presents a description of all solutions of Problem 1.2 in terms of linear fractional transformations. In Section 5 we treat the boundary Nevanlinna–Pick problem as a particular case of Problem 1.2 and present explicit formulas for coefficients of the corresponding linear transformation in terms of initial data.

2. The solvability criterion

In this section we establish the solvability criterion of Problem 1.2. First we establish some auxiliary results part of which can be considered as a multivariable operator analogue of the classical Julia–Carathéodory theorem.

Lemma 2.1. *Let* $S \in \mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$, $\beta \in \mathbb{T}^d$, $\mathbf{x} \in \mathcal{E}_*$ *and let* H_j $(j = 1, \ldots, d)$ *be* $\mathcal{L}(\mathcal{H}_j, \mathcal{E}_*)$*-valued functions from the representation* (1.1). *Then*:

(I) *The following three statements are equivalent*:

(1) *S is subject to*

$$\mathbf{L} := \sup_{0 \leqslant r < 1} \mathbf{x}^* \frac{I_{\mathcal{E}_*} - S(r\beta)S(r\beta)^*}{1 - r^2} \mathbf{x} < \infty.$$

(2) *The radial limit*

$$L := \lim_{r \to 1} \mathbf{x}^* \frac{I_{\mathcal{E}^*} - S(r\beta)S(r\beta)^*}{1 - r^2} \mathbf{x}$$

exists.

(3) *The radial limit*

$$\lim_{r \to 1} S(r\beta)^* \mathbf{x} = \mathbf{y} \tag{2.1}$$

exists in the strong sense and serves to define the vector $\mathbf{y} \in \mathcal{E}$. *Furthermore,*

$$\lim_{r \to 1} S(r\beta)\mathbf{y} = \mathbf{x}, \quad \|\mathbf{y}\| = \|\mathbf{x}\| \tag{2.2}$$

(*the limit is understood in the strong sense*), *and the radial limit*

$$\widetilde{L} = \lim_{r \to 1} \frac{\mathbf{y}^*\mathbf{y} - \mathbf{x}^* S(r\beta)\mathbf{y}}{1 - r} \tag{2.3}$$

exists.

(II) *Any two of the three equalities in* (2.1) *and* (2.2) *imply the third.*

(III) *If any of the three statements in part* (I) *holds true, then the radial limits*

$$T_j = \lim_{r\to 1} H_j(r\beta)^*\mathbf{x} \quad (j = 1, \ldots, d) \tag{2.4}$$

exist in the strong sense and

$$\sum_{j=1}^{d} T_j^* T_j = L = \widetilde{L} \leqslant \mathbf{L}. \tag{2.5}$$

Proof. For the proof of all the statements for the single-variable case ($d = 1$) see [8, Lemma 2.2] (all the statements but those related to T_j's and for finite-dimensional $\mathcal{E}$ and $\mathcal{E}_*$ are contained in [19, Lemma 8.3, Lemma 8.4, and Theo rem 8.5]). For the case $d \geqslant 2$, let us introduce the slice-functions

$$S_\beta(\zeta) := S(\beta\zeta) \quad \text{and} \quad H_\beta(\zeta) := H(\beta\zeta) \quad (\zeta \in \mathbb{D}), \tag{2.6}$$

the first of which clearly belongs to the classical Schur class $\mathcal{S}_1(\mathcal{E}, \mathcal{E}_*)$. Since $\beta \in \mathbb{T}^d$, it follows that $Z(\zeta\beta)Z(\omega\beta)^* = \zeta\bar{\omega}I_{\mathcal{H}}$ for every pair of $\zeta, \omega \in \mathbb{C}$ and thus, by (1.5),

$$I_{\mathcal{E}_*} - S_\beta(\zeta)S_\beta(\omega)^* = (1 - \zeta\bar{\omega})H_\beta(\zeta)H_\beta(\omega)^*.$$

All the statements of the lemma regard the boundary behavior of the function S_β near a boundary point $\zeta = 1$. Applying one-variable results to slice-functions S_β and H_β, returning to the original functions S and H and taking into account the block decomposition (1.4) of H, we obtain all the desired assertions. To see the third statement, note that by the one-variable result, there exists the strong limit

$$T := \lim_{r\to 1} H_\beta(r)^*\mathbf{x}, \tag{2.7}$$

which satisfies

$$T^*T = L = \widetilde{L} \leqslant \mathbf{L}. \tag{2.8}$$

Since, by (1.4) and (2.6),

$$H_\beta(r) = \big[H_1(r\beta) \ \ldots \ H_d(r\beta)\big],$$

we conclude from (2.7) that the strong limits $T_j = \lim_{r\to 1} H_j(r\beta)$ exist for $j = 1, \ldots, d$ and satisfy

$$T = \begin{bmatrix} T_1 \\ \vdots \\ T_d \end{bmatrix}, \qquad T^*T = \sum_{j=1}^{d} T_j^* T_j.$$

These identities together with (2.8) imply (2.5). $\square$

Theorem 2.2. *Problem* 1.2 *has a solution if and only if there exist* d *kernels* $\mathbb{P}_j(\xi,\mu):\Omega\times\Omega\to\mathcal{L}(\mathcal{M}_L)$ $(j=1,\dots,d)$ *such that*

$$\mathbb{P}_j(\xi,\mu)\succeq 0 \quad (\xi,\mu\in\Omega), \tag{2.9}$$

$$\sum_{j=1}^{d}\mathbb{P}_j(\xi,\xi)\leqslant\Psi(\xi) \quad (\xi,\mu\in\Omega), \tag{2.10}$$

and satisfying the generalized Stein identity

$$\sum_{j=1}^{d}\bigl(1-\sigma_j(\xi)\overline{\sigma_j(\mu)}\bigr)\mathbb{P}_j(\xi,\mu)=\mathbf{a}(\xi)^*\mathbf{a}(\mu)-\mathbf{c}(\xi)^*\mathbf{c}(\mu) \quad (\xi,\mu\in\Omega). \tag{2.11}$$

