1. Introduction

For a group $G$ with order 2 automorphism $\iota$, we consider irreducible representations $(\pi, V)$ of $G$ such that $\iota^* \pi \cong \hat{\pi}$, where $\hat{\pi}$ is the contragredient representation of $\pi$. This isomorphism gives rise to a nondegenerate bilinear form on $V$ which is $(\pi, \iota^* \pi)$-invariant, unique up to scalar, and consequently is either symmetric or skew-symmetric. For such a representation $\pi$, we study the question of when this form is symmetric and when it is skew-symmetric.

We begin with recalling a method of Prasad [13] in the case that $\iota$ is trivial, and $G$ is a finite group of Lie type. We apply a theorem of Klyachko to obtain results for all representations of many of the groups $\text{SL}(n, \mathbb{F}_q)$, improving the results of Prasad, which were for generic representations. In Section 3, we start by generalizing Lemma 1 to the case of any order 2 automorphism $\iota$ in Lemma 2. We are able to apply this to get another proof of a theorem of Gow [5] for the group $\text{GL}(n, \mathbb{F}_q)$ and $\iota$ the transpose-inverse automorphism, and in Section 4 we are also able to apply Lemma 2 to distinguished representations and Gelfand pairs. In Section 5 we find a generating set for a finite group $G$ given that we know certain information about the representations of $G$ satisfying $\iota^* \pi \cong \hat{\pi}$.

Finally, in Section 6, we turn to the case of a locally compact totally disconnected group. We adapt the methods of Section 2, along with approximation by compact open subgroups, to obtain an equivalent result of Theorem 4 of Gow for $\text{GL}(n, F)$, where $F$ is a nonarchimedean local field.

2. Prasad’s lemma and Klyachko’s theorem

Prasad’s main lemma for determining whether a self-dual representation of a finite group of Lie type is symplectic or orthogonal in [13, Lemma 1] is the following.

Lemma 1. Let $H$ be a subgroup of a finite group $G$. Let $s$ be an element of $G$ which normalizes $H$ and whose square belongs to the center of $G$. Let $\psi : H \to \mathbb{C}^\times$ be a one-dimensional representation of $H$ which is taken to its inverse by the inner conjugation action $s$ on $H$. Let $\pi$ be an irreducible representation of $G$ in which the character $\psi$ of $H$ appears with multiplicity 1. Then if $\pi$ is self-dual, it is orthogonal if and only if the element $s^2$ belonging to the center of $G$ operates by 1 on $\pi$. 

Prasad applies this lemma to the case that $G$ is the group of $\mathbb{F}_q$-rational points of a connected reductive algebraic group defined over $\mathbb{F}_q$, $H$ is a unipotent subgroup of $G$, $\psi$ is a nondegenerate character of the unipotent subgroup, and $s$ is an element of a torus that operates by $-1$ on all of the simple root spaces of the unipotent subgroup (this $s$ may or may not exist). The fact that the Gelfand-Graev representation is multiplicity free allows Lemma 1 to be applied to make conclusions on irreducible self-dual generic representations in many examples of finite groups of Lie type.

Klyachko [11] obtained a model for $GL(n, \mathbb{F}_q)$ with the following theorem, for which proofs are also given by Inglis and Saxl [9] and Howlett and Zworestine [8].

**Theorem 1.** Let $G = GL(n, \mathbb{F}_q)$ and consider the subgroups $G_k$, $0 \leq 2k \leq n$, whose elements are of the form $\left( \begin{array}{cc} U & \ast \\ A & \ast \end{array} \right)$, where $U$ is unipotent and $A \in Sp(2k, \mathbb{F}_q)$. Let $\theta : \mathbb{F}_q \to \mathbb{C}^\times$ be a nontrivial additive character, and define the character $\psi_k : G_k \to \mathbb{C}^\times$ by $\psi_k\left( \left( \begin{array}{cc} U & \ast \\ A & \ast \end{array} \right) \right) = \theta(\sum_i u_{i,i+1})$, with $U = (u_{i,j})$. Let $T_k = \text{Ind}_{G_k}^G(\psi_k)$. Then every irreducible representation of $G = GL(n, \mathbb{F}_q)$ occurs as a component with multiplicity one in exactly one of the $T_k$.

As Inglis and Saxl [9] point out, the components of each of the $T_k$ are invariant under choice of basis or $\theta$. A change in the nondegenerate character $\psi$ essentially amounts to a change in basis, and so Theorem 1 holds if we replace $\psi$ by any nondegenerate character.

