ON THE NUMBER OF IRREDUCIBLE REAL-VALUED CHARACTERS OF A FINITE GROUP

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Abstract. We prove that there exists an integer-valued function f on positive integers such that if a finite group G has at most k real-valued irreducible characters, then $|G/\operatorname{Sol}(G)| \leqslant f(k)$, where $\operatorname{Sol}(G)$ denotes the largest solvable normal subgroup of G. In the case $k = 5$, we further classify $G/\operatorname{Sol}(G)$. This partly answers a question of Iwasaki [15] on the relationship between the structure of a finite group and its number of real-valued irreducible characters.

1. Introduction

Analyzing fields of character values is a difficult problem in the representation theory of finite groups. Real-valued characters and rational-valued characters have received more attention than others.

It is well-known that a finite group G has a unique real/rational-valued irreducible character if and only if G has odd order. In [15], Iwasaki proposed to study the relationship between the structure of G and the number of real-valued irreducible characters of G, which we denote $k_{\mathbb{R}}(G)$. He showed that if $k_{\mathbb{R}}(G) = 2$, then G has a normal Sylow 2-subgroup which is either homocyclic or a so-called Suzuki 2-group of type A. Going further, Moretó and Navarro proved in [17] that if G has at most three irreducible real-valued characters, then G has a cyclic Sylow 2-subgroup or a normal Sylow 2-subgroup which is homocyclic, quaternion of order 8, or an iterated central extension of a Suzuki 2-group whose center is an elementary abelian 2-group. In particular, the groups with at most three irreducible real-valued characters must be solvable. Indeed, it was even proved in [18] that a finite group with at most three degrees of irreducible real-valued characters must be solvable.

In a more recent paper [23], the third author studied groups with four real-valued irreducible characters. Among other results, he proved that a nonsolvable group with exactly four real-valued irreducible characters must be the direct product of $\mathrm{SL}_3(2)$ and an odd-order group. Classifying finite groups with exactly five real-valued irreducible

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characters seems to be a difficult problem. In the next result, which we prove in Section 2, we control the nonsolvable part of those groups. We write $\mathrm{Sol}(G)$ to denote the solvable radical of G, i.e. the largest solvable normal subgroup of G.

Theorem A. *Suppose that a finite group G has at most five real-valued irreducible characters. Then $G/\mathrm{Sol}(G)$ is isomorphic to the trivial group, $\mathrm{SL}_3(2)$, A_5, $\mathrm{PSL}_2(8)\cdot 3$ or ${}^2\mathrm{B}_2(8)\cdot 3$.*

Theorem A and the aforementioned results suggest that the nonsolvable part of a finite group perhaps is bounded in terms of the number of real-valued irreducible characters of the group. We obtain the following result, proved in Section 4, which provides a partial answer to Iwasaki's problem.

Theorem B. *There exists an integer-valued function f on positive integers such that if G is a finite group with at most k real-valued irreducible characters, then $|G/\mathrm{Sol}(G)| \leqslant f(k)$.*

Our arguments would allow us to find explicit bounds in Theorem B, but these bounds perhaps are far from best possible. Therefore, for the sake of simplicity, we have not tried to find the best bounding function.

Our proof of Theorem B uses the classification of finite simple groups and the following statement for simple groups, proved in Section 3, which may be of independent interest.

Theorem C. *For a finite nonabelian simple group S and $S \trianglelefteq G \leqslant \mathrm{Aut}(S)$, let $k_{\mathbb{R}}(G|S)$ denote the number of real-valued irreducible characters of G whose kernels do not contain S. Then $k_{\mathbb{R}}(G|S) \to \infty$ as $|S| \to \infty$.*

There is no rational-valued analogue of Theorem B, as shown by the simple groups $\mathrm{PSL}_2(3^{2k+1})$ with $k \geqslant 1$. We also note that, by Brauer's permutation lemma, the number of real-valued irreducible characters and that of conjugacy classes of real elements in a finite group are always the same.

2. Nonsolvable groups with five real-valued irreducible characters

For a finite group G, we denote by $\mathrm{Re}(G)$ the set of all real elements of G, $\mathcal{E}(G)$ the set of orders of real elements of G and $\mathrm{Irr}_{\mathbb{R}}(G)$ the set of real-valued irreducible characters of G. Recall that the generalized Fitting subgroup $\mathbf{F}^*(G)$ of G is the central product of the layer $\mathbf{E}(G)$ of G (the subgroup of G generated by all quasisimple subnormal subgroups of G) and the Fitting subgroup $\mathbf{F}(G)$. Note that if G has a trivial solvable radical, that is, it has no nontrivial normal solvable subgroups, then $\mathbf{F}^*(G) = \mathbf{E}(G)$ is a direct product of nonabelian simple groups. Moreover, if $\mathbf{F}^*(G)$ is a nonabelian simple group, then G is an almost simple group with socle $\mathbf{F}^*(G)$.

Lemma 2.1. *Let G be a finite nonsolvable group with a trivial solvable radical. If $|\mathcal{E}(G)| \leqslant 5$, then G is an almost simple group.*

Proof. As the solvable radical of G is trivial, we see that $\mathbf{F}^*(G) = \mathbf{E}(G) = \prod_{i=1}^r S_i$ is a direct product of nonabelian simple groups S_i $(1 \leqslant i \leqslant r)$, for some integer $r \geqslant 1$. It suffices to show that $r = 1$.

Suppose by contradiction that $r \geqslant 2$. Let $M = S_1 \times S_2$. Since $\mathcal{E}(M) \subseteq \mathcal{E}(\mathbf{F}^*(G)) \subseteq \mathcal{E}(G)$, we deduce that $|\mathcal{E}(M)| \leqslant 5$. Observe that if $x_i \in \mathrm{Re}(S_i)$ for $i = 1, 2$, then $x_1x_2 \in \mathrm{Re}(M)$ and thus if x_1 and x_2 have coprime orders, then

$$o(x_1x_2) = o(x_1)o(x_2) \in \mathcal{E}(M).$$

We consider the following cases.

(i): S_1 or S_2 has no real element of order 4. Without loss, assume that S_1 has no real element of order 4. Then by [23, Proposition 3.2], S_1 is isomorphic to one of the following groups:

$$\begin{gathered}\mathrm{SL}_2(2^f)(f \geqslant 3), \mathrm{PSU}_3(2^f)(f \geqslant 2), {}^2\mathrm{B}_2(2^{2f+1})(f \geqslant 1);\\ \mathrm{PSL}_2(q)(5 \leqslant q \equiv 3, 5 \pmod 8), \mathrm{J}_1, {}^2\mathrm{G}_2(3^{2f+1})(f \geqslant 1).\end{gathered}$$

By [18, Theorem 3.1], S_1 contains real elements z_1 and z_2 of order p_1 and p_2, where $p_1 \neq p_2$ are odd primes. Since S_2 has a real element of order 2, we see that M has real elements of order $1, 2, p_1, p_2, 2p_1, 2p_2$, which is impossible.

