INVOLUTIONS ACTING ON REPRESENTATIONS

C. RYAN VINROOT

1. Introduction

For a group G with order 2 automorphism ι, we consider irreducible representations (π, V) of G such that ${}^{\iota}\pi \cong \hat{\pi}$, where $\hat{\pi}$ is the contragredient representation of π. This isomorphism gives rise to a nondegenerate bilinear form on V which is $(\pi, {}^{\iota}\pi)$-invariant, unique up to scalar, and consequently is either symmetric or skew-symmetric. For such a representation π, we study the question of when this form is symmetric and when it is skew-symmetric.

We begin with recalling a method of Prasad [13] in the case that ι is trivial, and G is a finite group of Lie type. We apply a theorem of Klyachko to obtain results for all representations of many of the groups $\mathrm{SL}(n, \mathbb{F}_q)$, improving the results of Prasad, which were for generic representations. In Section 3, we start by generalizing Lemma 1 to the case of any order 2 automorphism ι in Lemma 2. We are able to apply this to get another proof of a theorem of Gow [5] for the group $\mathrm{GL}(n, \mathbb{F}_q)$ and ι the transpose-inverse automorphism, and in Section 4 we are also able to apply Lemma 2 to distinguished representations and Gelfand pairs. In Section 5 we find a generating set for a finite group G given that we know certain information about the representations of G satisfying ${}^{\iota}\pi \cong \hat{\pi}$.

Finally, in Section 6, we turn to the case of a locally compact totally disconnected group. We adapt the methods of Section 2, along with approximation by compact open subgroups, to obtain an equivalent result of Theorem 4 of Gow for $\mathrm{GL}(n, F)$, where F is a nonarchimedean local field.

2. Prasad's lemma and Klyachko's theorem

Prasad's main lemma for determining whether a self-dual representation of a finite group of Lie type is symplectic or orthogonal in [13, Lemma 1] is the following.

Lemma 1. *Let H be a subgroup of a finite group G. Let s be an element of G which normalizes H and whose square belongs to the center of G. Let $\psi : H \to \mathbb{C}^{\times}$ be a one-dimensional representation of H which is taken to its inverse by the inner conjugation action s on H. Let π be an irreducible representation of G in which the character ψ of H appears with multiplicity 1. Then if π is self-dual, it is orthogonal if and only if the element s^2 belonging to the center of G operates by 1 on π.*

Prasad applies this lemma to the case that G is the group of $\mathbb{F}_q$-rational points of a connected reductive algebraic group defined over $\mathbb{F}_q$, H is a unipotent subgroup of G, ψ is a nondegenerate character of the unipotent subgroup, and s is an element of a torus that operates by -1 on all of the simple root spaces of the unipotent subgroup (this s may or may not exist). The fact that the Gelfand-Graev representation is multiplicity free allows Lemma 1 to be applied to make conclusions on irreducible self-dual generic representations in many examples of finite groups of Lie type.

Klyachko [11] obtained a model for $\mathrm{GL}(n, \mathbb{F}_q)$ with the following theorem, for which proofs are also given by Inglis and Saxl [9] and Howlett and Zworestine [8].

Theorem 1. *Let $G = \mathrm{GL}(n, \mathbb{F}_q)$ and consider the subgroups G_k, $0 \leq 2k \leq n$, whose elements are of the form $\begin{pmatrix} U & * \\ & A \end{pmatrix}$, where U is unipotent and $A \in \mathrm{Sp}(2k, \mathbb{F}_q)$. Let $\theta : \mathbb{F}_q \to \mathbb{C}^\times$ be a nontrivial additive character, and define the character $\psi_k : G_k \to \mathbb{C}^*$ by $\psi_k\left(\begin{pmatrix} U & * \\ & A \end{pmatrix}\right) = \theta(\sum_i u_{i,i+1})$, with $U = (u_{i,j})$. Let $T_k = \mathrm{Ind}_{G_k}^G(\psi_k)$. Then every irreducible representation of $G = \mathrm{GL}(n, \mathbb{F}_q)$ occurs as a component with multiplicity one in exactly one of the T_k.*

As Inglis and Saxl [9] point out, the components of each of the T_k are invariant under choice of basis or θ. A change in the nondegenerate character ψ essentially amounts to a change in basis, and so Theorem 1 holds if we replace ψ by any nondegenerate character.

We note that each of the subgroups G_k and characters ψ_k can also be applied to Lemma 1, where s is the element

$$\mathrm{diag}(1, -1, 1, -1, \ldots) = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \end{pmatrix}.$$

Since every irreducible character occurs in one of the representations T_k with multiplicity 1, and $s^2 = 1$, we conclude that any irreducible self-dual representation of $\mathrm{GL}(n, \mathbb{F}_q)$ is orthogonal. Prasad proves this using parabolic induction [13, Theorem 4], and this result is also implied by the main result of Gow in [5, Theorem 2].