Proof. Let S be a solution of Problem 1.2, that is, let it belong to $\mathcal{S}_d(\mathcal{E},\mathcal{E}_*)$ and satisfy interpolation conditions (1.8) and (1.9). Since S belongs to $\mathcal{S}_d(\mathcal{E},\mathcal{E}_*)$, the identity (1.1) holds for some $\mathcal{L}(\mathcal{H}_j,\mathcal{E}_*)$-valued functions H_j which are analytic on $\mathbb{B}^d$. Let $T_j(\xi)$ stand for the following strong limit

$$T_j(\xi):=\lim_{r\to1}H\bigl(r\sigma(\xi)\bigr)^*\mathbf{a}(\xi) \quad (j=1,\dots,d,\ \xi\in\Omega), \tag{2.12}$$

which exists at every point $\xi\in\Omega$, by Lemma 2.1, and satisfies

$$\sum_{j=1}^{d}T_j(\xi)^*T_j(\xi)=L(\xi):=\lim_{r\to1}\mathbf{a}(\xi)^*\,\frac{I_{\mathcal{E}_*}-S(r\sigma(\xi))S(r\sigma(\xi))^*}{1-r^2}\,\mathbf{a}(\xi) \tag{2.13}$$

for $j=1,\dots,d$ and each $\xi\in\Omega$. The kernels

$$\mathbb{P}_j(\xi,\mu)=T_j(\xi)^*T_j(\mu) \quad (\xi,\mu\in\Omega) \tag{2.14}$$

are clearly positive and satisfy (2.10) by (1.9) and (2.13). Setting $z=r\sigma(\xi)$ and $w=r\sigma(\mu)$ in (1.1) and multiplying both sides in the resulting identity by $\mathbf{a}(\xi)^*$ on the left and by $\mathbf{a}(\mu)$ on the right, we get

$$\begin{aligned}&\mathbf{a}(\xi)^*\bigl(I_{\mathcal{E}_*}-S\bigl(r\sigma(\xi)\bigr)S\bigl(r\sigma(\mu)\bigr)^*\bigr)\mathbf{a}(\mu)\\&\quad=\sum_{j=1}^{d}\bigl(1-r^2\sigma(\xi)\overline{\sigma(\mu)}\bigr)\mathbf{a}(\xi)^*H_j\bigl(r\sigma(\xi)\bigr)H_j\bigl(r\sigma(\mu)\bigr)^*\mathbf{a}(\mu).\end{aligned}$$

Taking the limit as $r\to1$ in the last identity and making use of (1.8), (2.12) and (2.14), we come to (2.11), which completes the proof of the necessity part of the theorem. The proof of the sufficiency part is postponed up to Section 4 where it will be obtained as a consequence of slightly stronger results. □

From now on we assume that the necessary conditions (2.9)–(2.11) for Problem 1.2 to have a solution are in force.

3. Solutions to the interpolation problem and unitary extensions

We recall that a d-variable colligation is defined as a quadruple

$$\mathcal{Q} = \left\{ \mathcal{H} = \bigoplus_{j=1}^{d} \mathcal{H}_j, \mathcal{E}, \mathcal{E}_*, \mathbf{U} \right\} \tag{3.1}$$

consisting of three Hilbert spaces $\mathcal{H}$ (*the state space*) which is specified to have a fixed d-fold orthogonal decomposition, $\mathcal{E}$ (*the input space*) and $\mathcal{E}_*$ (*the output space*), together with a connecting operator

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{E}_* \end{bmatrix}.$$

The colligation is said to be *unitary* if the connecting operator $\mathbf{U}$ is unitary. A colligation

$$\widetilde{\mathcal{Q}} = \left\{ \widetilde{\mathcal{H}} = \bigoplus_{j=1}^{d} \widetilde{\mathcal{H}}_j, \mathcal{E}, \mathcal{E}_*, \widetilde{\mathbf{U}} \right\} \tag{3.2}$$

is said to be *unitarily equivalent* to the colligation $\mathcal{Q}$ if there is a unitary operator $\alpha : \mathcal{H} \to \widetilde{\mathcal{H}}$ such that

$$\alpha P_j = \widetilde{P}_j \alpha \quad (j = 1, \dots, d) \quad \text{and} \quad \begin{bmatrix} \alpha & 0 \\ 0 & I_{\mathcal{E}_*} \end{bmatrix} \mathbf{U} = \widetilde{\mathbf{U}} \begin{bmatrix} \alpha & 0 \\ 0 & I_{\mathcal{E}} \end{bmatrix},$$

where P_j and $\widetilde{P}_j$ are orthogonal projections of $\mathcal{H}$ onto $\mathcal{H}_j$ and of $\widetilde{\mathcal{H}}$ onto $\widetilde{\mathcal{H}}_j$, respectively. The *characteristic function* of the colligation $\mathcal{Q}$ is defined as

$$S_{\mathcal{Q}}(z) = D + C\big(I_{\mathcal{H}} - Z(z)A\big)^{-1} Z(z) B, \tag{3.3}$$

where $Z(z)$ is defined as in (1.3). Thus, Theorem 1.1 asserts that a $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued function S analytic in $\mathbb{D}^d$ belongs to the class $\mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ if and only if it is the characteristic function of some d-variable unitary colligation (3.1).

Although the function Z depends on the d-fold decomposition (1.2) of the state space $\mathcal{H}$, we shall write $Z(z)$ rather than $Z_{\mathcal{H}}(z)$ if the state space and its decomposition will be clear from the context.

Remark 3.1. Unitary equivalent colligations have the same characteristic function.

In this section we associate a certain unitary colligation to Problem 1.2 for a fixed choice of kernels $\mathbb{P}_j$ satisfying conditions (2.9)–(2.11). It turns out that the characteristic function of this colligation is the transfer function of the Redheffer transform describing solutions of Problem 1.2 associated with this choice of $\mathbb{P}_j$. Assuming that the necessary conditions (2.9)–(2.11) for Problem 1.2 to have a solution are in force, let us consider the linear space $\mathcal{H}_0$ of $\mathcal{M}_L$-valued functions

$h(\xi)$ defined on Ω which take nonzero values at at most finitely many points. Let $X \in \mathcal{L}(\mathcal{H}_0, \mathcal{E}_*)$ and $Y \in \mathcal{L}(\mathcal{H}_0, \mathcal{E})$ be operators defined by

$$Xh = \sum_{\xi} \mathbf{a}(\xi) h(\xi), \qquad Yh = \sum_{\xi} \mathbf{c}(\xi) f(\xi), \tag{3.4}$$

and let $D_j(h, g)$ be the quadratic form on $\mathcal{H}_0 \times \mathcal{H}_0$ defined as

$$D_j(h, g) = \sum_{\xi_i, \xi_\ell} \big\langle \mathbb{P}_j(\xi_i, \xi_\ell) h(\xi_\ell), g(\xi_i) \big\rangle_{\mathcal{M}_L} \quad (j = 1, \ldots, d). \tag{3.5}$$

Then it follows from (2.11) that

$$\sum_{j=1}^{d} \big(D_j(h, g) - D_j(\bar{\sigma}_j h, \bar{\sigma}_j g) \big) = \langle Xh, Xg \rangle_{\mathcal{E}_*} - \langle Yh, Yg \rangle_{\mathcal{E}}. \tag{3.6}$$