We note that each of the subgroups $G_k$ and characters $\psi_k$ can also be applied to Lemma 1, where $s$ is the element

$$\text{diag}(1, -1, 1, -1, \ldots) = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \end{pmatrix}.$$  

Since every irreducible character occurs in one of the representations $T_k$ with multiplicity 1, and $s^2 = 1$, we conclude that any irreducible self-dual representation of $GL(n, \mathbb{F}_q)$ is orthogonal. Prasad proves this using parabolic induction [13, Theorem 4], and this result is also implied by the main result of Gow in [5, Theorem 2].

We have the following consequence of Theorem 1 for the group $SL(n, \mathbb{F}_q)$.

**Proposition 1.** Let $H = SL(n, \mathbb{F}_q)$, and let $G_k$, $0 \leq 2k \leq n$ be the same subgroups as in Theorem 1. Let $\varphi$ be some nondegenerate character of the unipotent subgroup, and define $\varphi_k$ to be the character for $G_k$ which acts as
\( \varphi \) on the unipotent factor of \( G_k \) and trivially elsewhere. Define \( R(k, \varphi) = \text{Ind}_{G_k}^H(\varphi_k) \). Then:

(i) For any \( \varphi \) and \( k \), \( R(k, \varphi) \) is multiplicity free.

(ii) For any irreducible representation \( \pi \) of \( H \), there exists a nondegenerate \( \varphi \) and a \( k \) such that \( \pi \) is a constituent of \( R(k, \varphi) \).

**Proof.** Let \( G = \text{GL}(n, \mathbb{F}_q) \). (i): From Theorem 1 and the comments after, for any nondegenerate \( \varphi \), \( \text{Ind}_H^G(R(k, \varphi)) \) is multiplicity free, and so \( R(k, \varphi) \) must also be multiplicity free.

(ii): Every irreducible representation \( \pi \) of \( H = \text{SL}(n, \mathbb{F}_q) \) occurs as a constituent of \( \text{Res}_H^G(\rho) \) for some irreducible representation \( \rho \) of \( G \). Since every irreducible of \( G \) occurs as constituent of one of the \( T_k \) by Theorem 1, then every irreducible of \( H \) occurs as a constituent of \( \bigoplus_k \text{Res}_H^G(T_k) = \bigoplus_k \text{Res}_H^G(\psi_k) \), where, for each \( x \in \mathbb{F}_q^\times \), \( x \psi_k \) acts on the unipotent factor of \( G_k \) as

\[
(x \psi_k)((u_{i,j})) = \theta(xu_{1,2} + \sum_{i>1} u_{i,i+1}),
\]

and acts trivially elsewhere. Then \( x \psi \) is a nondegenerate character, and every irreducible \( \pi \) of \( H \) occurs as a constituent of \( \text{Ind}_{G_k}^H(x \psi_k) = R(k, x \psi) \) for some \( k \) and some \( x \in \mathbb{F}_q^\times \).

**Theorem 2.** If \( q \) is even, or \( n = 2m + 1 \) or \( n = 4m \), then every irreducible self-dual representation of \( \text{SL}(n, \mathbb{F}_q) \) is orthogonal. If \( n = 4m + 2 \) and \( q \equiv 1(\text{mod } 4) \), then an irreducible self-dual representation of \( \text{SL}(n, \mathbb{F}_q) \) is orthogonal if and only if the element \(-I\) acts trivially.

**Proof.** For \( q \) even or \( n = 2m + 1 \), it follows from Theorem 3 of [13] that there exists an element \( s \) that satisfies Lemma 1, and this \( s \) also satisfies \( s^2 = I \). For \( n = 4m \), we may choose \( s = \text{diag}(1, -1, 1, -1, \ldots) \), as in [13, Section 6]. For \( n = 4m + 2 \) and \( q \equiv 1(\text{mod } 4) \), letting \( \alpha^2 = -1 \), again as in [13, Section 6], we take \( s = \text{diag}(\alpha, -\alpha, \alpha, -\alpha, \ldots) \). In each of these cases, we apply Lemma 1 with Proposition 1 to obtain results for all self-dual representations.