We have the following consequence of Theorem 1 for the group $\mathrm{SL}(n, \mathbb{F}_q)$.

Proposition 1. *Let $H = \mathrm{SL}(n, \mathbb{F}_q)$, and let G_k, $0 \leq 2k \leq n$ be the same subgroups as in Theorem 1. Let φ be some nondegenerate character of the unipotent subgroup, and define φ_k to be the character for G_k which acts as*

φ on the unipotent factor of G_k and trivially elsewhere. Define $R(k,\varphi) = \mathrm{Ind}_{G_k}^H(\varphi_k)$. Then:

(i) For any φ and k, $R(k,\varphi)$ is multiplicity free.

(ii) For any irreducible representation π of H, there exists a nondegenerate φ and a k such that π is a constituent of $R(k,\varphi)$.

Proof. Let $G = \mathrm{GL}(n,\mathbb{F}_q)$. (i): From Theorem 1 and the comments after, for any nondegenerate φ, $\mathrm{Ind}_H^G(R(k,\varphi))$ is multiplicity free, and so $R(k,\varphi)$ must also be multiplicity free.

(ii): Every irreducible representation π of $H = \mathrm{SL}(n,\mathbb{F}_q)$ occurs as a constituent of $\mathrm{Res}_H(\rho)$ for some irreducible representation ρ of G. Since every irreducible of G occurs as constituent of one of the T_k by Theorem 1, then every irreducible of H occurs as a constituent of

$$\bigoplus_k \mathrm{Res}_H(T_k) = \bigoplus_k \mathrm{Res}_H \mathrm{Ind}_{G_k}^G(\psi_k).$$

From Mackey theory, we have

$$\bigoplus_k \mathrm{Res}_H \mathrm{Ind}_{G_k}^G(\psi_k) = \bigoplus_k \bigoplus_{x\in\mathbb{F}_q^\times} \mathrm{Ind}_{G_k}^H({}^x\psi_k),$$

where, for each $x \in \mathbb{F}_q^\times$, ${}^x\psi_k$ acts on the unipotent factor of G_k as

$${}^x\psi((u_{i,j})) = \theta(xu_{1,2} + \sum_{i>1} u_{i,i+1}),$$

and acts trivially elsewhere. Then ${}^x\psi$ is a nondegenerate character, and every irreducible π of H occurs as a constituent of $\mathrm{Ind}_{G_k}^H({}^x\psi_k) = R(k,{}^x\psi)$ for some k and some $x \in \mathbb{F}_q^\times$. □

Theorem 2. *If q is even, or $n = 2m+1$ or $n = 4m$, then every irreducible self-dual representation of* $\mathrm{SL}(n,\mathbb{F}_q)$ *is orthogonal. If $n = 4m+2$ and $q \equiv 1 (\mathrm{mod}\ 4)$, then an irreducible self-dual representation of* $\mathrm{SL}(n,\mathbb{F}_q)$ *is orthogonal if and only if the element $-I$ acts trivially.*

Proof. For q even or $n = 2m+1$, it follows from Theorem 3 of [13] that there exists an element s that satisfies Lemma 1, and this s also satisfies $s^2 = I$. For $n = 4m$, we may choose $s = \mathrm{diag}(1,-1,1,-1,\dots)$, as in [13, Section 6]. For $n = 4m+2$ and $q \equiv 1 (\mathrm{mod}\ 4)$, letting $\alpha^2 = -1$, again as in [13, Section 6], we take $s = \mathrm{diag}(\alpha,-\alpha,\alpha,-\alpha,\dots)$. In each of these cases, we apply Lemma 1 with Proposition 1 to obtain results for all self-dual representations. □

Some of the cases of Theorem 2 are covered by Gow [4, Section 3], but by a different method. Turull [18] has found the Schur indices over the rationals of all of the finite special linear groups, while in the cases above, we have just described which real-valued characters have Schur index 1 or 2 over the reals. We find it worthwhile, however, to see this application of Klyachko's Theorem 1 to the finite special linear groups and Prasad's Lemma 1.