(ii): Both S_1 and S_2 have real elements of order 4. By [7, Proposition 6.4], S_1 contains a real element of order p, where p is an odd prime. Since S_2 has real elements of order 2 and 4, M has real elements of order $1, 2, 4, p, 2p$ and $4p$, which is impossible again. □

Next, we classify all finite nonabelian simple groups S with $|\mathcal{E}(S)| \leqslant 5$. Recall that a finite group G is called a (C)-group if the centralizer of every involution of G has a normal Sylow 2-subgroup. By [6, Lemma 2.7], G is a (C)-group if and only if G has no real element of order $2m$ with $m > 1$ being odd.

Lemma 2.2. *Let S be a nonabelian simple group. Then $|\mathcal{E}(S)| \leqslant 5$ if and only if S is isomorphic to one of the following simple groups:*

(1) $\mathsf{A}_5 \cong \mathrm{PSL}_2(4)$, $\mathrm{SL}_3(2)$, $\mathrm{PSL}_3(3)$, *or* $\mathrm{PSU}_3(3)$*;*
(2) $\mathrm{PSL}_2(8)$, $\mathsf{A}_6 \cong \mathrm{PSL}_2(9), \mathrm{PSL}_2(11), \mathrm{PSL}_2(27), \mathrm{PSU}_3(4), \mathrm{PSL}_3(4), {}^2\mathrm{B}_2(8)$*;*
(3) $\mathrm{PSL}_2(3^f)$*, where $f \geqslant 7$ is an odd prime, $3^f + 1 = 4r$, $3^f - 1 = 2s$, and r, s are distinct odd primes.*

Proof. By [23, Theorem B], we have $4 \leqslant |\mathcal{E}(S)| \leqslant 5$. If $S \cong \mathrm{PSL}_3(3)$, $\mathrm{PSU}_3(3)$ or $\mathrm{SL}_3(2)$, then $|\mathcal{E}(S)| = 5$ and these groups appear in part (1). Assume that S is not isomorphic to one of these groups. By [18, Theorem 3.1], $\mathcal{E}(S)$ contains at least two distinct odd primes, say p_1 and p_2.

Assume first that $|\mathcal{E}(S)| = 4$. Then $\mathcal{E}(S) = \{1, 2, p_1, p_2\}$. It follows that S is a (C)-group and thus $S \cong \mathsf{A}_5$ by [23, Theorem 3.1].

Assume next that $|\mathcal{E}(S)| = 5$. Assume that $4 \in \mathcal{E}(S)$. Then $\mathcal{E}(S) = \{1, 2, 4, p_1, p_2\}$. In particular, S has no real element of order $2m$ with $m > 1$ odd. By [20, Theorem 1], S is isomorphic to $\mathrm{PSL}_2(p)$ where p is a Fermat or a Mersenne prime; A_6; or $\mathrm{PSL}_2(q), {}^2\mathrm{B}_2(q), \mathrm{PSU}_3(q)$, or $\mathrm{PSL}_3(q)$ where $q > 2$ is a power of 2. Assume that

$4 \notin \mathcal{E}(S)$. By [23, Proposition 3.2] S is isomorphic to one of the following simple groups:

$$\mathrm{SL}_2(2^f)(f \geqslant 3), \mathrm{PSU}_3(2^f)(f \geqslant 2), {}^2\mathrm{B}_2(2^{2f+1})(f \geqslant 1);$$
$$\mathrm{PSL}_2(q)(5 \leqslant q \equiv 3, 5 (\mathrm{mod}\ 8)), \mathrm{J}_1, {}^2\mathrm{G}_2(3^{2f+1})(f \geqslant 1).$$

We can check that A_6 has five distinct real element orders but J_1 has more than five distinct real element order. So we may assume that S is not one of these two groups. We now consider the following cases.

Case 1: $S \cong \mathrm{SL}_2(2^f), f \geqslant 3$. If $3 \leqslant f \leqslant 6$, then we can check that only $\mathrm{SL}_2(8)$ has exactly five distinct real element orders. So, assume $f > 6$. Since $\mathrm{SL}_2(2^f)$ contains real elements x and y of order $2^f - 1$ and $2^f + 1$, respectively, together with real elements of order 1 and 2, we deduce that one of the numbers $2^f \pm 1$ is an odd prime and the other is a square of an odd prime. Since $f \geqslant 6$, we can check that this cannot occur.

Case 2: $S \cong \mathrm{PSL}_2(q)$, $q = p^f$, where $f \geqslant 1$ and $p > 2$ is a prime. Using [9], if $q \leqslant 37$, then $q \in \{9, 11, 27\}$. Assume that $q \equiv \epsilon \bmod 4$, $\epsilon = \pm 1$. In this case, S has real elements of order order $(q-\epsilon)/2$ and $(q+\epsilon)/2$, respectively. Note that $(q-\epsilon)/2$ is even.

Assume that $(q-\epsilon)/2 = 2^a$ for some integer $a \geqslant 1$. Since $q > 37$, $2^a \geqslant 16$ and thus S has real elements of order $1, 2, 4, 8, 16$ and $(q+\epsilon)/2$, which is a contradiction. Thus $(q-\epsilon)/2$ is divisible by $2r$ for some odd prime r. Let s be a prime divisor of $(q+\epsilon)/2$. Then $\{1, 2, r, s, 2r\} \subseteq \mathcal{E}(S)$ and since $|\mathcal{E}(S)| = 5$, $(q-\epsilon)/2 = 2r$ and $(q+\epsilon)/2 = s$, where r, s are distinct odd primes. If $p > 3$, then since $3 \mid q^2 - 1$, we must have $r = 3$ or $s = 3$, which is not the case as $q > 37$. Therefore $p = 3$ and $q = 3^f > 37$ so $f \geqslant 4$. If f is even, then $q \equiv 1 \bmod 8$ and thus $(q-1)/2$ is divisible by 4 which is impossible. Thus $f \geqslant 5$ is odd and so $\epsilon = -1$. Hence $(3^f + 1)/2 = 2r$ and $(3^f - 1)/2 = s$. The latter equation forces f to be a prime. This is part (3) of the lemma. Direct calculation shows that $f \geqslant 7$.