Some of the cases of Theorem 2 are covered by Gow [4, Section 3], but by a different method. Turull [18] has found the Schur indices over the rationals of all of the finite special linear groups, while in the cases above, we have just described which real-valued characters have Schur index 1 or 2 over the reals. We find it worthwhile, however, to see this application of Klyachko’s Theorem 1 to the finite special linear groups and Prasad’s Lemma 1.
3. A twisted version of Prasad’s lemma

We now consider the situation when we have an irreducible representation \((\pi, V)\) of a finite group \(G\) such that \(\iota \pi \cong \pi\) for some order 2 automorphism \(\iota\) of \(G\), where \(\iota \pi(\gamma) = \pi(\iota(\gamma))\). Let \(\hat{V}\) be the dual space of \(V\). The isomorphism \(\iota \pi \cong \pi\) gives rise to a intertwining operator \(\Phi : V \rightarrow \hat{V}\) satisfying \(\Phi \circ \iota \pi(g) = \pi(g) \circ \Phi\) for every \(g \in G\). Defining \(B_i(v, w) = \Phi(v)(w)\), we have that \(B_i\) is a bilinear form satisfying

\[
B_i(\pi(g)v, \iota \pi(g)w) = B_i(v, w) \quad \text{for all } g \in G \text{ and } v, w \in V.
\]

This bilinear form is unique up to scalar by Schur’s lemma, and so it must be either symmetric or skew-symmetric. We also write

\[
B_i(v, w) = \varepsilon_i(\pi)B_i(w, v),
\]

where \(\varepsilon_i(\pi) = 1\) if \(B_i\) is symmetric and \(\varepsilon_i(\pi) = -1\) if it is skew-symmetric. If \(\iota\) is trivial, we write \(\varepsilon_i(\pi) = \varepsilon(\pi)\), and if \(\iota \pi \cong \pi\), we define \(\varepsilon_i(\pi) = 0\). We have the following generalization of Lemma 1.

**Lemma 2.** Let \(H\) be a subgroup of a finite group \(G\) with an order 2 automorphism \(\iota\) which fixes \(H\). Let \(\psi : H \rightarrow \mathbb{C}^*\) be a one-dimensional representation of \(H\) such that \(\iota \psi = \psi^{-1}\). Let \(\pi\) be an irreducible representation of \(G\) in which the character \(\psi\) of \(H\) appears with multiplicity 1. Then if \(\iota \pi \cong \pi\), the associated bilinear form is symmetric, that is \(\varepsilon_i(\pi) = 1\).

**Proof.** Let \(W \subset V\) be the one-dimensional representation space for \(\psi\), and \(w\) a nonzero vector in \(W\). When restricting the representation \(\iota \pi\) to \(H\), the one-dimensional \(\iota \psi = \psi^{-1}\) appears with multiplicity 1. Let \((\iota \rho, U)\) be any \(H\)-irreducible \(H\)-subrepresentation of \((\iota \pi, V)\) disjoint from \((\iota \psi, W)\). Then \(B_i\) restricted to \(W \times U\) must be 0, otherwise we would obtain a nonzero intertwining operator for \(\iota \psi\) and \(\rho\), but then \(\iota \rho \cong \psi^{-1} = \iota \psi\). This is impossible since \(\iota \psi\) appears with multiplicity 1.

Since \(B_i\) is nondegenerate on \(V\), but zero when restricted to \(W \times U\) for any \(U\) as above, we must have \(B_i\) nondegenerate on the space \(W\). Now \(B_i\) must be symmetric on the one-dimensional space \(W\), since for any \(\lambda_1, \lambda_2 \in \mathbb{C}\), we have

\[
B_i(\lambda_1 w, \lambda_2 w) = \lambda_1 \lambda_2 B_i(w, w) = B_i(\lambda_2 w, \lambda_1 w).
\]

Since \(B_i\) is either symmetric or skew-symmetric on all of \(V\), and is symmetric on a nondegenerate subspace, then \(B_i\) must be symmetric, so we have \(\varepsilon_i(\pi) = 1\).

Let \(\omega_\pi\) denote the central character of the irreducible representation \(\pi\). The following proposition is what is needed to see how Lemma 1 follows from Lemma 2.

**Proposition 2.** Let \(G\) be a finite group with \(\kappa\) an order 2 automorphism. Let \(s \in G\) be such that \(s^{-1}s = z\) for some order 2 central element \(z \in G\). Define
σ to be the order 2 automorphism of G such that \( \sigma g = \kappa(s^{-1}gs) \). Then for any irreducible representation \( \pi \) of G, we have \( \varepsilon_\sigma(\pi) = \omega_\pi(z)\varepsilon_\kappa(\pi) \).