3. A twisted version of Prasad's lemma

We now consider the situation when we have an irreducible representation (π, V) of a finite group G such that ${}^{\iota}\pi \cong \hat{\pi}$ for some order 2 automorphism ι of G, where ${}^{\iota}\pi(g) = \pi({}^{\iota}g)$. Let $\hat{V}$ be the dual space of V. The isomorphism ${}^{\iota}\pi \cong \hat{\pi}$ gives rise to an intertwining operator $\Phi : V \to \hat{V}$ satisfying $\Phi \circ {}^{\iota}\pi(g) = \hat{\pi}(g) \circ \Phi$ for every $g \in G$. Defining $B_{\iota}(v, w) = \Phi(v)(w)$, we have that B_{ι} is a bilinear form satisfying

$$B_{\iota}(\pi(g)v,\ {}^{\iota}\pi(g)w) = B_{\iota}(v, w) \text{ for all } g \in G \text{ and } v, w \in V.$$

This bilinear form is unique up to scalar by Schur's lemma, and so it must be either symmetric or skew-symmetric. We also write

$$B_{\iota}(v, w) = \varepsilon_{\iota}(\pi) B_{\iota}(w, v),$$

where $\varepsilon_{\iota}(\pi) = 1$ if B_{ι} is symmetric and $\varepsilon_{\iota}(\pi) = -1$ if it is skew-symmetric. If ι is trivial, we write $\varepsilon_{\iota}(\pi) = \varepsilon(\pi)$, and if ${}^{\iota}\pi \not\cong \hat{\pi}$, we define $\varepsilon_{\iota}(\pi) = 0$. We have the following generalization of Lemma 1.

Lemma 2. *Let H be a subgroup of a finite group G with an order 2 automorphism ι which fixes H. Let $\psi : H \to \mathbb{C}^*$ be a one-dimensional representation of H such that ${}^{\iota}\psi = \psi^{-1}$. Let π be an irreducible representation of G in which the character ψ of H appears with multiplicity 1. Then if ${}^{\iota}\pi \cong \hat{\pi}$, the associated bilinear form is symmetric, that is $\varepsilon_{\iota}(\pi) = 1$.*

Proof. Let $W \subset V$ be the one-dimensional representation space for ψ, and w a nonzero vector in W. When restricting the representation ${}^{\iota}\pi$ to H, the one-dimensional ${}^{\iota}\psi = \psi^{-1}$ appears with multiplicity 1. Let $({}^{\iota}\rho, U)$ be any H-irreducible H-subrepresenation of $({}^{\iota}\pi, V)$ disjoint from $({}^{\iota}\psi, W)$. Then B_{ι} restricted to $W \times U$ must be 0, otherwise we would obtain a nonzero intertwining operator for ${}^{\iota}\psi$ and $\hat{\rho}$, but then ${}^{\iota}\rho \cong \psi^{-1} = {}^{\iota}\psi$. This is impossible since ${}^{\iota}\psi$ appears with multiplicity 1.

Since B_{ι} is nondegenerate on V, but zero when restricted to $W \times U$ for any U as above, we must have B_{ι} nondegenerate on the space W. Now B_{ι} must be symmetric on the one-dimensional space W, since for any $\lambda_1, \lambda_2 \in \mathbb{C}$, we have

$$B_{\iota}(\lambda_1 w, \lambda_2 w) = \lambda_1 \lambda_2 B_{\iota}(w, w) = B_{\iota}(\lambda_2 w, \lambda_1 w).$$

Since B_{ι} is either symmetric or skew-symmetric on all of V, and is symmetric on a nondegenerate subspace, then B_{ι} must be symmetric, so we have $\varepsilon_{\iota}(\pi) = 1$. $\square$

Let ω_{π} denote the central character of the irreducible representation π. The following proposition is what is needed to see how Lemma 1 follows from Lemma 2.

Proposition 2. *Let G be a finite group with κ an order 2 automorphism. Let $s \in G$ be such that $s\,{}^{\kappa}s = z$ for some order 2 central element $z \in G$. Define*

σ to be the order 2 automorphism of G such that ${}^{\sigma}g = {}^{\kappa}(s^{-1}gs)$. Then for any irreducible representation π of G, we have $\varepsilon_\sigma(\pi) = \omega_\pi(z)\varepsilon_\kappa(\pi)$.

Proof. Since ${}^{\sigma}\pi \cong {}^{\kappa}\pi$, then $\varepsilon_\sigma(\pi) \neq 0$ if and only if $\varepsilon_\kappa(\pi) \neq 0$. So we may assume they are both nonzero. Suppose we have the two bilinear forms B_σ and B_κ, unique up to scalar, with the properties as described at the beginning of the section. Define a new bilinear form B'_σ as

$$B'_\sigma(u,v) = B_\kappa(u, \pi({}^{\kappa}s)v).$$

Then we have

$$B'_\sigma(v,u) = B_\kappa(v, \pi({}^{\kappa}s)u) = \omega_\pi(z)B_\kappa(\pi({}^{\kappa}s)v, u) = \omega_\pi(z)\varepsilon_\kappa(\pi)B'_\sigma(u,v).$$

But also we have

$$\begin{aligned} B'_\sigma(u,v) &= B_\kappa(\pi(g)u, {}^{\kappa}\pi(g)\pi({}^{\kappa}s)v) = B_\kappa(\pi(g)u, \pi({}^{\kappa}s)\pi({}^{\sigma}g)v) \\ &= B'_\sigma(\pi(g)u, {}^{\sigma}\pi(g)v). \end{aligned}$$

Therefore, we must have that B'_σ is a scalar multiple of B_σ, and so $\varepsilon_\sigma(\pi) = \omega_\pi(z)\varepsilon_\kappa(\pi)$. □

Now it is clear that Lemma 1 follows from Lemma 2 by taking κ to be trivial and σ to be ι in Proposition 2. We also note that the proofs of Lemma 2 and Proposition 2 do not depend on the group G being finite, and in fact the same proofs allow for G to be compact. Proposition 2 also holds, without change in the proof, for the case that G is a locally compact totally disconnected group and π is an irreducible admissible representation.