Case 3: $S \cong \mathrm{PSU}_3(2^f), f \geqslant 2$. In this case, S has a subgroup $T \cong \mathrm{SL}_2(2^f)$. From Case 1, we must have $f = 2$ or 3. However $|\mathcal{E}(\mathrm{PSU}_3(8))| = 6$ so $S \cong \mathrm{PSU}_3(4)$.

Case 4: $S \cong {}^2\mathrm{B}_2(2^{2f+1}), f \geqslant 1$. If $f = 1$, then we can check that S satisfies the hypothesis of the lemma. Assume $f \geqslant 2$. By [21, Theorem 9 and Proposition 16], S has three nontrivial real elements of odd distinct orders, which are $2^{2f+1} - 1$ and $2^{2f+1} \pm 2^{f+1} + 1$. As $|\mathcal{E}(S)| = 5$, all of these numbers must be primes. Hence

$$4^{2f+1} + 1 = (2^{2f+1} + 2^{f+1} + 1)(2^{2f+1} - 2^{f+1} + 1)$$

is a product of two distinct primes. Since 5 divides $4^{2f+1} + 1$, we deduce that

$$2^{2f+1} - 2^{f+1} + 1 = 5$$

which is impossible as $f \geqslant 2$.

Case 5: $S \cong \mathrm{PSL}_3(2^f), f \geqslant 2$. As S contains a subgroup isomorphic to $\mathrm{SL}_2(2^f)$, we deduce that $f = 2$ or 3. Using [9], only $\mathrm{PSL}_3(4)$ satisfies the hypothesis of the lemma.

Case 6: $S \cong {}^2\mathrm{G}_2(3^{2f+1}), f \geqslant 1$. In this case, S contains subgroups isomorphic to $\mathrm{PSL}_2(3^{2f+1})$ and $\mathrm{PSL}_2(8)$. Since $|\mathcal{E}(\mathrm{PSL}_2(8))| = 5$, we deduce that

$$\mathcal{E}(\mathrm{PSL}_2(3^{2f+1})) \subseteq \mathcal{E}(\mathrm{PSL}_2(8)) = \{1, 2, 3, 7, 9\}.$$

Thus $(3^{2f+1}+1)/2 \leqslant 9$ as $\mathrm{PSL}_2(3^{2f+1})$ has a real element of order $(3^{2f+1}+1)/2$. Therefore, $3^{2f+1} \leqslant 17$ which is impossible as $f \geqslant 1$.

Conversely, if S is one of the simple groups in (1)-(3), then we can check that S has at most 5 distinct real element orders. □

Lemma 2.3. *Let G be an almost simple group with a nonabelian simple socle S. Then*

1. *G has exactly four real-valued irreducible characters if and only if $G \cong \mathrm{SL}_3(2)$.*
2. *G has exactly five real-valued irreducible characters if and only if G is isomorphic to $\mathsf{A}_5, \mathrm{PSL}_2(8)\cdot 3$ or ${}^2\mathrm{B}_2(8)\cdot 3$.*

Proof. Part (1) follows from [23, Theorem 3.3]. Assume that G is an almost simple group with a nonabelian simple socle S and that G has exactly five real-valued irreducible characters. By Brauer's Lemma on character tables, G has exactly five conjugacy classes of real elements and thus $|\mathcal{E}(G)| \leqslant 5$. Hence $|\mathcal{E}(S)| \leqslant 5$ as $\mathcal{E}(S) \subseteq \mathcal{E}(G)$. Therefore S is one of the simple groups appear in the conclusion of Lemma 2.2. If S is one of the groups in $(1)-(2)$ of Lemma 2.2, then by using [9], G is isomorphic to $\mathsf{A}_5, \mathrm{PSL}_2(8)\cdot 3$ or ${}^2\mathrm{B}_2(8)\cdot 3$.

Now assume that $S \cong \mathrm{PSL}_2(q)$ with $q = 3^f$, where $f \geqslant 7$ is a prime, $3^f+1 = 4r$ and $3^f-1 = 2s$, where r, s are distinct odd primes. Let $x \in S$ be a real element of order s. Then $\langle x\rangle$ is a Sylow s-subgroup of S and its normalizer in S is a dihedral group of order $2s$. It follows that S has $(s-1)/2$ conjugacy classes of real elements of order s. Since $|\mathrm{Out}(S)| = 2f$, we see that G has at least $(s-1)/(4f)$ conjugacy classes of real elements of order s. Since G already has 4 conjugacy classes of real elements of orders $1, 2, r, 2r$, we deduce that G must have exactly one conjugacy class of real element of order s. Since $(s-1)/(4f) = (3^f-3)/(8f)$ and $f \geqslant 7$ is a prime, we can check that $(3^f-3)/(8f) > 1$. Thus this case cannot occur. □

The next theorem proves Theorem A, and provides additional information.

Theorem 2.4. *Let G be a finite group. Assume that G has at most five real-valued irreducible characters. Then G is either solvable or $G/\mathrm{Sol}(G) \cong \mathrm{SL}_3(2), \mathsf{A}_5, \mathrm{PSL}_2(8)\cdot 3$, ${}^2\mathrm{B}_2(8)\cdot 3$. Moreover, if $|\mathrm{Irr}_{\mathbb{R}}(G/\mathrm{Sol}(G))| = 5$, then one of the following holds.*

1. *$G \cong \mathsf{A}_5 \times K$, where K is of odd order.*
2. *$G \cong (L \times K)\cdot 3$, where $L \cong \mathrm{PSL}_2(8)$ or ${}^2\mathrm{B}_2(8)$ and K is of odd order.*

Proof. We may assume that G is nonsolvable and $|\mathrm{Irr}_{\mathbb{R}}(G)| \leqslant 5$. Then $|\mathrm{Irr}_{\mathbb{R}}(G/\mathrm{Sol}(G))| \leqslant 5$ and thus $|\mathcal{E}(G/\mathrm{Sol}(G))| \leqslant 5$. By Lemma 2.1, $G/\mathrm{Sol}(G)$ is an almost simple group. It follows from [23, Theorem B] that $G/\mathrm{Sol}(G)$ has at least four real-valued irreducible characters; hence $4 \leqslant |\mathrm{Irr}_{\mathbb{R}}(G/\mathrm{Sol}(G))| \leqslant 5$. Now Lemma 2.3 yields the first part of the theorem.