**Proof.** Since \( \sigma \pi \cong \kappa \pi \), then \( \varepsilon_\sigma(\pi) \neq 0 \) if and only if \( \varepsilon_\kappa(\pi) \neq 0 \). So we may assume they are both nonzero. Suppose we have the two bilinear forms \( B_\sigma \) and \( B_\kappa \), unique up to scalar, with the properties as described at the beginning of the section. Define a new bilinear form \( B'_\sigma \) as

\[
B'_\sigma(u, v) = B_\kappa(u, \pi(\kappa s)v).
\]

Then we have

\[
B'_\sigma(v, u) = B_\kappa(v, \pi(\kappa s)u) = \omega_\pi(z)B_\kappa(\pi(\kappa s)v, u) = \omega_\pi(z)\varepsilon_\kappa(\pi)B'_\sigma(u, v).
\]

But also we have

\[
B'_\sigma(u, v) = B_\kappa(\pi(g)u, \pi(\kappa s)\pi(\kappa s)v) = B_\kappa(\pi(g)u, \pi(\kappa s)\pi(\kappa g)v) = B'_\sigma(\pi(g)u, \pi(\kappa g)v).
\]

Therefore, we must have that \( B'_\sigma \) is a scalar multiple of \( B_\sigma \), and so \( \varepsilon_\sigma(\pi) = \omega_\pi(z)\varepsilon_\kappa(\pi) \). \( \Box \)

Now it is clear that Lemma 1 follows from Lemma 2 by taking \( \kappa \) to be trivial and \( \sigma \) to be \( \iota \) in Proposition 2. We also note that the proofs of Lemma 2 and Proposition 2 do not depend on the group \( G \) being finite, and in fact the same proofs allow for \( G \) to be compact. Proposition 2 also holds, without change in the proof, for the case that \( G \) is a locally compact totally disconnected group and \( \pi \) is an irreducible admissible representation.

Consider the example \( G = \text{GL}(n, \mathbb{F}_q) \), and define \( \iota \) to be the automorphism

\[
\iota g = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \begin{pmatrix} t & & \\ & \ddots & \\ & & 1 \end{pmatrix}.
\]

Since \( g \) is conjugate to \( \iota g \) for every \( g \in G \), then we have that every irreducible representation \( \pi \) of \( G \) satisfies \( \iota \pi \cong \pi \). We apply Lemma 2 with \( H \) the unipotent subgroup, and \( \psi \) any nondegenerate character, and we obtain that every generic representation \( \pi \) of \( \text{GL}(n, \mathbb{F}_q) \) satisfies \( \varepsilon_\iota(\pi) = 1 \). In fact, we can obtain this result for every irreducible representation of \( \text{GL}(n, \mathbb{F}_q) \) by applying the following result of Zelevinsky [21, Proposition 12.5].

**Theorem 3.** Let \( G = \text{GL}(n, \mathbb{F}_q) \) and \( H \) the subgroup of upper triangular unipotent matrices. For any irreducible representation \( \pi \) of \( G \), there is some one-dimensional character \( \psi \) of \( H \), possibly degenerate, such that \( \pi \) appears with multiplicity 1 in \( \text{Ind}^G_H(\psi) \).

If we redefine \( \iota \) as \( \iota g = \iota g^{-1} \), then by Proposition 2, the \( \varepsilon_\iota(\pi) \) remain unchanged. We may also obtain from Theorem 3, as we did in the previous section from Klyachko’s Theorem 1, that \( \varepsilon(\pi) = 0 \) or 1 for every \( \pi \) of \( G \). It is a short computation to check that these two facts together, that is, \( \varepsilon(\pi) \geq 0 \).
and \( \varepsilon_\iota(\pi) = 1 \) for every \( \pi \) of \( G = \text{GL}(n, \mathbb{F}_q) \), is the equivalent information of the following result of Gow [5, Theorem 2].

**Theorem 4.** Let \( G = \text{GL}(n, \mathbb{F}_q) \), and define \( G^+ \) to be the split extension of \( G \) by the transpose-inverse involution. That is,
\[
G^+ = \langle G, \tau \mid \tau^2 = 1, \tau^{-1}g\tau = \tau^{-1}g^{-1} \text{ for every } g \in G \rangle.
\]
Then every irreducible representation of \( G^+ \) is self-dual and orthogonal.

So, Lemma 2 and Theorem 3 give us a short alternative proof to Gow’s Theorem 4. However, Gow’s proof of Theorem 4 does not depend at all on the character theory of \( \text{GL}(n, \mathbb{F}_q) \), but rather on the structure of certain subgroups of \( G^+ \). Theorem 4 also follows from Klyachko’s Theorem 1, but in the more indirect manner of applying a formula for the twisted indicators given in [10].