Consider the example $G = \mathrm{GL}(n, \mathbb{F}_q)$, and define ι to be the automorphism

$${}^{\iota}g = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} {}^{t}g^{-1} \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Since g is conjugate to ${}^{t}g$ for every $g \in G$, then we have that every irreducible representation π of G satisfies ${}^{\iota}\pi \cong \hat{\pi}$. We apply Lemma 2 with H the unipotent subgroup, and ψ any nondegenerate character, and we obtain that every generic representation π of $\mathrm{GL}(n, \mathbb{F}_q)$ satisfies $\varepsilon_\iota(\pi) = 1$. In fact, we can obtain this result for every irreducible representation of $\mathrm{GL}(n, \mathbb{F}_q)$ by applying the following result of Zelevinsky [21, Proposition 12.5].

Theorem 3. *Let $G = \mathrm{GL}(n, \mathbb{F}_q)$ and H the subgroup of upper triangular unipotent matrices. For any irreducible representation π of G, there is some one-dimensional character ψ of H, possibly degenerate, such that π appears with multiplicity 1 in $\mathrm{Ind}_H^G(\psi)$.*

If we redefine ι as ${}^{\iota}g = {}^{t}g^{-1}$, then by Proposition 2, the $\varepsilon_\iota(\pi)$ remain unchanged. We may also obtain from Theorem 3, as we did in the previous section from Klyachko's Theorem 1, that $\varepsilon(\pi) = 0$ or 1 for every π of G. It is a short computation to check that these two facts together, that is, $\varepsilon(\pi) \geq 0$

and $\varepsilon_\iota(\pi) = 1$ for every π of $G = \mathrm{GL}(n, \mathbb{F}_q)$, is the equivalent information of the following result of Gow [5, Theorem 2].

Theorem 4. *Let $G = \mathrm{GL}(n, \mathbb{F}_q)$, and define G^+ to be the split extension of G by the transpose-inverse involution. That is,*

$$G^+ = \langle G, \tau \mid \tau^2 = 1, \tau^{-1} g \tau = {}^t g^{-1} \text{ for every } g \in G \rangle.$$

Then every irreducible representation of G^+ is self-dual and orthogonal.

So, Lemma 2 and Theorem 3 give us a short alternative proof to Gow's Theorem 4. However, Gow's proof of Theorem 4 does not depend at all on the character theory of $\mathrm{GL}(n, \mathbb{F}_q)$, but rather on the structure of certain subgroups of G^+. Theorem 4 also follows from Klyachko's Theorem 1, but in the more indirect manner of applying a formula for the twisted indicators given in [10].

4. Distinguished representations and Gelfand pairs

A representation (π, V) of a group G is distinguished with respect to a subgroup H if there is a nontrivial space of H-invariant linear forms on (π, V), and we are usually concerned with the space being one-dimensional. Consider the case when $G = \mathbf{G}(\mathbb{F}_{q^2})$ and $H = \mathbf{G}(\mathbb{F}_q)$, where $\mathbf{G}$ is a connected algebraic group defined over $\mathbb{F}_q$. Then, the question becomes whether or not V has an H-invariant vector. A representation of G is called stable if its character values are constant on conjugacy classes which become the same in $\mathbf{G}(\overline{\mathbb{F}}_q)$. Let σ be the automorphism of $\mathbf{G}(\mathbb{F}_{q^2})$ that acts as the Frobenius in $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ on each entry. The following result is proved in [14, Theorem 2]. Related results appear in papers of Gow [6] and Lusztig [12].

Theorem 5. *Let (π, V) be a stable representation of $\mathbf{G}(\mathbb{F}_{q^2})$. Then V has a $\mathbf{G}(\mathbb{F}_q)$-invariant vector if and only if ${}^\sigma\pi \cong \hat{\pi}$.*

Then we may immediately apply Lemma 2 to Theorem 5 and say that the stable distinguished representations of $\mathbf{G}(\mathbb{F}_{q^2})$ are those stable representations satisfying $\varepsilon_\sigma(\pi) = 1$, and $\varepsilon_\sigma(\pi) = 1$ if and only if ${}^\sigma\pi \cong \hat{\pi}$. This is also noted in [16] to come out of the proof of Theorem 5. Kawanaka and Matsuyama [10] prove results like Theorem 5 in a less general form, and note there that the twisted indicators are 1, but the proof is quite different than the proof of Theorem 5 in [14].