We say that $h_1 \sim h_2$ if and only if $D_j(h_1 - h_2, y) = 0$ for all $y \in \mathcal{H}_0$ and denote $[h]_{D_j}$ the equivalence class of h with respect to the above equivalence. The linear space of equivalence classes endowed with the inner product

$$\big\langle [h], [y] \big\rangle = D_j(h, y) \tag{3.7}$$

is a pre-Hilbert space, whose completion we denote by $\widehat{\mathcal{H}}_j$. Rewriting (3.6) as

$$\begin{aligned} &\sum_{j=1}^{d} \big\langle [f]_{D_j}, [f]_{D_j} \big\rangle_{\widehat{\mathcal{H}}_j} + \langle Yf, Yf \rangle_{\mathcal{E}} \\ &\quad = \sum_{j=1}^{d} \big\langle [\bar{\sigma}_j f]_{D_j}, [\bar{\sigma}_j f]_{D_j} \big\rangle_{\widehat{\mathcal{H}}_j} + \langle Xf, Xf \rangle_{\mathcal{E}_*} \end{aligned} \tag{3.8}$$

and setting

$$\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_1 \oplus \cdots \oplus \widehat{\mathcal{H}}_d,$$

we conclude that the linear map

$$\mathbf{V}_{\mathbb{P}_1, \ldots, \mathbb{P}_d} : \begin{bmatrix} [f]_{D_1} \\ \vdots \\ [f]_{D_d} \\ Yf \end{bmatrix} \to \begin{bmatrix} [\bar{\sigma}_1 f]_{D_1} \\ \vdots \\ [\bar{\sigma}_d f]_{D_d} \\ Xf \end{bmatrix} \tag{3.9}$$

is an isometry from

$$\mathcal{D}_{\mathbf{V}} = \operatorname{Clos} \left\{ \begin{bmatrix} [f]_{D_1} \\ \vdots \\ [f]_{D_d} \\ Yf \end{bmatrix}, \ f \in \mathcal{H}_0 \right\} \subset \begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E} \end{bmatrix} \tag{3.10}$$

onto

$$\mathcal{R}_{\mathbf{V}} = \mathrm{Clos}\left\{ \begin{bmatrix} [\overline{\sigma}_1 f]_{D_1} \\ \vdots \\ [\overline{\sigma}_d f]_{D_d} \\ Xf \end{bmatrix}, \ f \in \mathcal{H}_0 \right\} \subset \begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix}. \tag{3.11}$$

Theorems 3.2 and 3.4 below establish a correspondence between solutions S to Problem 1.2 and unitary extensions of the partially defined isometry $\mathbf{V}_{\mathbb{P}_1,\ldots,\mathbb{P}_d}$ given in (3.9).

Theorem 3.2. *Let S be a solution to Problem* 1.2. *Then there exist d kernels $\mathbb{P}_1, \ldots, \mathbb{P}_d$ on Ω satisfying conditions* (2.9)–(2.11) *such that S is a characteristic function of a unitary colligation*

$$\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} : \begin{bmatrix} \widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix}, \tag{3.12}$$

which is an extension of the isometry $\mathbf{V}_{\mathbb{P}_1,\ldots,\mathbb{P}_d}$ given in (3.9).

Proof. Let S be a solution to Problem 1.2. In particular, S belongs to $\mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ and, by Theorem 1.1, it is the characteristic function of some unitary colligation $\mathcal{Q}$ of the form (3.1). In other words, S admits a unitary realization (1.6) with the state space $\mathcal{H}$ and the equality (1.1) holds for functions H_j's defined via (1.7). The functions H_j's are analytic and take $\mathcal{L}(\mathcal{H}_j, \mathcal{E}_*)$ values on $\mathbb{D}^d$. The function $H(z)$ defined as in (1.4) is analytic and $\mathcal{L}(\mathcal{H}, \mathcal{E}_*)$-valued on $\mathbb{D}^d$. It also can be represented in terms of the realization (1.6) as in (1.7) and thus leads to the following representation

$$S(z) = D + H(z)Z(z)B \tag{3.13}$$

of S, which is equivalent to (1.6).

The interpolation conditions (1.8) and (1.9), which are assumed to be satisfied by S, force certain restrictions on the connecting operator

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

By Lemma 2.1 and in view of (1.4), the following strong limit exists:

$$\lim_{r \to 1} H\big(r\sigma(\xi)\big)^* \mathbf{a}(\xi) = \begin{bmatrix} T_1(\xi) \\ \vdots \\ T_d(\xi) \end{bmatrix} =: T(\xi) \quad (\xi \in \Omega). \tag{3.14}$$

Substituting (3.13) into (1.8) we get

$$\lim_{r \to 1} \Big(D^* + B^* Z\big(r\sigma(\xi)\big)^* H\big(r\sigma(\xi)\big)^*\Big)\mathbf{a}(\xi) = \mathbf{c}(\xi) \quad (\xi \in \Omega),$$

where the limit is understood in the strong sense. It also follows from (1.5) that $C + H(z)Z(z)A = H(z)$ and, therefore, that (strongly)

$$C^*\mathbf{a}(\xi) + \lim_{r\to 1} A^*Z\bigl(r\sigma(\xi)\bigr)^*H\bigl(r\sigma(\xi)\bigr)^*\mathbf{a}(\xi) = \lim_{r\to 1} H\bigl(r\sigma(\xi)\bigr)^*\mathbf{a}(\xi).$$

By (1.8) and (3.14), the two last (displayed) equalities are equivalent to

$$D^*\mathbf{a}(\xi) + B^*Z\bigl(\sigma(\xi)\bigr)^*T(\xi) = \mathbf{c}(\xi) \tag{3.15}$$

and

$$C^*\mathbf{a}(\xi) + A^*Z\bigl(\sigma(\xi)\bigr)^*T(\xi) = T(\xi), \tag{3.16}$$

which can be written in matrix form as

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\begin{bmatrix} Z(\sigma(\xi))^*T(\xi) \\ \mathbf{a}(\xi) \end{bmatrix} = \begin{bmatrix} T(\xi) \\ \mathbf{c}(\xi) \end{bmatrix} \quad (\xi \in \Omega).$$

Since the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is unitary, we conclude from the last equality that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} T(\xi) \\ \mathbf{c}(\xi) \end{bmatrix} = \begin{bmatrix} Z(\sigma(\xi))^*T(\xi) \\ \mathbf{a}(\xi) \end{bmatrix}. \tag{3.17}$$

Let $\mathbb{P}_1, \ldots, \mathbb{P}_d$ be defined as in (2.14), let $\mathbf{V}_{\mathbb{P}_1,\ldots,\mathbb{P}_d}$ be the isometry given in (3.9) and let $\mathbf{T}_j : \mathcal{H}_0 \to \mathcal{H}_j$ be the operator given by

$$\mathbf{T}_j h = \sum_{\xi} T_j(\xi)h(\xi) \quad (j = 1, \ldots, d). \tag{3.18}$$