4. **DISTINGUISHED REPRESENTATIONS AND GELFAND PAIRS**

A representation \((\pi, V)\) of a group \( G \) is distinguished with respect to a subgroup \( H \) if there is a nontrivial space of \( H \)-invariant linear forms on \((\pi, V)\), and we are usually concerned with the space being one-dimensional. Consider the case when \( G = \text{GL}(\mathbb{F}_{q^2}) \) and \( H = \text{GL}(\mathbb{F}_q) \), where \( G \) is a connected algebraic group defined over \( \mathbb{F}_q \). Then, the question becomes whether or not \( V \) has an \( H \)-invariant vector. A representation of \( G \) is called stable if its character values are constant on conjugacy classes which become the same in \( \text{GL}(\mathbb{F}_{q^2}) \). Let \( \sigma \) be the automorphism of \( \text{GL}(\mathbb{F}_{q^2}) \) that acts as the Frobenius in \( \text{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) \) on each entry. The following result is proved in [14, Theorem 2]. Related results appear in papers of Gow [6] and Lusztig [12].

**Theorem 5.** Let \((\pi, V)\) be a stable representation of \( \text{GL}(\mathbb{F}_{q^2}) \). Then \( V \) has a \( \text{GL}(\mathbb{F}_q) \)-invariant vector if and only if \( \sigma \pi \cong \hat{\pi} \).

Then we may immediately apply Lemma 2 to Theorem 5 and say that the stable distinguished representations of \( \text{GL}(\mathbb{F}_{q^2}) \) are those stable representations satisfying \( \varepsilon_\sigma(\pi) = 1 \), and \( \varepsilon_\sigma(\pi) = 1 \) if and only if \( \sigma \pi \cong \hat{\pi} \). This is also noted in [16] to come out of the proof of Theorem 5. Kawanaka and Matsuyama [10] prove results like Theorem 5 in a less general form, and note there that the twisted indicators are 1, but the proof is quite different than the proof of Theorem 5 in [14].

Kawanaka and Matsuyama [10] also prove that if \( G \) is a group of odd order with order 2 automorphism \( \iota \), and \( H \) is the subgroup of \( \iota \)-fixed elements, then \( \text{Ind}_H^G(1) \) is multiplicity free and consists of representations \( \pi \) of \( G \) such that \( \varepsilon_\iota(\pi) = 1 \). There is a much less complicated proof of the multiplicity free part of this statement given by Balmaceda [1] using the double coset method of obtaining a Gelfand pair. The next proposition shows that we can still conclude information about the twisted indicators from this proof. The following is a twisted version of [15, Lemma 3], and we follow the proof that Prasad gives there.
Proposition 3. Let $G$ be a compact group with order 2 automorphism $\iota$, and let $H$ be a closed subgroup such that the anti-involution $g \mapsto g^{-1}$ takes every $(H,H)$-double coset in $G$ to itself. Then every irreducible representation $\pi$ of $G$ that has an $H$-fixed vector satisfies $\iota \pi \cong \hat{\pi}$, and moreover $\varepsilon_\iota(\pi) = 1$.

Proof. The double coset condition implies that $(G,H)$ is a Gelfand pair, or that for every irreducible $\pi$ of $G$, the space of $H$-fixed vectors has dimension at most 1. If we can show that any irreducible representation whose space of $H$-fixed vectors has dimension 1 satisfies $\iota \pi \cong \hat{\pi}$, then it follows from Lemma 2 that $\varepsilon_\iota(\pi) = 1$.

Let $[\cdot,\cdot]$ be a $G$-invariant Hermitian form for $\pi$, and let $v$ be an $H$-fixed vector. Define the matrix coefficient $f$ by

$$f(g) = [\pi(g)v,v].$$

Then $f$ is constant on $(H,H)$-double cosets since $v$ is $H$-fixed. Since the anti-involution $g \mapsto g^{-1}$ fixes $(H,H)$-double cosets, we have $f(g) = f((g^{-1})$. Now $f(g^{-1})$ is a matrix coefficient of $\iota \hat{\pi}$, and matrix coefficients of representations are either orthogonal, or the representations are isomorphic. So now we have $\pi \cong \iota \hat{\pi}$, or $\iota \pi \cong \hat{\pi}$. $\square$

There are many examples of Gelfand compact pairs which are obtained by the double coset method, as in [7], all of which are examples where Proposition 3 may be applied.

5. A generating set

Wang and Grove [19, Theorem 3.3] prove that if every irreducible representation of a finite group $G$ is orthogonal, then $G$ is generated by involutions. In this section we generalize this result. The following proposition is a twisted version of the Frobenius-Schur involution formula. It is implicit in the paper of Kawanaka and Matsuyama [10], and is proven in a more general form by Bump and Ginzburg [3, Proposition 1].