Kawanaka and Matsuyama [10] also prove that if G is a group of odd order with order 2 automorphism ι, and H is the subgroup of ι-fixed elements, then $\mathrm{Ind}_H^G(\mathbf{1})$ is multiplicity free and consists of representations π of G such that $\varepsilon_\iota(\pi) = 1$. There is a much less complicated proof of the multiplicity free part of this statement given by Balmaceda [1] using the double coset method of obtaining a Gelfand pair. The next proposition shows that we can still conclude information about the twisted indicators from this proof. The following is a twisted version of [15, Lemma 3], and we follow the proof that Prasad gives there.

Proposition 3. *Let G be a compact group with order 2 automorphism ι, and let H be a closed subgroup such that the anti-involution $g \mapsto {}^{\iota}g^{-1}$ takes every (H,H)-double coset in G to itself. Then every irreducible representation π of G that has an H-fixed vector satisfies ${}^{\iota}\pi \cong \hat{\pi}$, and moreover $\varepsilon_{\iota}(\pi) = 1$.*

Proof. The double coset condition implies that (G,H) is a Gelfand pair, or that for every irreducible π of G, the space of H-fixed vectors has dimension at most 1. If we can show that any irreducible representation whose space of H-fixed vectors has dimension 1 satisfies ${}^{\iota}\pi \cong \hat{\pi}$, then it follows from Lemma 2 that $\varepsilon_{\iota}(\pi) = 1$.

Let $[\cdot,\cdot]$ be a G-invariant Hermitian form for π, and let v be an H-fixed vector. Define the matrix coefficient f by

$$f(g) = [\pi(g)v, v],$$

Then f is constant on (H,H)-double cosets since v is H-fixed. Since the anti-involution $g \mapsto {}^{\iota}g^{-1}$ fixes (H,H)-double cosets, we have $f(g) = f({}^{\iota}g^{-1})$. Now $f({}^{\iota}g^{-1})$ is a matrix coefficient of ${}^{\iota}\hat{\pi}$, and matrix coefficients of representations are either orthogonal, or the representations are isomorphic. So now we have $\pi \cong {}^{\iota}\hat{\pi}$, or ${}^{\iota}\pi \cong \hat{\pi}$. $\square$

There are many examples of Gelfand compact pairs which are obtained by the double coset method, as in [7], all of which are examples where Proposition 3 may be applied.

5. A generating set

Wang and Grove [19, Theorem 3.3] prove that if every irreducible representation of a finite group G is orthogonal, then G is generated by involutions. In this section we generalize this result. The following proposition is a twisted version of the Frobenius-Schur involution formula. It is implicit in the paper of Kawanaka and Matsuyama [10], and is proven in a more general form by Bump and Ginzburg [3, Proposition 1].

Proposition 4. *Let G be a finite group with an order 2 automorphism ι, let z be a central element of G satisfying $z^2 = 1$, and let $\mathrm{Irr}(G)$ be the collection of irreducible characters of G. Then*

$$\sum_{\chi \in \mathrm{Irr}(G)} \varepsilon_{\iota}(\chi)\chi(z) = \left|\{g \in G \mid g\,{}^{\iota}g = z\}\right|.$$

In particular, $\varepsilon_{\iota}(\pi) = \omega_{\pi}(z)$ for every irreducible representation π of G if and only if

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1) = \left|\{g \in G \mid g\,{}^{\iota}g = z\}\right|.$$

The proof of the following theorem uses the same main idea as Wang and Grove of applying Clifford's theorem on the restriction of characters to a normal subgroup.

Theorem 6. *Let G be a finite group, let ι be an order 2 automorphism of G, and let z be a central element of G such that $z^2 = 1$. If $\varepsilon_\iota(\pi) = \omega_\pi(z)$ for every irreducible representation π of G, then*

$$G = \langle g \in G \mid g\, {}^\iota g = z \rangle.$$

Proof. First consider the case $z = 1$. Let $N = \langle g \in G \mid g\, {}^\iota g = 1 \rangle$, so ι fixes N. We first show that N is normal in G. Let g be such that $g\, {}^\iota g = 1$. Then for any $x \in G$, ${}^\iota x g x^{-1}$ has the same property, as

$${}^\iota x g x^{-1}\, {}^\iota({}^\iota x g x^{-1}) = 1.$$

Now note that the element $x\, {}^\iota x^{-1}$ also has the same property. So $x\, {}^\iota x^{-1}$ and ${}^\iota x g x^{-1}$ are both in N, and so their product xgx^{-1} is in N. Since the generators of N remain in N under G-conjugacy, then N is normal.