Upon making subsequent use of (3.7), (3.5), (2.14) and (3.18), we get

$$\begin{aligned}
\bigl\langle [h]_{D_j}, [y]_{D_j} \bigr\rangle_{\widehat{\mathcal{H}}_j} = D_j(h, y) &= \sum_{\xi_i,\xi_\ell} \bigl\langle \mathbb{P}_j(\xi_i, \xi_\ell)h(\xi_\ell), y(\xi_i) \bigr\rangle_{\mathcal{M}_L} \\
&= \sum_{\xi_i,\xi_\ell} \bigl\langle T_j(\xi_\ell)h(\xi_\ell), T_j(\xi_i)y(\xi_i) \bigr\rangle_{\mathcal{M}_L} \\
&= \left\langle \sum_{\xi_\ell} T_j(\xi_\ell)h(\xi_\ell), \sum_{\xi_i} T_j(\xi_i)y(\xi_i) \right\rangle_{\mathcal{M}_L} \\
&= \langle \mathbf{T}_j h, \mathbf{T}_j y \rangle_{\widehat{\mathcal{H}}_j}.
\end{aligned}$$

Therefore, the linear transformation $U_j : \mathcal{H}_0 \to \widehat{\mathcal{H}}_j$ defined by the rule

$$U_j : \mathbf{T}_j f \to [f]_{D_j} \quad (f \in \mathcal{H}_0) \tag{3.19}$$

can be extended to the unitary map (which still is denoted by U_j) from $\overline{\operatorname{Ran}\mathbf{T}_j}$ onto $\widehat{\mathcal{H}}_j$. Noticing that $\overline{\operatorname{Ran}\mathbf{T}_j}$ is a subspace of $\mathcal{H}_j$ and setting

$$\mathcal{N}_j := \mathcal{H}_j \ominus \overline{\operatorname{Ran}\mathbf{T}_j} \quad \text{and} \quad \widetilde{\mathcal{H}}_j := \widehat{\mathcal{H}}_j \oplus \mathcal{N}_j,$$

we define the unitary map $\widetilde{U}_j : \mathcal{H}_j \to \widetilde{\mathcal{H}}_j$ by the rule

$$\widetilde{U}_j g = \begin{cases} U_j g & \text{for } g \in \overline{\operatorname{Ran} \mathbf{T}_j}, \\ g & \text{for } g \in \mathcal{N}_j. \end{cases} \tag{3.20}$$

The operator

$$\widetilde{U} := \bigoplus_{j=1}^{d} \widetilde{U}_j : \mathcal{H} \to \widetilde{\mathcal{H}} := \bigoplus_{j=1}^{d} \widetilde{\mathcal{H}}_j \tag{3.21}$$

is unitary and satisfies

$$\widetilde{U} P_j = \widetilde{P}_j \widetilde{U} \quad (j = 1, \dots, d),$$

where P_j and $\widetilde{P}_j$ are orthogonal projections of $\mathcal{H}$ onto $\mathcal{H}_j$ and of $\widetilde{\mathcal{H}}$ onto $\widetilde{\mathcal{H}}_j$, respectively. Introducing the operators

$$\widetilde{A} = \widetilde{U} A \widetilde{U}^*, \qquad \widetilde{B} = \widetilde{U} B, \qquad \widetilde{C} = C \widetilde{U}^*, \qquad \widetilde{D} = D \tag{3.22}$$

we construct the colligation $\widetilde{\mathcal{Q}}$ via (3.2) and (3.12). By definition, $\widetilde{\mathcal{Q}}$ is unitarily equivalent to the initial colligation $\mathcal{Q}$ defined in (3.1). By Remark 3.1, $\widetilde{\mathcal{Q}}$ has the same characteristic function as $\mathcal{Q}$, that is, $S(z)$. It remains to check that the connecting operator of $\widetilde{\mathcal{Q}}$ is an extension of $\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}$, that is,

$$\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} [f]_{D_1} \\ \vdots \\ [f]_{D_d} \\ Yf \end{bmatrix} = \begin{bmatrix} [\overline{\sigma}_1 f]_{D_1} \\ \vdots \\ [\overline{\sigma}_d f]_{D_d} \\ Xf \end{bmatrix}, \quad f \in \mathcal{H}_0. \tag{3.23}$$

To this end, note that by (3.18)–(3.20),

$$\widetilde{U}_j^* \big([f]_{D_j}\big) = \mathbf{T}_j f = \sum_{\xi} T_j(\xi) f(\xi)$$

and

$$\widetilde{U}_j \left(\sum_{\xi} \overline{\sigma_j(\xi)} T_j(\xi) f(\xi) \right) = \widetilde{U}_j \mathbf{T}_j (\overline{\sigma}_j f) = [\overline{\sigma}_j f]_{D_j}$$

for $j = 1, \dots, d$ and for every $f \in \mathcal{H}_0$. Taking into account the diagonal structure (3.21) of $\widetilde{U}$, we now get from the two last equalities that

$$\widetilde{U}^* \left(\begin{bmatrix} [f]_{D_1} \\ \vdots \\ [f]_{D_d} \end{bmatrix} \right) = \sum_{\xi} T(\xi) f(\xi) \tag{3.24}$$

and

$$\widetilde{U} \left(\sum_{\xi} Z\big(\sigma(\xi)\big)^* T(\xi) f(\xi) \right) = \begin{bmatrix} [\overline{\sigma}_1 f]_{D_1} \\ \vdots \\ [\overline{\sigma}_d f]_{D_d} \end{bmatrix}. \tag{3.25}$$

Thus, making subsequent use of (3.22), (3.24), (3.17) and (3.25), we get

$$
\begin{aligned}
\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix}
\begin{bmatrix} [f]_{D_1} \\ \vdots \\ [f]_{D_d} \\ Yf \end{bmatrix}
&= \begin{bmatrix} \widetilde{U} & 0 \\ 0 & I \end{bmatrix}
\begin{bmatrix} A & B \\ C & D \end{bmatrix}
\begin{bmatrix} \widetilde{U}^* & 0 \\ 0 & I \end{bmatrix}
\begin{bmatrix} [f]_{D_1} \\ \vdots \\ [f]_{D_d} \\ Yf \end{bmatrix} \\
&= \begin{bmatrix} \widetilde{U} & 0 \\ 0 & I \end{bmatrix}
\begin{bmatrix} A & B \\ C & D \end{bmatrix}
\left(\sum_{\xi} \begin{bmatrix} T(\xi) \\ \mathbf{c}(\xi) \end{bmatrix} f(\xi) \right) \\
&= \begin{bmatrix} \widetilde{U} & 0 \\ 0 & I \end{bmatrix}
\left(\sum_{\xi} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T(\xi) \\ \mathbf{c}(\xi) \end{bmatrix} f(\xi) \right) \\
&= \begin{bmatrix} \widetilde{U} & 0 \\ 0 & I \end{bmatrix}
\left(\sum_{\xi} \begin{bmatrix} Z(\sigma(\xi))^* T(\xi) \\ \mathbf{a}(\xi) \end{bmatrix} f(\xi) \right) \\
&= \begin{bmatrix} [\bar{\sigma}_1 f]_{D_1} \\ \vdots \\ [\bar{\sigma}_d f]_{D_d} \\ Xf \end{bmatrix},
\end{aligned}
\tag{3.26}
$$

which proves (3.23) and completes the proof of the lemma. □

The converse statement will be proved in Theorem 3.4 below. We start with some auxiliary results (for the proof see [8, Section 2]).