Proposition 4. Let $G$ be a finite group with an order 2 automorphism $\iota$, let $z$ be a central element of $G$ satisfying $z^2 = 1$, and let $\text{Irr}(G)$ be the collection of irreducible characters of $G$. Then

$$\sum_{\chi \in \text{Irr}(G)} \varepsilon_\iota(\chi) \chi(z) = |\{g \in G \mid g \iota g = z\}|.$$

In particular, $\varepsilon_\iota(\pi) = \omega_\pi(z)$ for every irreducible representation $\pi$ of $G$ if and only if

$$\sum_{\chi \in \text{Irr}(G)} \chi(1) = |\{g \in G \mid g \iota g = z\}|.$$

The proof of the following theorem uses the same main idea as Wang and Grove of applying Clifford’s theorem on the restriction of characters to a normal subgroup.
Theorem 6. Let $G$ be a finite group, let $\iota$ be an order 2 automorphism of $G$, and let $z$ be a central element of $G$ such that $z^2 = 1$. If $\varepsilon_i(\pi) = \omega_\pi(z)$ for every irreducible representation $\pi$ of $G$, then $G = \langle g \in G \mid g^zg = z \rangle$.

Proof. First consider the case $z = 1$. Let $N = \langle g \in G \mid g^zg = 1 \rangle$, so $\iota$ fixes $N$. We first show that $N$ is normal in $G$. Let $g$ be such that $g^zg = 1$. Then for any $x \in G$, $^txgx^{-1}$ has the same property, as $^txgx^{-1}i(^txgx^{-1}) = 1$.

Now note that the element $x^zix^{-1}$ also has the same property. So $x^zix^{-1}$ and $^txgx^{-1}$ are both in $N$, and so their product $xgx^{-1}$ is in $N$. Since the generators of $N$ remain in $N$ under $G$-conjugacy, then $N$ is normal.

Let $\psi_1, \ldots, \psi_k \in \text{Irr}(N)$ be representatives of the $G$-conjugate orbits of characters of $N$, where $\psi_1 = 1_N$, let $\chi_1, \ldots, \chi_k \in \text{Irr}(G)$ be such that $\psi_i$ is a constituent of $\chi_i|_N$, and let $\{\theta_{ij}\}_{j=1}^k$ be the orbit under $G$-conjugation of $\psi_i$, where $\theta_{i1} = \psi_i$. Since $N$ is normal in $G$, Clifford’s theorem says that

$$\chi_i|_N = \langle \chi_i|_N, \theta \rangle \sum_{j=1}^{t_i} \theta_{ij}.$$ 

From Proposition 4, since $\varepsilon_i(\pi) = 1$ for every irreducible $\pi$ of $G$, we have

$$\left| \{ g \in G \mid g^i g = 1 \} \right| = \sum_{\chi \in \text{Irr}(G)} \chi(1).$$

Also we have

$$\sum_{\chi \in \text{Irr}(G)} \chi(1) \geq \sum_{i=1}^k \chi_i(1) = \sum_{i=1}^k \langle \chi_i|_N, \psi_i \rangle \sum_{j=1}^{t_i} \theta_{ij}(1)$$

from Clifford’s theorem, since $\chi_i(1) = \chi_i|_N(1)$. Since the union of the $G$-conjugate orbits of the characters of $N$ is all of the characters of $N$, and $\langle \chi_i|_N, \psi_i \rangle \geq 1$, we have

$$\sum_{i=1}^k \langle \chi_i|_N, \psi_i \rangle \sum_{j=1}^{t_i} \theta_{ij}(1) \geq \sum_{\theta \in \text{Irr}(N)} \theta(1) \geq \sum_{\theta \in \text{Irr}(N)} \varepsilon_i(\theta)\theta(1)$$

$$= \left| \{ g \in N \mid g^i g = 1 \} \right| = \left| \{ g \in G \mid g^i g = 1 \} \right|,$$

which is where we started the string of inequalities, and so we have equality throughout. In particular, $\chi_i|_N = \psi_i$ for every $i$, and the number of orbit representatives of $N$ is equal to the number of irreducibles of $G$, and it follows that we must have $G = N = \langle g \in G \mid g^zg = 1 \rangle$.