Assume that $|\mathrm{Irr}_{\mathbb{R}}(G/\mathrm{Sol}(G))| = 5$. By Lemma 2.3, $G/\mathrm{Sol}(G) \cong \mathsf{A}_5, \mathrm{PSL}_2(8)\cdot 3$ or ${}^2\mathrm{B}_2(8)\cdot 3$. In all cases, $G/\mathrm{Sol}(G)$ has 3 conjugacy classes of nontrivial real elements of odd orders. As real elements of odd order of $G/\mathrm{Sol}(G)$ lift to real elements of odd order of G by [13, Lemma 2.2], G has three conjugacy classes of nontrivial real elements of odd orders. It follows that $\mathrm{Sol}(G)$ has no nontrivial real element of odd order and thus $\mathrm{Sol}(G)$ has a normal Sylow 2-subgroup by [7, Proposition 6.4]. Moreover, as $|\operatorname{Irr}_{\mathbb{R}}(G)| \leqslant 5$, the above argument shows that G has no real element of order $2m$ with $m > 1$ being odd, so G is a (C)-group and has no real element of order 4. By [23, Theorem 2.3],

$$L = \mathbf{O}^{2'}(G) \cong \mathrm{SL}_2(2^f)(f \geqslant 2) \text{ or } {}^2\mathrm{B}_2(2^{2f+1})(f \geqslant 1).$$

It follows that $L \cong \mathsf{A}_5, \mathrm{PSL}_2(8)$ or ${}^2\mathrm{B}_2(8)$ as these are the only possible nonabelian composition factors of G.

Let $K := \mathbf{C}_G(L)$. Then $K \cap L = 1$ and $K \times L \trianglelefteq G$. Since G is a (C)-group, we deduce that $|K|$ is odd. Now G/K is isomorphic to a subgroup of $\mathrm{Aut}(L)$ and $|\operatorname{Irr}_{\mathbb{R}}(G/K)| \leqslant 5$, we conclude that either $G = K \times \mathsf{A}_5$ or $G = (L \times K) \cdot 3$, where $L \cong \mathrm{PSL}_2(8)$ or ${}^2\mathrm{B}_2(8)$, as claimed. □

3. Real-valued characters of almost simple groups

In this section we prove Theorem C. We begin with the following observation:

Lemma 3.1. *Keep the notation as in Theorem C. Then* $k_{\mathbb{R}}(G|S) = k_{\mathbb{R}}(G) - k_{\mathbb{R}}(G/S)$, $k_{\mathbb{R}}(G) \geqslant k_{\mathbb{R}}(S)/|\mathrm{Out}(S)|$, *and* $k_{\mathbb{R}}(G/S) \leqslant |\mathrm{Out}(S)|$. *In particular, we have*

$$k_{\mathbb{R}}(G|S) \geqslant k_{\mathbb{R}}(S)/|\mathrm{Out}(S)| - |\mathrm{Out}(S)|.$$

For S a finite nonabelian simple group, we will write

$$\mathfrak{K}(S) := k_{\mathbb{R}}(S)/|\mathrm{Out}(S)| - |\mathrm{Out}(S)|.$$

Hence, to prove Theorem C, it suffices to show that $\mathfrak{K}(S) \to \infty$ as $|S| \to \infty$. We remark that due to the nature of the statement of Theorem C, we may disregard a finite number of simple groups. Recall that we may view $k_{\mathbb{R}}(S)$ as either the number of real-valued irreducible characters or the number of real conjugacy classes of S.

3.1. Initial Considerations.

Throughout, when q is a power of a prime p, we will write $\nu(q)$ for the positive integer such that $q = p^{\nu(q)}$.

Lemma 3.2. *Let S be a simple group isomorphic to the alternating group A_n for $n \geqslant 5$, or a simple group of Lie type* ${}^2\mathrm{B}_2(q)$, ${}^2\mathrm{G}_2(q)$, $\mathrm{G}_2(q)$, ${}^3\mathrm{D}_4(q)$, $\mathrm{D}_4(q)$, ${}^2\mathrm{F}_4(q)$, $\mathrm{F}_4(q)$, *or* ${}^2\mathrm{D}_{2n}(q)$ *with $n \geqslant 2$ for q a power of a prime. Or assume* $q \not\equiv 3 \pmod 4$ *and that S is a simple group of Lie type* $\mathrm{B}_n(q)$ *with $n \geqslant 3$,* $\mathrm{C}_n(q)$ *with $n \geqslant 1$, or* $\mathrm{D}_{2n}(q)$ *with $n \geqslant 3$. Then $\mathfrak{K}(S) \to \infty$ as $|S| \to \infty$.*

Proof. Note that every conjugacy class of the symmetric group S_n is real, and that a class in S_n yields exactly one class in A_n if and only if the cycle type of the elements in the classes contain an even cycle or two cycles of the same length. Since the number of cycle types of this form is increasing with n and $|\mathrm{Out}(\mathsf{A}_n)| = 2$ for $n \geqslant 7$, we see that the statement holds if $S = \mathsf{A}_n$.

If S is $\mathrm{G}_2(q)$, ${}^2\mathrm{G}_2(q)$, ${}^2\mathrm{B}_2(q)$, or ${}^2\mathrm{F}_4(q)$, then observing the generic character tables available in CHEVIE [10], we see that all except two, six, two, or twelve, respectively, of the characters are real-valued. If S is $\mathrm{F}_4(q)$, then all except four of the characters are real-valued, using [24, Theorem 4.1]. If S is one of the remaining simple groups of Lie type listed, then [22, Theorem 1.2] yields that every element of S is real.

Further, $\mathrm{Out}(S) \leqslant 24\nu(q)$ where $q = p^{\nu(q)}$ for a prime p. Hence since the number of classes in S can be written as a polynomial in p whose exponents are in terms of n and $\nu(q)$, the statement also holds in these cases. □

3.2. **Fixed Parameters and Classical Groups.**

Proposition 3.3. *Let $\mathbb{S}$ be a family of simple groups of Lie type with the same type and rank. That is, there is some generic reductive group $\mathbb{G}$ as in* [2, Section 2.1] *of simply connected type such that for each $S \in \mathbb{S}$, S is of the form $G/Z(G)$ where $G = \mathbb{G}(q)$ for some prime power q. Then for $S \in \mathbb{S}$, we have $\mathfrak{K}(S) \to \infty$ as $q \to \infty$.*

Proof. Let q be a power of a prime p and let $G = \mathbb{G}(q)$ be the fixed points $\mathbf{G}^F$ of a simple simply connected algebraic group $\mathbf{G}$ over $\overline{\mathbb{F}}_q$ under a Frobenius morphism F. Let $\mathbf{T}$ be a rational maximal torus of $\mathbf{G}$ and let Φ and Δ be a root system and set of simple roots, respectively, for $\mathbf{G}$ with respect to $\mathbf{T}$. Let $|\Delta| = n$. We use the notation as in [12] for the Chevalley generators. In particular, note that $\mathbf{T}$ is generated by $h_\alpha(t)$ for $t \in \overline{\mathbb{F}}_q^\times$ and $\alpha \in \Phi$, and $\mathbf{N_G}(\mathbf{T})$ is generated by $\mathbf{T}$ and the $n_\alpha(1)$ for $\alpha \in \Phi$.