Lemma 3.3. *Let A be a contraction on a Hilbert space $\mathcal{H}$. Then the following strong limits*

$$
\begin{gathered}
R := \lim_{r \to 1} (1 - r)(I_{\mathcal{H}} - rA)^{-1}, \qquad Q := \lim_{r \to 1} (I_{\mathcal{H}} - rA)^{-1}(I_{\mathcal{H}} - A), \\
\lim_{r \to 1} (1 - r)^2 (I_{\mathcal{H}} - rA^*)^{-1}(I_{\mathcal{H}} - A^*A)(I_{\mathcal{H}} - rA)^{-1} = 0
\end{gathered}
\tag{3.27}
$$

exist. Moreover, R and Q are in fact orthogonal projection onto $\operatorname{Ker}(I_{\mathcal{H}} - A)$ *and* $\overline{\operatorname{Ran}(I_{\mathcal{H}} - A^*)}$, *respectively.*

Theorem 3.4. *Let $\mathbb{P}_1, \ldots, \mathbb{P}_d$ be kernels on Ω satisfying conditions* (2.9)–(2.11) *and let $\widetilde{\mathbf{U}}$ of the form* (3.12) *be a unitary extension of the partially defined isometry $\mathbf{V}_{\mathbb{P}_1,\ldots,\mathbb{P}_d}$ given in* (3.9). *Then the characteristic function S of the unitary colligation $\widetilde{\mathcal{Q}}$ defined via* (3.2) *is a solution of Problem* 1.2.

Proof. We start with factorizations (2.14) of the kernels $\mathbb{P}_k$ and let, according to (3.14),

$$T(\xi) = \begin{bmatrix} T_1(\xi) \\ \vdots \\ T_d(\xi) \end{bmatrix}. \tag{3.28}$$

Furthermore, we define the unitary map $\widetilde{U}$ via (3.18)–(3.20). Then relations (3.22) hold by construction and, therefore, the operator

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \widetilde{U}^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} \widetilde{U} & 0 \\ 0 & I \end{bmatrix}$$

satisfies (3.17) (or, equivalently, (3.15) and (3.16)), which can be easily seen from (3.26). By Remark 3.1, the colligations $\widehat{\mathcal{Q}}$ and $\widetilde{\mathcal{Q}}$ defined in (3.1) and (3.2) have the same characteristic functions and thus S can be taken in the form (1.6). Let $H_k(z)$ and $H(z)$ be given by (1.7).

By (3.27) (with A replaced by $A^*Z(\sigma(\xi))^*$), it follows that

$$\lim_{r\to 1}(1-r)^2\big(I - rZ\big(\sigma(\xi)\big)A\big)^{-1}Z\big(\sigma(\xi)\big)(I - AA^*)Z\big(\sigma(\xi)\big)^* \\ \times \big(I - rA^*Z\big(\sigma(\xi)\big)^*\big)^{-1} = 0,$$

which is equivalent, since $AA^* + BB^* = I$, to

$$\lim_{r\to 1}(1-r)^2\big(I - rZ\big(\sigma(\xi)\big)A\big)^{-1}Z\big(\sigma(\xi)\big)BB^*Z\big(\sigma(\xi)\big)^* \\ \times \big(I - rA^*Z\big(\sigma(\xi)\big)^*\big)^{-1} = 0.$$

Therefore,

$$\lim_{r\to 1}(1-r)B^*Z\big(\sigma(\xi)\big)^*\big(I - rA^*Z\big(\sigma(\xi)\big)^*\big)^{-1}x = 0 \\ \text{(for every } x \in \mathcal{H}). \tag{3.29}$$

Using (1.6) and expressions for $D^*\mathbf{a}(\xi)$ and $C^*\mathbf{a}(\xi)$ derived from (3.15) and (3.16), respectively, we get

$$\begin{aligned} S\big(r\sigma(\xi)\big)^*\mathbf{a}(\xi) &= D^*\mathbf{a}(\xi) + rB^*Z\big(\sigma(\xi)\big)^*\big(I - rA^*Z\big(\sigma(\xi)\big)^*\big)^{-1}C^*\mathbf{a}(\xi) \\ &= \mathbf{c}(\xi) - B^*Z\big(\sigma(\xi)\big)^*T(\xi) + rB^*Z\big(\sigma(\xi)\big)^* \\ &\quad \times \big(I - rA^*Z\big(\sigma(\xi)\big)^*\big)^{-1}\big(I - rA^*Z\big(\sigma(\xi)\big)^*\big)T(\xi) \\ &= \mathbf{c}(\xi) - (1-r)B^*Z\big(\sigma(\xi)\big) \\ &\quad \times \big(I - rA^*Z\big(\sigma(\xi)\big)^*\big)^{-1}T(\xi). \end{aligned} \tag{3.30}$$

Taking limits in the last identity as r tends to one and taking into account (3.29), we come to (1.8). Furthermore, by (3.30),

$$\frac{\mathbf{c}(\xi)^*\mathbf{c}(\xi) - \mathbf{c}(\xi)^*S(r\sigma(\xi))^*\mathbf{a}(\xi)}{1-r} = \mathbf{c}(\xi)^*B^*Z\big(\sigma(\xi)\big)^*\big(I - rA^*Z\big(\sigma(\xi)\big)^*\big)^{-1}T(\xi). \tag{3.31}$$

It follows from (3.17) that

$$AT(\xi) + B\mathbf{c}(\xi) = Z\big(\sigma(\xi)\big)^*T(\xi)$$

and, therefore,

$$\mathbf{c}(\xi)^*B^* = T(\xi)^*\big(Z\big(\sigma(\xi)\big) - A^*\big).$$

Substituting the latter equality into (3.31) and taking into account that $Z(\sigma(\xi)) \times Z(\sigma(\xi))^* = I$, we get

$$\frac{\mathbf{c}(\xi)^*\mathbf{c}(\xi) - \mathbf{c}(\xi)^*S(r\sigma(\xi))^*\mathbf{a}(\xi)}{1-r} = T(\xi)^*\big(I - A^*Z\big(\sigma(\xi)^*\big)\big)\big(I - rA^*Z\big(\sigma(\xi)\big)^*\big)^{-1}T(\xi).$$

Taking limits in the last identity as r tends to one and applying Lemma 3.3 (with A replaced by $A^*Z(\sigma(\xi))^*$), we conclude that the following weak limit exists:

$$\lim_{r\to 1}\frac{\mathbf{c}(\xi)^*\mathbf{c}(\xi) - \mathbf{c}(\xi)^*S(r\sigma(\xi))^*\mathbf{a}(\xi)}{1-r} = T(\xi)^*\mathbf{P}_{\overline{\mathrm{Ran}(I_{\mathcal{H}} - Z(\sigma(\xi))A)}}T(\xi).$$