For general $z$, choose an $h \in G$ such that $h^zz = z$; such an $h$ exists since the sum of the degrees of the characters of $G$ is equal to the number of such $h$, by Proposition 4. Define the order 2 automorphism $\sigma$ of $G$ by
σg = ^h^{-1}gh). Then by Proposition 2, \( \varepsilon_\sigma(\pi) = \omega_\pi(z)\varepsilon_\iota(\pi) = 1 \). By the case above, we have
\[
G = \langle g \in G \mid g \sigma g = 1 \rangle = \langle g \in G \mid gh ^h(gh) = z \rangle.
\]
If \( x \) satisfies \( xh ^h(xh) = z \), then since \( ^h = z \), we have
\[
x \in \langle g \in G \mid g ^h g = z \rangle.
\]
Since a set of generators for \( G \) is in this group, we finally have
\[
G = \langle g \in G \mid g ^h g = z \rangle.
\]

6. \( \text{GL}(n) \) over a local field

If \( G \) is a locally compact totally disconnected group with order 2 automorphism \( \iota \), and \( \pi \) is an irreducible admissible representation of \( G \) satisfying \( ^\iota \pi \cong \hat{\pi} \), where \( \hat{\pi} \) now denotes the smooth contragredient of \( \pi \), then we may define \( \varepsilon_\iota(\pi) \) just as we did for finite or compact groups. The following is the adaptation of Lemma 2 to this situation.

**Lemma 3.** Let \( G \) be a locally compact totally disconnected group with \( H \) a compact open subgroup and \( \iota \) an order 2 automorphism which fixes \( H \). Let \( \psi : H \to \mathbb{C}^\times \) be a one-dimensional representation of \( H \) such that \( ^\iota \psi = \psi^{-1} \).

Let \( \pi \) be an irreducible admissible representation of \( G \) in which the character \( \psi \) of \( H \) appears with multiplicity 1. If \( ^\iota \pi \cong \hat{\pi} \), then we have \( \varepsilon_\iota(\pi) = 1 \).

**Proof.** When we restrict the representation \( ^\iota \pi \) to the compact open subgroup \( H \), we have complete reducibility, and \( ^\iota \pi \) becomes a direct sum of irreducible representations where \( ^\iota \psi \) appears with multiplicity 1. The proof for Lemma 2 then applies without change.

We now concentrate on the case that \( G = \text{GL}(n,F) \), where \( F \) is a nonarchimedean local field. We let \( \mathcal{O} \) be the ring of integers of \( F \), \( p \) the maximal ideal of \( \mathcal{O} \), and \( \omega \) a uniformizing parameter. Let \( T \) be the set of diagonal matrices of \( G \), \( U \) the set of upper triangular unipotent elements, and \( U^- \) the set of lower triangular unipotent elements. Let \( \theta : F \to \mathbb{C}^\times \) be a nontrivial additive character. Define a character \( \psi \) of \( U \) by \( \psi((u_{ij})) = \theta(\sum_{i=1}^{n-1} \alpha_i u_{i,i+1}) \), where each \( \alpha_i \) is either 1 or 0. If \( \alpha_i = 1 \) for every \( i \), then \( \psi \) is nondegenerate, and otherwise \( \psi \) is degenerate. We recall the following result of Zelevinsky in [20, Corollary 8.3], which is the \( p \)-adic version of his result in Theorem 3 above.

**Theorem 7.** Let \( \pi \) be an irreducible admissible representation of \( G = \text{GL}(n,F) \). Then there is some one dimensional \( \psi \) of \( U \), possibly degenerate, such that \( \dim \mathbb{C} \text{Hom}_G(\pi, \text{Ind}_U^G(\psi)) = 1 \).

Now let \( G_0 = \text{GL}(n,\mathcal{O}) \) be the maximal compact subgroup of \( G \), and for each \( m \geq 1 \), we define \( G_m = \{ g \in G_0 \mid g \equiv I(\text{mod } p^m) \} \). Then for each \( m \), we denote \( T_m = T \cap G_m \), \( U_m = U \cap G_m \), and \( U^-_m = U^- \cap G_m \). Then the Iwahori factorization gives us \( G_m = U^-_m T_m U_m \) for each \( m \geq 1 \). Fix the
additive character \( \theta : F \to \mathbb{C}^\times \) so that ker(\( \theta \)) = \( \mathcal{O} \). We now describe the idea of compact approximation of Whittaker models as given by Rodier [17], but we make appropriate changes to include degenerate characters. For each \( i \) such that \( 1 \leq i \leq n - 1 \), we choose \( \alpha_i = 0 \) or \( 1 \). For \( g \in G_m \), write \( g = u^{-}tu \) according to the Iwahori factorization. Define the character \( \chi_m \) of \( G_m \) as

\[
\chi_m(u^{-}tu) = \theta\left( \varpi^{-2m} \sum_{i=1}^{n-1} \alpha_i u_{i,i+1} \right),
\]

where \( u = (u_{ij}) \in U_m \). Now define \( d = \text{diag}(1, \varpi^2, \varpi^4, \ldots, \varpi^{2m}) \), and let \( H_m = d^m G_m d^{-m} \). We define the character \( \psi_m \) of \( H_m \) as

\[
\psi_m(h) = \chi_m(d^{-m}hd^m).
\]

As before, let \( \psi \) be the character of \( U \) defined as \( \psi((u_{ij})) = \theta(\sum_{i=1}^{n-1} \alpha_i u_{i,i+1}) \).