Let $\psi_1, \ldots, \psi_k \in \mathrm{Irr}(N)$ be representatives of the G-conjugate orbits of characters of N, where $\psi_1 = \mathbf{1}_N$, let $\chi_1, \ldots, \chi_k \in \mathrm{Irr}(G)$ be such that ψ_i is a constituent of $\chi_i|_N$, and let $\{\theta_{ij}\}_{j=1}^{t_i}$ be the orbit under G-conjugation of ψ_i, where $\theta_{i1} = \psi_i$. Since N is normal in G, Clifford's theorem says that

$$\chi_i|_N = \langle \chi_i|_N, \theta \rangle \sum_{j=1}^{t_i} \theta_{ij}.$$

From Proposition 4, since $\varepsilon_\iota(\pi) = 1$ for every irreducible π of G, we have

$$\left|\{g \in G \mid g\, {}^\iota g = 1\}\right| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1).$$

Also we have

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1) \geq \sum_{i=1}^{k} \chi_i(1) = \sum_{i=1}^{k} \langle \chi_i|_N, \psi_i \rangle \sum_{j=1}^{t_i} \theta_{ij}(1)$$

from Clifford's theorem, since $\chi_i(1) = \chi_i|_N(1)$. Since the union of the G-conjugate orbits of the characters of N is all of the characters of N, and $\langle \chi_i|_N, \psi_i \rangle \geq 1$, we have

$$\sum_{i=1}^{k} \langle \chi_i|_N, \psi_i \rangle \sum_{j=1}^{t_i} \theta_{ij}(1) \geq \sum_{\theta \in \mathrm{Irr}(N)} \theta(1) \geq \sum_{\theta \in \mathrm{Irr}(N)} \varepsilon_\iota(\theta)\theta(1)$$

$$= \left|\{g \in N \mid g\, {}^\iota g = 1\}\right| = \left|\{g \in G \mid g\, {}^\iota g = 1\}\right|,$$

which is where we started the string of inequalities, and so we have equality throughout. In particular, $\chi_i|_N = \psi_i$ for every i, and the number of orbit representatives of N is equal to the number of irreducibles of G, and it follows that we must have $G = N = \langle g \in G \mid g\, {}^\iota g = 1 \rangle$.

For general z, choose an $h \in G$ such that $h\, {}^\iota h = z$; such an h exists since the sum of the degrees of the characters of G is equal to the number of such h, by Proposition 4. Define the order 2 automorphism σ of G by

${}^{\sigma}g = {}^{\iota}(h^{-1}gh)$. Then by Proposition 2, $\varepsilon_{\sigma}(\pi) = \omega_{\pi}(z)\varepsilon_{\iota}(\pi) = 1$. By the case above, we have

$$G = \langle g \in G \mid g\, {}^{\sigma}g = 1 \rangle = \langle g \in G \mid gh\, {}^{\iota}(gh) = z \rangle.$$

If x satisfies $xh\, {}^{\iota}(xh) = z$, then since $h\, {}^{\iota}h = z$, we have

$$x \in \langle g \in G \mid g\, {}^{\iota}g = z \rangle.$$

Since a set of generators for G is in this group, we finally have

$$G = \langle g \in G \mid g\, {}^{\iota}g = z \rangle. \ \square$$

6. GL(n) over a local field

If G is a locally compact totally disconnected group with order 2 automorphism ι, and π is an irreducible admissible representation of G satisfying ${}^{\iota}\pi \cong \hat{\pi}$, where $\hat{\pi}$ now denotes the smooth contragredient of π, then we may define $\varepsilon_{\iota}(\pi)$ just as we did for finite or compact groups. The following is the adaptation of Lemma 2 to this situation.

Lemma 3. *Let G be a locally compact totally disconnected group with H a compact open subgroup and ι an order 2 automorphism which fixes H. Let $\tilde{\psi} : H \to \mathbb{C}^{\times}$ be a one-dimensional representation of H such that ${}^{\iota}\tilde{\psi} = \tilde{\psi}^{-1}$. Let π be an irreducible admissible representation of G in which the character $\tilde{\psi}$ of H appears with multiplicity 1. If ${}^{\iota}\pi \cong \hat{\pi}$, then we have $\varepsilon_{\iota}(\pi) = 1$.*

Proof. When we restrict the representation ${}^{\iota}\pi$ to the compact open subgroup H, we have complete reducibility, and ${}^{\iota}\pi$ becomes a direct sum of irreducible representations where ${}^{\iota}\tilde{\psi}$ appears with multiplicity 1. The proof for Lemma 2 then applies without change. $\square$

We now concentrate on the case that $G = \mathrm{GL}(n, F)$, where F is a nonarchimedean local field. We let $\mathcal{O}$ be the ring of integers of F, $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$, and ϖ a uniformizing parameter. Let T be the set of diagonal matrices of G, U the set of upper triangular unipotent elements, and U^{-} the set of lower triangular unipotent elements. Let $\theta : F \to \mathbb{C}^{\times}$ be a nontrivial additive character. Define a character ψ of U by $\psi((u_{ij})) = \theta(\sum_{i=1}^{n-1} \alpha_i u_{i,i+1})$, where each α_i is either 1 or 0. If $\alpha_i = 1$ for every i, then ψ is nondegenerate, and otherwise ψ is degenerate. We recall the following result of Zelevinsky in [20, Corollary 8.3], which is the p-adic version of his result in Theorem 3 above.