Making use of (1.8) and (1.10) we conclude now by Lemma 2.1 that

$$\lim_{r\to 1}\mathbf{a}(\xi)^*\frac{I_{\mathcal{E}_*} - S(r\sigma(\xi))S(r\sigma(\xi))^*}{1-r^2}\mathbf{a}(\xi) = \lim_{r\to 1}\frac{\mathbf{c}(\xi)^*\mathbf{c}(\xi) - \mathbf{c}(\xi)^*S(r\sigma(\xi))^*\mathbf{a}(\xi)}{1-r}, \tag{3.32}$$

where the limit on the left-hand side in (3.32) is meant in the weak sense (as well as the limit on the right-hand side). Comparing the two last equalities and making use of (2.14), (2.10) and (3.28), we get that for every $\xi \in \Omega$

$$\lim_{r\to 1}\mathbf{a}(\xi)^*\frac{I_{\mathcal{E}_*} - S(r\sigma(\xi))S(r\sigma(\xi))^*}{1-r^2}\mathbf{a}(\xi) = T(\xi)^*\mathbf{P}_{\overline{\mathrm{Ran}(I_{\mathcal{H}} - Z(\sigma(\xi))A)}}T(\xi) \leqslant T(\xi)^*T(\xi) = \sum_{j=1}^{d}T_j(\xi)^*T_j(\xi) = \sum_{j=1}^{d}\mathbb{P}_j(\xi,\xi) \leqslant \Psi(\xi),$$

which proves (1.9) and completes the proof of theorem. □

4. The universal unitary colligation associated with the interpolation problem

A general result of Arov and Grossman (see [6,7]) describes how to parametrize the set of all unitary extensions of a given partially defined isometry $\mathbf{V}$. Their result has been extended to the multivariable case in [14] and can be applied to the present setting.

Let $\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d} : \mathcal{D}_{\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}} \to \mathcal{R}_{\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}}$ be the isometry given in (3.9) with $\mathcal{D}_{\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}}$ and $\mathcal{R}_{\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}}$ given in (3.10) and (3.11). Introduce the defect spaces

$$\Delta = \begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E} \end{bmatrix} \ominus \mathcal{D}_{\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}} \quad \text{and} \quad \Delta_* = \begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix} \ominus \mathcal{R}_{\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}},$$

and let $\widetilde{\Delta}$ be another copy of Δ and $\widetilde{\Delta}_*$ be another copy of Δ_* with unitary identification maps

$$i : \Delta \to \widetilde{\Delta} \quad \text{and} \quad i_* : \Delta_* \to \widetilde{\Delta}_*.$$

Define a unitary operator $\mathbf{U}_0$ from $\mathcal{D}_{\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}} \oplus \Delta \oplus \widetilde{\Delta}_*$ onto $\mathcal{R}_{\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}} \oplus \Delta_* \oplus \widetilde{\Delta}$ by the rule

$$\mathbf{U}_0 x = \begin{cases} \mathbf{V}x & \text{if } x \in \mathcal{D}_{\mathbf{V}}, \\ i(x) & \text{if } x \in \Delta, \\ i_*^{-1}(x) & \text{if } x \in \widetilde{\Delta}_*. \end{cases} \tag{4.1}$$

Identifying $\begin{bmatrix} \mathcal{D}_{\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}} \\ \Delta \end{bmatrix}$ with $\begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E} \end{bmatrix}$ and $\begin{bmatrix} \mathcal{R}_{\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}} \\ \Delta_* \end{bmatrix}$ with $\begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix}$, we decompose $\mathbf{U}_0$ defined by (4.1) according to

$$\mathbf{U}_0 = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & 0 \end{bmatrix} : \begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E} \\ \widetilde{\Delta}_* \end{bmatrix} \to \begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E}_* \\ \widetilde{\Delta} \end{bmatrix}.$$

The “33” block in this decomposition is zero, since (by definition (4.1)), for every $x \in \widetilde{\Delta}_*$, the vector $\mathbf{U}_0 x$ belongs to Δ, which is a subspace of $\begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix}$ and, therefore, is orthogonal to $\widetilde{\Delta}$ (in other words, $\mathbf{P}_{\widetilde{\Delta}} \mathbf{U}_0|_{\widetilde{\Delta}_*} = 0$, where $\mathbf{P}_{\widetilde{\Delta}}$ stands for the orthogonal projection of $\mathcal{R}_{\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}} \oplus \Delta_* \oplus \widetilde{\Delta}$ onto $\widetilde{\Delta}$).

The unitary operator $\mathbf{U}_0$ is the connecting operator of the unitary colligation

$$\Omega_0 = \left\{ \widehat{\mathcal{H}}, \begin{bmatrix} \mathcal{E} \\ \widetilde{\Delta}_* \end{bmatrix}, \begin{bmatrix} \mathcal{E}_* \\ \widetilde{\Delta} \end{bmatrix}, \mathbf{U}_0 \right\},$$

which is called *the universal unitary colligation* associated with the interpolation problem. According to (3.3), the characteristic function of this colligation is given by

$$\begin{aligned} \Sigma(z) &= \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} \\ &= \begin{bmatrix} U_{22} & U_{23} \\ U_{32} & 0 \end{bmatrix} + \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} \big(I_n - Z(z)U_{11}\big)^{-1} Z(z)[U_{12}\ U_{13}] \end{aligned} \tag{4.2}$$

and belongs to the class $\mathcal{S}_d(\mathcal{E} \oplus \widetilde{\Delta}_*, \mathcal{E}_* \oplus \widetilde{\Delta})$, by Theorem 1.1.

Theorem 4.1. *Let* $\mathbf{V}$ *be the isometry defined in* (3.9), *let* Σ *be the function constructed as above, and let* S *be a* $\mathcal{L}(\mathcal{E},\mathcal{E}_*)$*-valued function. Then the following are equivalent*:

(1) *S is a characteristic function of a colligation*

$$\Omega=\left\{\widehat{\mathcal{H}}\oplus\widetilde{\mathcal{H}},\mathcal{E},\mathcal{E}_*,\begin{bmatrix} A & B\\ C & D\end{bmatrix}\right\}$$

with the connecting operator being a unitary extension of $\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}$.

(2) *S is of the form*

$$S(z)=\Sigma_{11}(z)+\Sigma_{12}(z)\big(I_{\mathcal{E}_*\oplus\widetilde{\Delta}}-\mathcal{T}(z)\Sigma_{22}(z)\big)^{-1}\mathcal{T}(z)\Sigma_{21}(z),\qquad(4.3)$$

where $\mathcal{T}$ is a function from the class $\mathcal{S}_d(\mathcal{E}\oplus\widetilde{\Delta}_*,\mathcal{E}_*\oplus\widetilde{\Delta})$.

This result (which has been proved in [14]) together with Theorems 3.2 and 3.4 leads to a description of all solutions of Problem 1.2.

The following corollary on parametrization of particular subclasses of solutions implies in particular the sufficiency part of Theorem 2.2.