Note that we may assume that $\mathbb{S}$ is not one of the families considered in Lemma 3.2. Hence if F is twisted, we may assume that $F = \tau F_q$ where $|\tau| = 2$ and either $\mathbf{G}$ is type A_n or $n > 4$. Here F_q denotes the standard Frobenius morphism induced from the map $x \mapsto x^q$ on $\overline{\mathbb{F}}_q$, and τ denotes a graph automorphism of $\mathbf{G}$.

First assume F is not twisted, so $\tau = 1$. Fix some $\alpha \in \Delta$. Then for $t \in \mathbb{F}_q^\times$, we know $s := h_\alpha(t)$ is real in G, with reversing element $n_\alpha(1)$. Further, s lies in the maximally split torus $T := \mathbf{T}^F$. By [5, Cor. 0.12], we know that $\mathbf{N}_G(\mathbf{T})$ controls fusion in $\mathbf{T}$, so if $s = h_\alpha(t)$ and $s' := h_\alpha(t')$ for $t, t' \in \mathbb{F}_q^\times$ are conjugate, then they are conjugate in $\mathbf{N_G}(\mathbf{T})$. In particular, this means there is some product $x := \prod_{\beta \in J} n_\beta(1)$ with $J \subseteq \Phi$ that conjugates s to s'.

Then by the properties of the Chevalley generators from [12, 1.12.1], we see this is impossible unless $t' = \pm t^{\pm 1}$. Indeed, $h_\alpha(t)^{n_\beta(1)} = h_{r_\beta(\alpha)}(\pm t)$, so we may write $h_\alpha(t)^x = h_{r(\alpha)}(\pm t)$ where $r := \prod_{\beta \in J} r_\beta$ is the corresponding composition of reflections. Then if $h_\alpha(t)^x = h_\alpha(t')$, we have $h_{r(\alpha)}(\pm t) = h_\alpha(t')$. But if $\widecheck{r(\alpha)} = \sum_{i=1}^n c_i \check{\alpha}_i$, then $h_{r(\alpha)}(\pm t) = \prod_{i=1}^n h_{\alpha_i}(\pm t^{c_i})$. Here $\Delta := \{\alpha_1, \ldots, \alpha_n\}$ and for any $\beta \in \Phi$ we write $\check{\beta} = 2\beta/(\beta, \beta)$. Further, since $\mathbf{G}$ is simply connected, there is an isomorphism $(\overline{\mathbb{F}}_q^\times)^n \to \mathbf{T}$ given by $(t_1, \ldots t_n) \mapsto h_{\alpha_1}(t_1) \cdots h_{\alpha_n}(t_n)$ (see [12, 1.12.5]). Then this yields $c_i = 0$ for $\alpha_i \neq \alpha$, and hence $\widecheck{r(\alpha)} = c\check{\alpha}$ for some integer c and $t' = \pm t^c$. Then since $(r(\alpha), r(\alpha)) = (\alpha, \alpha)$, we have $r(\alpha) = c\alpha$ and $c = \pm 1$.

Now by [12, 1.12.6], we see $s \notin Z(G)$ for $\delta \neq \pm 1$. This yields that $k_{\mathbb{R}}(G) \geqslant (q-3)/2$, and since (except for a finite number of exceptions) $|Z(G)| \leqslant n+1$ and $|\mathrm{Out}(S)| \leqslant 2(n+1)\nu(q)$, we see $\mathfrak{K}(S) \geqslant \frac{(q-3)}{8(n+1)^2\nu(q)} - 2(n+1)\nu(q) \to \infty$ as $q \to \infty$.

If $\tau \neq 1$, we may argue similarly, taking $\alpha \in \Delta$ to be fixed by τ, unless $\mathbf{G}$ is type A_n with n even. In the latter case, we may instead take $h_{\alpha_1}(t)h_{\alpha_n}(t^q)$ with $t \in \mathbb{F}_{q^2}^\times$. Then in each case, the element being considered lies in $T = \mathbf{T}^F$ (see [12, 2.4.7]) and similar arguments to above show that we still have $\mathfrak{K}(S) \to \infty$ as $n \to \infty$. □

Corollary 3.4. *Let $S(q)$ be a simple group $\mathrm{E}_6(q)$, ${}^2\mathrm{E}_6(q)$, $\mathrm{E}_7(q)$, $\mathrm{E}_8(q)$, $\mathrm{A}_n(q)$, or ${}^2\mathrm{A}_n(q)$, with n a fixed positive integer. Then $\mathfrak{K}(S(q)) \to \infty$ as $q \to \infty$.*

Lemma 3.5. *Let q be a fixed power of a prime and let S_n be a simple group of Lie type of classical type: $\mathrm{A}_n(q)$, ${}^2\mathrm{A}_n(q)$, $\mathrm{B}_n(q)$, $\mathrm{C}_n(q)$, $\mathrm{D}_n(q)$, or ${}^2\mathrm{D}_n(q)$. Then $\mathfrak{K}(S_n) \to \infty$ as $n \to \infty$.*

Proof. Consider the unipotent characters of S_n. By [16], these characters are real-valued. For S_n of the form $\mathrm{A}_n(q)$ or ${}^2\mathrm{A}_n(q)$, these characters are indexed by partitions of $n+1$ (see [3, 13.8]). Since the number of these behaves asymptotically like $\frac{\exp(\pi\sqrt{2(n+1)/3})}{4(n+1)\sqrt{3}}$ as $n \to \infty$ (see [1, (5.1.2)]), and $|\mathrm{Out}(S)| \leqslant 2(q+1)\nu(q)$, we have $\mathfrak{K}(S_n(q)) \to \infty$ as $n \to \infty$.