**Proposition 5.** Let \( \pi \) be an irreducible admissible representation of \( G \), and suppose that \( \text{Hom}_G(\pi, \text{Ind}_G^G(\psi)) \) is finite dimensional. Then there is an integer \( m_0 \) such that for all \( m \geq m_0 \), we have

\[
\dim \text{Hom}_G(\pi, \text{Ind}_G^G(\psi_{m})) = \dim \text{Hom}_G(\pi, \text{Ind}_G^G(\psi)).
\]

**Proof.** If \( \psi \) is nondegenerate, that is if \( \alpha_i = 1 \) for each \( i \), then this statement is the content of the results of Rodier in [17, Corollaries 2 and 3]. When \( G \) is any split reductive \( p \)-adic group, Rodier has restrictions on the characteristic and residue characteristic of \( F \), but when \( G = \text{GL}(n,F) \), there are no such restrictions, as noted in [17, Section VI]. The only time the fact that \( \psi \) is nondegenerate is used in [17] is in the application of the results to the uniqueness of Whittaker models. That is to say, the proposition in fact follows directly from the work of Rodier.

Finally, we are able to prove the following, which is an equivalent form of Theorem 4 for the general linear group over a nonarchimedean local field. We note that in [15], it is shown that every self-dual generic representation of \( \text{GL}(n,F) \) is orthogonal.

**Theorem 8.** Let \( G = \text{GL}(n,F) \) and let \( \iota \) be the transpose-inverse automorphism of \( G \). Every irreducible admissible representation \( \pi \) of \( G \) satisfies \( \varepsilon_\iota(\pi) = 1 \), and every self-dual irreducible admissible representation of \( G \) is orthogonal.

**Proof.** Let \( \pi \) be any irreducible admissible representation of \( G \). First, choose a character \( \psi \) of \( U \) such that the space \( \text{Hom}_G(\pi, \text{Ind}_G^G(\psi)) \) is one-dimensional, whose existence is guaranteed by Theorem 7. Now let \( m \geq m_0 \), where \( m_0 \) is the integer coming from Proposition 5, and let \( \tilde{\psi} = \psi_m \). For the moment, redefine \( \iota \) to be transpose-inverse composed with conjugation by a matrix with 1’s on the antidiagonal and 0’s elsewhere. Then \( \iota \) preserves the Iwahori factorization \( G_m = U_m^{-T_m}U_m \), and \( \tilde{\psi} \) satisfies \( \iota \tilde{\psi} = \tilde{\psi}^{-1} \). From
Proposition 5 and Frobenius reciprocity, we have
\[ \dim_{\mathbb{C}} \text{Hom}_{H}(\pi, \tilde{\psi}) = 1, \]
where \( H = H_m \) is a compact open subgroup of \( G \). It is a result of I.M. Gelfand and Kazhdan that any irreducible admissible representation \( \pi \) of \( G \) satisfies \( \iota \pi \cong \hat{\pi} \) (a proof appears in [2, Theorem 7.3]). We may then apply Lemma 3 to conclude that \( \varepsilon_i(\pi) = 1 \). Now \( \iota \) can be taken to be the transpose-inverse automorphism by Proposition 2 to obtain the desired result.

If \( \pi \) is a self-dual irreducible admissible representation, we choose \( m \) and \( \tilde{\psi} \) just as before, but now we define the automorphism \( \sigma \) to be conjugation by the element \( s = \text{diag}(1, -1, 1, -1, \ldots) \). Then \( H = H_m \) is fixed by \( \sigma \), \( \tilde{\psi} \) satisfies \( \sigma \tilde{\psi} = \tilde{\psi}^{-1} \), and \( s^2 = I \). By applying Lemma 3 and Proposition 2, we conclude that \( \pi \) is orthogonal.

Acknowledgements. The author thanks Dipendra Prasad for very useful suggestions and discussion, U. K. Anandavardhanan for several helpful conversations, and the referee for comments and a careful reading.

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