Corollary 4.2. *Let* $\mathbb{P}_1,\dots,\mathbb{P}_d$ *be d kernels satisfying conditions* (2.9)–(2.11), *and let*

$$\Sigma(z)=\begin{bmatrix}\Sigma_{11}(z) & \Sigma_{12}(z)\\ \Sigma_{21}(z) & \Sigma_{22}(z)\end{bmatrix}$$

be the characteristic function as in (4.2) *of the unitary colligation* $\mathbf{U}_0$ *constructed from* $\mathbf{V}_{\mathbb{P}_1,\dots,\mathbb{P}_d}$ *as in* (4.1). *Then the set of all solutions S of Problem* 1.2 *satisfying the auxiliary side conditions*

$$T_j(\xi)^*T_j(\mu)=\mathbb{P}_j(\xi,\mu)\quad\text{for }\xi,\mu\in\Omega\text{ and }j=1,\dots,d$$

(*where* $T_j(\xi)=\lim_{r\to1}H_j(r\sigma(\xi))^*\mathbf{a}(\xi)$ *for some choice of* $H_1(z),\dots,H_d(z)$ *for which* (1.1) *holds*) *is given by* (4.3) *where $\mathcal{T}$ is a function from the class* $\mathcal{S}_d(\mathcal{E}\oplus\widetilde{\Delta}_*,\mathcal{E}_*\oplus\widetilde{\Delta})$.

5. Boundary Nevanlinna–Pick interpolation problem

In this section we consider a boundary Nevanlinna–Pick problem for the class $\mathcal{S}_d(\mathcal{E},\mathcal{E}_*)$. We are given an auxiliary Hilbert space $\mathcal{M}_L$, the set $\mathcal{Z}=\{z^{(1)},\dots,z^{(n)}\}\subset\mathbb{T}^d$, $2n$ operators

$$x_1,\dots,x_n\in\mathcal{L}(\mathcal{M}_L,\mathcal{E}_*),\qquad y_1,\dots,y_n\in\mathcal{L}(\mathcal{M}_L,\mathcal{E})$$

and n positive semidefinite operators

$$\gamma_1, \dots, \gamma_n \in \mathcal{L}(\mathcal{M}_L), \quad \gamma_j \geqslant 0 \quad (j = 1, \dots, n).$$

Problem 5.1. *Find all functions $S \in \mathcal{S}_d(\mathcal{E}, \mathcal{E}_*)$ such that*

$$\lim_{r \to 1} S(rz^{(j)})^* x_j = y_j \quad (\xi \in \Omega) \tag{5.1}$$

and

$$\lim_{r \to 1} x_j^* \frac{I_{\mathcal{E}_*} - S(rz^{(j)}) S(rz^{(j)})^*}{1 - r^2} x_j \leqslant \gamma_j \quad (j = 1, \dots, n), \tag{5.2}$$

where the limits in (5.1) *and* (5.2) *are understood in the strong and in the weak sense, respectively.*

Problem 5.1 is a particular case of Problem 1.2 corresponding to the following choice of interpolation data:

$$\Omega = \mathcal{Z}, \qquad \sigma(\xi) = \xi, \qquad \Psi\big(z^{(j)}\big) = \gamma_j, \qquad \mathbf{a}\big(z^{(j)}\big) = x_j,$$
$$\mathbf{c}\big(z^{(j)}\big) = y_j \quad (j = 1, \dots, n). \tag{5.3}$$

In the present context Theorem 2.2 takes the following form.

Theorem 5.2. *Problem* 5.1 *has a solution if and only if there exist d positive semidefinite block matrices $P_k = [p_{ij}^k]_{i,j=1}^n \geqslant 0$ $(j = 1, \dots, d)$ with block entries $p_{ij}^k \in \mathcal{L}(\mathcal{M}_L)$ subject to*

$$\sum_{k=1}^d p_{jj}^k \leqslant \gamma_k \quad (j = 1, \dots, n), \tag{5.4}$$

which satisfy the generalized Stein equality

$$\sum_{k=1}^d \big(P_k - N_k^* P_k N_k\big) = X^* X - Y^* Y, \tag{5.5}$$

where

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{bmatrix} \quad \textit{and}$$

$$N_k = \begin{bmatrix} z_k^{(1)} I_{\mathcal{M}_L} & & 0 \\ & \ddots & \\ 0 & & z_k^{(n)} I_{\mathcal{M}_L} \end{bmatrix} \quad (k = 1, \dots, d). \tag{5.6}$$

Indeed, since Ω is a finite set of points, Stein equation (2.11) is equivalent to equality of Hermitian forms

$$\begin{aligned}&\sum_{i,j=1}^{n}\left\langle \sum_{k=1}^{d}\mathbb{P}_k\big(z^{(i)},z^{(j)}\big)g_j,\,g_i\right\rangle\\ &\qquad-\sum_{i,j=1}^{n}\left\langle \sum_{k=1}^{d}\mathbb{P}_j\big(z^{(i)},z^{(r)}\big)\overline{\sigma\big(z^{(j)}\big)}g_j,\,\overline{\sigma\big(z^{(i)}\big)}g_i\right\rangle\\ &\quad=\sum_{i,j=1}^{n}\big\langle \mathbf{a}\big(z^{(j)}\big)g_j,\mathbf{a}\big(z^{(i)}\big)g_i\big\rangle-\sum_{i,j=1}^{n}\big\langle \mathbf{c}\big(z^{(j)}\big)g_j,\mathbf{c}\big(z^{(i)}\big)g_i\big\rangle\quad (g_i\in\mathcal{M}_L)\end{aligned}$$

which, in turn, is equivalent to the operator identity (5.5) with

$$P_k=\big[\mathbb{P}_k\big(z^{(i)},z^{(j)}\big)\big]_{i,j=1}^{n}\quad (k=1,\dots,d).\tag{5.7}$$

Since the kernels $\mathbb{P}_k$ are positive on Ω, the matrices P_k are positive semidefinite. Finally, condition (5.4) follows from (2.10).