For $S_n(q)$ of the form $\mathrm{B}_n(q)$, $\mathrm{C}_n(q)$, $\mathrm{D}_n(q)$, or ${}^2\mathrm{D}_n(q)$, the unipotent characters are indexed by symbols as in [3, 13.8], the number of which is at least the number of partitions of $n-1$. Then since $|\mathrm{Out}(S_n)| \leqslant 8\nu(q)$ for $n \geqslant 5$, we have $\mathfrak{K}(S_n(q)) \to \infty$ as $n \to \infty$ again in this case. □

Proposition 3.6. *Let $S_n(q) := \Omega_{2n+1}(q)$, $\mathrm{PSp}_{2n}(q)$, or $\mathrm{P}\Omega^{\pm}_{2n}(q)$, with $n \geqslant 5$. Then $\mathfrak{K}(S_n(q)) \to \infty$ as $nq \to \infty$.*

Proof. Write $S = S_n(q)$ as $G/Z(G)$, where $G = \mathbf{G}^F$ is the set of fixed points of a simple, simply connected algebraic group $\mathbf{G}$ over $\overline{\mathbb{F}}_q$ under a Frobenius morphism F. Notice that $|Z(G)| \leqslant 4$.

Let $\mathbf{T}$ be a rational maximal torus of $\mathbf{G}$ and let Φ and Δ be a root system and set of simple roots, respectively, for $\mathbf{G}$ with respect to $\mathbf{T}$. Here we have $|\Delta| = n$ and Φ is of type B_n, C_n, or D_n. We use the notation as in [12, 1.12.1] for the Chevalley generators. Recall that $\mathbf{T}$ is generated by $h_\alpha(t)$ for $t \in \overline{\mathbb{F}}_q^\times$ and $\alpha \in \Phi$ and that $\mathbf{N_G}(\mathbf{T})$ is generated by $\mathbf{T}$ and the $n_\alpha(1)$ for $\alpha \in \Phi$.

We will use the standard model as in [12, Remark 1.8.8] for the members of Δ. Namely, let $\{e_1, \ldots e_n\}$ be an orthonormal basis for the n-dimensional Euclidean space and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. Note that for $1 \leqslant i \leqslant n-1$, we have $\alpha_i := e_i - e_{i+1}$. Further, since $\mathbf{G}$ is simply connected, there is an isomorphism $(\overline{\mathbb{F}}_q^\times)^n \to \mathbf{T}$ given by $(t_1, \ldots t_n) \mapsto h_{\alpha_1}(t_1)\cdots h_{\alpha_n}(t_n)$ (see [12, 1.12.5]).

Using Lemmas 3.2 and 3.5, we may suppose $q \geqslant 3$. For each $1 \neq \delta \in \mathbb{F}_q^\times$, we let $s_0(\delta) := h_{\alpha_1}(\delta)$, $s_1(\delta) := h_{\alpha_1}(\delta)h_{\alpha_3}(\delta)$, and in general for $0 \leqslant m \leqslant \lceil \frac{n-4}{2} \rceil$, let $s_m(\delta) := \prod_{k=0}^m h_{\alpha_{2k+1}}(\delta)$. Our choices of m ensure that $s_m(\delta) \in \mathbf{G}$ is fixed by F, since in the case of ${}^2\mathrm{D}_n(q) = \mathrm{P}\Omega^-_{2n}(q)$, $s_m(\delta)$ is fixed by the graph automorphism and Frobenius F_q. Hence $s_m(\delta) \in G = \mathbf{G}^F$. Further, since $(\alpha_i, \alpha_j) = 0$ for $|j-i| > 1$, we see that each s_m is real in G, with reversing element $\prod_{k=0}^m n_{\alpha_{2k+1}}(-1)$.

Recall that $\mathbf{N_G}(\mathbf{T})$ controls fusion in $\mathbf{T}$. Hence if $s_m(\delta)$ and $s_{m'}(\delta')$ are conjugate in G, then there is some $w \in W := \mathbf{N_G}(\mathbf{T})/\mathbf{T}$ such that $s_m(\delta)^w = s_{m'}(\delta')$. But $W \leqslant C_2 \wr \mathsf{S}_n$

(with C_2 the group of order 2) where the generators of the base subgroup C_2^n act via $e_i \mapsto -e_i$ and the copy of S_n permutes the e_i's. Then by the properties of the Chevalley generators from [12, 1.12.1], we see this is impossible unless $m = m'$ and $\delta = \delta'$.

Note that we have $(q-3)/2$ choices for $\delta \neq \pm 1$, since $s_m(\delta)$ is conjugate to $s_m(\delta^{-1})$, giving $(q-3)/2+1 = (q-1)/2$ elements in this form for a fixed m. Further, since $\delta \neq 1$ and $s_m(\delta)$ has no factor $h_{\alpha_n}(\delta)$ nor $h_{\alpha_{n-1}}(\delta)$, we see by [12, 1.12.6] that $s_m(\delta) \notin Z(G)$, so we have $k_{\mathbb{R}}(S) > \frac{1}{4}\lceil\frac{n-4}{2}\rceil\left(\frac{q-1}{2}\right)$. Further, $|\mathrm{Out}(S)| \leqslant 8\nu(q)$, so

$$\mathfrak{K}(S_n(q)) > \frac{(n-4)(q-1)}{16 \cdot 8\nu(q)} - 8\nu(q)$$

which tends to ∞ as $nq \to \infty$. □

3.3. **Linear and Unitary Groups.** We write $\mathrm{SL}_n^\epsilon(q)$ with $\epsilon \in \{\pm 1\}$ to denote $\mathrm{SL}_n(q)$ for $\epsilon = 1$ and $\mathrm{SU}_n(q)$ for $\epsilon = -1$, and similarly for $\mathrm{GL}_n^\epsilon(q)$ and $\mathrm{PSL}_n^\epsilon(q)$. Throughout this section, we also write $\widetilde{G} = \mathrm{GL}_n^\epsilon(q)$, $G = \mathrm{SL}_n^\epsilon(q) = [\widetilde{G}, \widetilde{G}]$, and $S = G/Z(G) = \mathrm{PSL}_n^\epsilon(q)$. Note that $\widetilde{G} \cong \widetilde{G}^*$ in this case, where $\widetilde{G}^*$ denotes the dual group, and we make this identification.