Theorem 7. *Let π be an irreducible admissible representation of $G = \mathrm{GL}(n, F)$. Then there is some one dimensional ψ of U, possibly degenerate, such that* $\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G(\psi)) = 1$.

Now let $G_0 = \mathrm{GL}(n, \mathcal{O})$ be the maximal compact subgroup of G, and for each $m \geq 1$, we define $G_m = \{g \in G_0 \mid g \equiv I (\mathrm{mod}\ \mathfrak{p}^m)\}$. Then for each m, we denote $T_m = T \cap G_m$, $U_m = U \cap G_m$, and $U_m^{-} = U^{-} \cap G_m$. Then the Iwahori factorization gives us $G_m = U_m^{-} T_m U_m$ for each $m \geq 1$. Fix the additive character $\theta : F \to \mathbb{C}^\times$ so that $\ker(\theta) = \mathcal{O}$. We now describe the idea of compact approximation of Whittaker models as given by Rodier [17], but we make appropriate changes to include degenerate characters. For each i such that $1 \leq i \leq n-1$, we choose $\alpha_i = 0$ or 1. For $g \in G_m$, write $g = u^- tu$ according to the Iwahori factorization. Define the character χ_m of G_m as

$$\chi_m(u^- tu) = \theta\big(\varpi^{-2m} \sum_{i=1}^{n-1} \alpha_i u_{i,i+1}\big),$$

where $u = (u_{ij}) \in U_m$. Now define $d = \mathrm{diag}(1, \varpi^2, \varpi^4, \ldots, \varpi^{2n})$, and let $H_m = d^m G_m d^{-m}$. We define the character ψ_m of H_m as

$$\psi_m(h) = \chi_m(d^{-m} h d^m).$$

As before, let ψ be the character of U defined as $\psi((u_{ij})) = \theta(\sum_{i=1}^{n-1} \alpha_i u_{i,i+1})$.

Proposition 5. *Let π be an irreducible admissible representation of G, and suppose that $\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G(\psi))$ is finite dimensional. Then there is an integer m_0 such that for all $m \geq m_0$, we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, \mathrm{Ind}_{H_m}^G(\psi_m)) = \dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G(\psi)).$$

Proof. If ψ is nondegenerate, that is if $\alpha_i = 1$ for each i, then this statement is the content of the results of Rodier in [17, Corollaries 2 and 3]. When G is any split reductive p-adic group, Rodier has restrictions on the characteristic and residue characteristic of F, but when $G = \mathrm{GL}(n, F)$, there are no such restrictions, as noted in [17, Section VI]. The only time the fact that ψ is nondegenerate is used in [17] is in the application of the results to the uniqueness of Whittaker models. That is to say, the proposition in fact follows directly from the work of Rodier. $\square$

Finally, we are able to prove the following, which is an equivalent form of Theorem 4 for the general linear group over a nonarchimedean local field. We note that in [15], it is shown that every self-dual generic representation of $\mathrm{GL}(n, F)$ is orthogonal.

Theorem 8. *Let $G = \mathrm{GL}(n, F)$ and let ι be the transpose-inverse automorphism of G. Every irreducible admissible representation π of G satisfies $\varepsilon_\iota(\pi) = 1$, and every self-dual irreducible admissible representation of G is orthogonal.*

Proof. Let π be any irreducible admissible representation of G. First, choose a character ψ of U such that the space $\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G(\psi))$ is one-dimensional, whose existence is guaranteed by Theorem 7. Now let $m \geq m_0$, where m_o is the integer coming from Proposition 5, and let $\tilde{\psi} = \psi_m$. For the moment, redefine ι to be transpose-inverse composed with conjugation by a matrix with 1's on the antidiagonal and 0's elsewhere. Then ι preserves the Iwahori factorization $G_m = U_m^- T_m U_m$, and $\tilde{\psi}$ satisfies ${}^\iota\tilde{\psi} = \tilde{\psi}^{-1}$. From

Proposition 5 and Frobenius reciprocity, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi, \tilde{\psi}) = 1,$$

where $H = H_m$ is a compact open subgroup of G. It is a result of I.M. Gelfand and Kazhdan that any irreducible admissible representation π of G satisfies ${}^{\iota}\pi \cong \hat{\pi}$ (a proof appears in [2, Theorem 7.3]). We may then apply Lemma 3 to conclude that $\varepsilon_\iota(\pi) = 1$. Now ι can be taken to be the transpose-inverse automorphism by Proposition 2 to obtain the desired result.