For every choice of positive semidefinite matrices $P_1,\dots,P_d$ satisfying (5.4) and (5.5), the set of solutions of Problem 5.1 associated with this choice is parametrized by Theorem 4.1 in terms of a Redheffer linear fractional transformation (4.3). Moreover, in this case one can get explicit formulas for coefficients $\Sigma_{ij}(z)$ of this transformation in terms of interpolation data. Such formulas have been established in [5] for nonboundary bitangential problem. Since the formulas depend only on the entries in the identity (5.5) (in contrast to the boundary problem, the matrices P_j for the nonboundary problem are prescribed), they are still true for the present context. We present these formulas for the sake of completeness:

$$\begin{aligned}\Sigma_{11}(z)&=X\Delta(z)^{[-1]}Y^*,\\ \Sigma_{12}(z)&=\big(X\Delta(z)^{[-1]}W_1^*Z(z),\,I_{\mathcal{E}_*}\big)T_2,\\ \Sigma_{21}(z)&=T_1^*\begin{pmatrix}Z(z)W_2\Delta(z)^{[-1]}Y^*\\ I_{\mathcal{E}}\end{pmatrix},\\ \Sigma_{22}(z)&=T_1^*\begin{pmatrix}I\\0\end{pmatrix}\big(I+Z(z)W_2\Delta(z)^{[-1]}W_1^*\big)(I,0)Z(z)T_2,\end{aligned}$$

where

$$W_1=\begin{bmatrix}P_1^{1/2}\\ \vdots\\ P_d^{1/2}\end{bmatrix},\qquad W_2=\begin{bmatrix}P_1^{1/2}N_1\\ \vdots\\ P_d^{1/2}N_d\end{bmatrix},$$

T_1 and T_2 are isometric operators such that

$$[W_1^*\ Y^*]T_1=0\quad\text{and}\quad [W_2^*\ X^*]T_2=0,$$

and where $\Delta(z)^{[-1]}$ stands for the Moore–Penrose generalized inverse of the function

$$\Delta(z) = \sum_{k=1}^{d} P_k(I - z_k N_k) + Y^* Y.$$

References

[1] J. Agler, On the representation of certain holomorphic functions defined on a polydisk, in: L. de Branges, I. Gohberg, J. Rovnyak (Eds.), Topics in Operator Theory: Ernst D. Hellinger Memorial Volume, in: Operator Theory: Advances and Applications, Vol. 48, Birkhäuser, Basel, 1990, pp. 47–66.

[2] J. Agler, Some interpolation theorems of Nevanlinna–Pick type, preprint.

[3] J. Agler, J.E. McCarthy, Nevanlinna–Pick interpolation on the bidisk, J. Reine Angew. Math. 506 (1999) 191–204.

[4] N. Akhiezer, M.G. Kreĭn, Some Questions in the Theory of Moments, in: Translation of Mathematical Monographs, American Mathematical Society, 1962, Article II.

[5] D. Alpay, J.A. Ball, V. Bolotnikov, On the bitangential interpolation problem for contractive valued functions in the polydisk, J. Operator Theory 44 (2000) 277–301.

[6] D.Z. Arov, L.Z. Grossman, Scattering matrices in the theory of unitary extensions of isometric operators, Soviet Math. Dokl. 270 (1983) 17–20.

[7] D.Z. Arov, L.Z. Grossman, Scattering matrices in the theory of unitary extensions of isometric operators, Math. Nachr. 157 (1992) 105–123.

[8] J.A. Ball, V. Bolotnikov, A bitangential interpolation problem on the closed unit ball for multipliers of the Arveson space, Integral Equations Operator Theory, to appear.

[9] J.A. Ball, Interpolation problems of Pick–Nevanlinna and Loewner type for meromorphic matrix functions, Integral Equations Operator Theory 6 (1983) 804–840.

[10] J.A. Ball, I. Gohberg, L. Rodman, Boundary Nevanlinna–Pick interpolation for rational matrix functions, J. Math. Systems Estim. Control 1 (1991) 131–164.

[11] J.A. Ball, I. Gohberg, L. Rodman, Interpolation of Rational Matrix Functions, Birkhäuser, Basel, 1990.

[12] J.A. Ball, J.W. Helton, Interpolation problems of Pick–Nevanlinna and Loewner types for meromorphic matrix-functions: parametrization of the set of all solutions, Integral Equations Operator Theory 9 (1986) 155–203.

[13] J.A. Ball, W.S. Li, D. Timotin, T.T. Trent, A commutant lifting theorem on the polydisc, Indiana Univ. Math. J. 48 (1999) 653–675.

[14] J.A. Ball, T. Trent, Unitary colligations, reproducing kernel Hilbert spaces and Nevanlinna–Pick interpolation in several variables, J. Funct. Anal. 157 (1998) 1–61.

[15] V. Bolotnikov, H. Dym, On degenerate interpolation maximum entropy and extremal problems for matrix Schur functions, Integral Equations Operator Theory 32 (1998) 367–435.

[16] V. Bolotnikov, H. Dym, On boundary interpolation for matrix Schur functions, preprint MCS99-22, Department of Mathematics, The Weizmann Institute of Science, Israel.

[17] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, in: SIAM Studies in Applied Mathematics, Vol. 15, SIAM, Philadelphia, 1994.

[18] P. Dewilde, H. Dym, Lossless inverse scattering, digital filters and estimation theory, IEEE Trans. Inform. Theory 30 (1984) 644–662.

[19] H. Dym, *J* Contractive Matrix Functions, Reproducing Kernel Spaces and Interpolation, in: CBMS Lecture Notes, Vol. 71, American Mathematical Society, Providence, RI, 1989.

[20] L. El Ghaoui, S.-I. Niculescu (Eds.), Advances in Linear Matrix Inequality Methods in Control, Advances in Design and Control, SIAM, Philadelphia, 2000.
[21] J.W. Helton, Some adaptive control problems which convert to a “classical” problem in several complex variables, IEEE Trans. Automat. Control 46 (2001) 2038–2093.
[22] V. Katsnelson, A. Kheifets, P. Yuditskii, An abstract interpolation problem and the extension theory of isometric operators, in: H. Dym, B. Fritzsche, V. Katsnelson, B. Kirstein (Eds.), Topics in Interpolation Theory, in: Operator Theory: Advances and Applications, Vol. OT 95, Birkhäuser, Basel, 1997, pp. 283–297; Translated from V.A. Marchenko (Ed.), Operators in Function Spaces and Problems in Function Theory, Naukova Dumka, Kiev, 1987, pp. 83–96.
[23] A. Kheifets, Hamburger moment problem: Parseval equality and A-singularity, J. Funct. Anal. 141 (1996) 374–420.
[24] A. Kheifets, The abstract interpolation problem and applications, in: D. Sarason, S. Axler, J. McCarthy (Eds.), Holomorphic Spaces, Cambridge University Press, Cambridge, 1998, pp. 351–379.
[25] I.V. Kovalishina, Carathéodory–Julia theorem for matrix-functions, Teor. Funktsii Funktsional. Anal. i Prilozhen. 43 (1985) 70–82; English translation: J. Soviet Math. 48 (1990) 176–186.
[26] I.V. Kovalishina, A multiple boundary interpolation problem for contracting matrix-valued functions in the unit circle, Teor. Funktsii Funktsional. Anal. i Prilozhen. 51 (1989) 38–55; English translation: J. Soviet Math. 52 (1990) 3467–3481.
[27] R. Nevanlinna, Über beschränkte Funktionen, Ann. Acad. Sci. Fenn. Ser. A 32 (1939).
[28] L.J. Patton, Interpolation of Nevanlinna–Pick type on the boundary of the bidisc, Complex Variables Theory Appl. 24 (1994) 107–119.
[29] D. Sarason, Nevanlinna–Pick interpolation with boundary data, Integral Equations Operator Theory 30 (1998) 231–250.