If s is a semisimple element of $\widetilde{G}$, there exists a unique semisimple character $\widetilde{\chi}_s$ associated to the $\widetilde{G}$-conjugacy class of s, and $\widetilde{\chi}_{s^{-1}}$ is the complex conjugate character of $\widetilde{\chi}_s$. Hence $\widetilde{\chi}_s$ is real if s is. If further $s \in G = [\widetilde{G}, \widetilde{G}]$, then $\widetilde{\chi}_s$ is trivial on $Z(\widetilde{G})$, using [19, Lemma 4.4]. Furthermore, the number of irreducible constituents of $\chi := \widetilde{\chi}_s|_G$ is exactly the number of irreducible characters $\theta \in \mathrm{Irr}(\widetilde{G}/G)$ satisfying $\widetilde{\chi}_s\theta = \widetilde{\chi}_s$, and we have $\mathrm{Irr}(\widetilde{G}/G) = \{\widetilde{\chi}_z \mid z \in Z(\widetilde{G})\}$. Also, for such $z \in Z(\widetilde{G})$, if we take the product with $\widetilde{\chi}_z$ of each character in the Lusztig series for $\widetilde{G}$ indexed by s, we obtain the Lusztig series indexed by sz, by [5, 13.30]. Then χ is irreducible if and only if s is not $\widetilde{G}$-conjugate to sz for any nontrivial $z \in Z(\widetilde{G})$. Further, if s and s' are two such elements, an application of Gallagher's theorem [14, Corollary 6.17] together with the above reasoning yields that if $\widetilde{\chi}_s|_G = \widetilde{\chi}_{s'}|_G$, then s is conjugate to $s'z$ for some $z \in Z(\widetilde{G})$.

Hence we aim to construct a collection X of real semisimple elements of G such that two elements $s, s' \in X$ satisfy that s and $s'z$ for $z \in Z(\widetilde{G})$ are $\widetilde{G}$-conjugate if and only if $s' = s$ and $z = 1$ and such that $|X|/|\mathrm{Out}(S)| - |\mathrm{Out}(S)|$ tends to ∞ as $nq \to \infty$.

Proposition 3.7. *Let $S_n(q) := \mathrm{PSL}_n^\pm(q)$. Then $\mathfrak{K}(S_n(q)) \to \infty$ as $nq \to \infty$.*

Proof. By Corollary 3.4 and Lemma 3.5, we may assume that q and n are sufficiently large. Write $\bar{n} := \lfloor n/4 \rfloor$.

Recall that the semisimple elements of $\widetilde{G}$ are completely determined by their eigenvalues. Consider a semisimple element

$$s = s(\lambda_1, \ldots, \lambda_{\bar{n}}) := \mathrm{diag}(\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}, \ldots, \lambda_{\bar{n}}, \lambda_{\bar{n}}^{-1}, I_{n-2\bar{n}})$$

in G, where each λ_i is an element of the cyclic subgroup $C_{q-\epsilon}$ of $\mathbb{F}_{q^2}^\times$ and not all of the λ_i are in $\{\pm 1\}$.

We see by the dimension of $\ker(s-1)$ that s is not conjugate to sz for any $1 \neq z = \mu I_n \in Z(\widetilde{G})$, since otherwise $1 = \lambda_i\mu = \lambda_i^{-1}\mu$ for each i, implying that $\lambda_i^2 = 1$ for each

i, contradicting our assumption that not all λ_i are in $\{\pm 1\}$. Similarly, if s' is another semisimple element of this form, defined by λ_i' for $1 \leqslant i \leqslant \bar{n}$, such that s' is conjugate to sz with $z = \mu I_n \in Z(\widetilde{G})$, then it must be that $\mu = 1$ and s is conjugate to s'.

Then by considering the elements of the form

$$s(\lambda_1, 1, \ldots, 1), s(\lambda_1, \lambda_1, 1, \ldots, 1), ..., s(\lambda_1, \ldots, \lambda_1),$$

together with those of the form

$$s(\lambda_1, \lambda_2, 1, \ldots, 1), s(\lambda_1, \lambda_2, \lambda_2, 1, \ldots, 1), \ldots, s(\lambda_1, \lambda_2, \ldots, \lambda_2)$$

and

$$s(\lambda_1, \lambda_2, \lambda_3, 1, \ldots, 1), s(\lambda_1, \lambda_2, \lambda_3, \lambda_3, 1, \ldots, 1), \ldots, s(\lambda_1, \lambda_2, \lambda_3 \ldots, \lambda_3)$$

with $\lambda_1, \lambda_2, \lambda_3$ and their inverses all distinct, we see

$$\begin{aligned} k_{\mathbb{R}}(S_n(q)) &\geqslant \bar{n}(q-3)/2 + (\bar{n}-1)(q-3)(q-5)/4 + (\bar{n}-2)(q-3)(q-5)(q-7)/8 \\ &> \bar{n}(q-5)^3/8 - (q-5)(q-3)(q-6)/4 \\ &> \bar{n}(q-5)^3/8 - 2(q-3)^3/8 \\ &= \frac{(\bar{n}-2)(q-5)^3 - 12(q-5)^2 - 24(q-5) - 16}{8}. \end{aligned}$$

So

$$\begin{aligned} \mathfrak{K}(S_n(q)) &\geqslant \frac{(\bar{n}-2)(q-5)^3 - 12(q-5)^2 - 24(q-5) - 16}{16(q+1)\nu(q)} - 2(q+1)\nu(q) \\ &= \frac{(\bar{n}-2)(p^{\nu(q)}-5)^3 - 12(p^{\nu(q)}-5)^2 - 24(p^{\nu(q)}-5) - 16 - 4(p^{\nu(q)}+1)^2\nu(q)^2}{16(p^{\nu(q)}+1)\nu(q)}, \end{aligned}$$

which tends toward ∞ as $nq \to \infty$. □

Theorem C now follows by combining Lemmas 3.2 and 3.5 with Propositions 3.3, 3.6, and 3.7.

4. Proof of Theorem B

We start with a well-known observation.

Lemma 4.1. *Let S be a finite nonabelian simple group. Then there exists a non-principal irreducible character of S that is extendible to a rational-valued character of* $\mathrm{Aut}(S)$.