If π is a self-dual irreducible admissible representation, we choose m and $\tilde{\psi}$ just as before, but now we define the automorphism σ to be conjugation by the element $s = \mathrm{diag}(1, -1, 1, -1, \ldots)$. Then $H = H_m$ is fixed by σ, $\tilde{\psi}$ satisfies ${}^{\sigma}\tilde{\psi} = \tilde{\psi}^{-1}$, and $s^2 = I$. By applying Lemma 3 and Proposition 2, we conclude that π is orthogonal. $\square$

Acknowledgements. The author thanks Dipendra Prasad for very useful suggestions and discussion, U. K. Anandavardhanan for several helpful conversations, and the referee for comments and a careful reading.

References

1. J.M.P. Balmaceda, A note on multiplicity-free permutation characters, Graph Theory and Combinatorics (Manila, 1991), *Discrete Math.* **151** (1996), no. 1–3, 55–58.
2. J. Bernstein and A.V. Zelevinsky, Representations of the group $GL(n, F)$ where F is a non-archimedean local field, *Russian Math. Surveys* **3** (1976), 1–68.
3. D. Bump and D. Ginzburg, Generalized Frobenius-Schur numbers, *J. Algebra* **278** (2004), no. 1, 294–313.
4. R. Gow, On the Schur indices of characters of finite classical groups, *J. London Math. Soc.* (2) **24** (1981), no. 1, 135–147.
5. R. Gow, Properties of the characters of the finite general linear group related to the transpose-inverse involution, *Proc. London Math. Soc. (3)* **47** (1983), no. 3, 493–506.
6. R. Gow, Two multiplicity-free representations of the general linear group $\mathrm{GL}(n, q^2)$, *Math. Z.* **188** (1984), no. 1, 45–54.
7. B. Gross, Some applications of Gelfand pairs to number theory, *Bull. Amer. Math. Soc. (N.S.)* **24** (1991), no. 2, 277–301.
8. R.B. Howlett and C. Zworestine, On Klyachko's model for the representations of finite general linear groups, In *Representations and quantizations (Shanghai, 1998)*, pages 229–245, China High. Educ. Press, Beijing, 2000.
9. N.F.J. Inglis and J. Saxl, An explicit model for the complex representations of the finite general linear groups, *Arch. Math. (Basel)* **57** (1991), no. 5, 424–431.
10. N. Kawanaka and H. Matsuyama, A twisted version of the Frobenius-Schur indicator and multiplicity-free representations, *Hokkaido Math. J.* **19** (1990), no. 3, 495–508.
11. A. A. Klyachko, Models for complex representations of the groups $\mathrm{GL}(n,\, q)$, *Mat. Sb. (N.S.)* **120(162)** (1983), no. 3, 371–386.
12. G. Lusztig, $G(F_q)$-invariants in irreducible $G(F_{q^2})$-modules, *Represent. Theory* **4** (2000), 446–465.
13. D. Prasad, On the self-dual representations of finite groups of Lie type, *J. Algebra* **210** (1998), no. 1, 298–310.
14. D. Prasad, Distinguished representations for quadratic extensions, *Compositio Math.* **119** (1999), no. 3, 335–345.

15. D. Prasad, On the self-dual representations of a p-adic group, *Internat. Math. Res. Notices* (1999), no. 8, 443–452.
16. D. Prasad and D. Ramakrishnan, On the self-dual representations of division algebras over local fields, Preprint, 2004.
17. F. Rodier, Modèle de Whittaker et charactères de représentations, in Noncommutative Harmonic Analysis (Actes Colloq., Marseille-Luminy, 1974), *Lecture Notes in Mathematics* **466**, Springer–Verlag, Berlin, 1975, 151–171.
18. A. Turull, The Schur indices of the irreducible characters of the special linear groups, *J. Algebra* **235** (2001), no. 1, 275–314.
19. K. S. Wang and L. C. Grove, Realizability of representations of finite groups, *J. Pure Appl. Algebra* 54 (1988), no. 2–3, 299–310.
20. A.V. Zelevinsky, Induced representations of reductive $\mathfrak{p}$-adic groups II. On irreducible representations of $\mathrm{GL}(n)$, *Ann. Sci. École Norm. Sup. (4)* **13** (1980), no. 2, 165–210.
21. A.V. Zelevinsky, Representations of finite classical groups. A Hopf algebra approach, *Lecture Notes in Mathematics* **869**, Springer–Verlag, Berlin–New York, 1981.

Department of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India

E-mail address: vinroot@math.tifr.res.in