Proof. For each $n \geqslant 5$, consider the irreducible character of the symmetric group S_n labeled by the partition $(n-1, 1)$. This character restricts irreducibly to the alternating group A_n. As it is well known that every character of S_n is rational-valued, the lemma is proved for the alternating groups. For the sporadic simple groups and the Tits group, one can check the statement directly by using [4]. Finally, when S is a simple group of

Lie type, the Steinberg character of S extends to a rational-valued character of $\mathrm{Aut}(S)$, see [8] for instance. □

Proposition 4.2. *Assume that $N = S_1 \times S_2 \times \cdots \times S_n$, a direct product of copies of a finite nonabelian simple group $S \cong S_i$, is a normal subgroup of G. Then the number of rational-valued irreducible characters of G is at least n.*

Proof. Modding out $\mathbf{C}_G(N)$ if necessary, we may assume that $\mathbf{C}_G(N) = 1$ so that $N \trianglelefteq G \leqslant \mathrm{Aut}(N)$. By Lemma 4.1, there exists $\theta \in \mathrm{Irr}(S)$ that is extendible to a rational-valued character, say λ, of $\mathrm{Aut}(S)$. For each $1 \leqslant j \leqslant n$, set

$$\psi_j := \theta \otimes \cdots \otimes \theta \otimes 1_{S_{j+1}} \otimes \cdots \otimes 1_{S_n} \in \mathrm{Irr}(N).$$

Since $\mathrm{Aut}(N)$ acts transitively on the direct factors S_i's of N, the $\mathrm{Aut}(N)$-orbit of ψ_j consists of characters of the form $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n$ where $\alpha_i \in \{1_{S_i}, \theta\}$ for every $1 \leqslant i \leqslant n$ and the number of times that θ appears in the tensor product is precisely equal to j. This means that the size of the $\mathrm{Aut}(N)$-orbit containing ψ_j is $n!/j!(n-j)!$. On the other hand, we see that ψ_j is invariant under

$$(\mathrm{Aut}(S) \wr \mathsf{S}_j) \times (\mathrm{Aut}(S) \wr \mathsf{S}_{n-j}),$$

and

$$|\,\mathrm{Aut}(N) : (\mathrm{Aut}(S) \wr \mathsf{S}_j) \times (\mathrm{Aut}(S) \wr \mathsf{S}_{n-j})| = n!/j!(n-j)!.$$

We therefore deduce that $\mathrm{Aut}(S) \wr \mathsf{S}_j \times \mathrm{Aut}(S) \wr \mathsf{S}_{n-j}$ is the inertia subgroup of ψ_j in $\mathrm{Aut}(N)$.

Recall that θ extends to the rational-valued character $\lambda \in \mathrm{Irr}(\mathrm{Aut}(S))$. Let V be a $\mathbb{C}\,\mathrm{Aut}(S)$-module affording λ. Then $\mathrm{Aut}(S)^j$ acts naturally on $V^{\otimes j}$, with the character $\lambda \otimes \cdots \otimes \lambda$, and S_j permutes the j tensor factors of $V^{\otimes j}$. So $V^{\otimes j}$ becomes a tensor-induced module for $\mathrm{Aut}(S^j) = \mathrm{Aut}(S) \wr \mathsf{S}_j$. Let μ be the character afforded by this module. Then, as λ is rational-valued, the formula for the tensor-induced character (see [11] for instance) implies that μ is also rational-valued. We have seen that θ^j extends to the rational-valued character $\mu \in \mathrm{Irr}(\mathrm{Aut}(S^j))$. It follows that ψ_j extends to a rational-valued character of $I_{\mathrm{Aut}(N)}(\psi_j)$. In particular, ψ_j extends to a rational-valued character, say ν_j, of $I_G(\psi_j) = G \cap I_{\mathrm{Aut}(N)}(\psi_j)$. The Clifford correspondence now produces n different rational-valued irreducible characters, namely ν_j^G, for $1 \leqslant j \leqslant n$, of G, and the proposition is proved. □

We are now ready to prove Theorem B.

Proof of Theorem B. Since $k_{\mathbb{R}}(G/\mathrm{Sol}(G)) \leqslant k_{\mathbb{R}}(G)$ and $\mathrm{Sol}(G/\mathrm{Sol}(G))$ is trivial, we may assume with no loss that $\mathrm{Sol}(G)$ is trivial. The generalized Fitting subgroup of G, denoted by $\mathbf{F}^*(G)$, is then the direct product of the minimal normal subgroups of G, each of which is a product of copies of a nonabelian simple group. Therefore $\mathbf{C}_G(\mathbf{F}^*(G)) = 1$ and $G \leqslant \mathrm{Aut}(\mathbf{F}^*(G))$.

Let S be a simple direct factor of $\mathbf{F}^*(G)$ and assume that the number of times that S appears in $\mathbf{F}^*(G)$ is n. By Proposition 4.2, we know that n is bounded by k. It remains to prove that $|S|$ is bounded in terms of k. Notice that if $|S|$ is bounded in terms of k, then the number of choices for S appearing in $\mathbf{F}^*(G)$ is bounded, and therefore $\mathbf{F}^*(G)$ is bounded, which in turn implies that $|G|$ is bounded in terms of k.

Let $N := S_1 \times S_2 \times \cdots \times S_n$ where each S_i is isomorphic to S.

We have $\mathbf{C}_{\mathbf{N}_G(S_1)/\mathbf{C}_G(S_1)}(N\mathbf{C}_G(S_1)/\mathbf{C}_G(S_1)) = 1$, and hence

$$S_1 \cong N\mathbf{C}_G(S_1)/\mathbf{C}_G(S_1) \trianglelefteq \mathbf{N}_G(S_1)/\mathbf{C}_G(S_1) \leqslant \mathrm{Aut}(S_1).$$

Assume, to the contrary, that $|S| = |S_1|$ can be arbitrarily large while k is fixed. Using Theorem C, we then can choose S_1 so that $\mathbf{N}_G(S_1)/\mathbf{C}_G(S_1)$ has at least $k^2 + 1$ real-valued irreducible characters whose kernels do not contain S_1. Let λ be one of these characters.

Let θ be an irreducible constituent of $\lambda\downarrow_{S_1}$, and set $\psi := \theta \otimes 1_{S_2} \otimes \cdots \otimes 1_{S_n}$. Since $S_1 \nsubseteq \mathrm{Ker}(\lambda)$, we see that θ is nontrivial, and hence the inertia subgroup $I_G(\psi)$ is contained in $\mathbf{N}_G(S_1)$. The Clifford correspondence then implies that, as λ (considered as a character of $\mathbf{N}_G(S_1)$) lies over ψ, λ^G is an irreducible character of G. Moreover, λ^G is real-valued since λ is.

We have shown that, for each λ among $k^2 + 1$ real-valued irreducible characters of $\mathbf{N}_G(S_1)$ whose kernels do not contain S_1, there corresponds the real-valued irreducible character λ^G of G. On the other hand, as

$$|G : \mathbf{N}_G(S_1)| \leqslant |\,\mathrm{Aut}(\mathbf{F}^*(G)) : \mathbf{N}_{\mathrm{Aut}(\mathbf{F}^*(G))}(S_1)| = n \leqslant k,$$

each real-valued irreducible character of G lies above at most k irreducible characters of $\mathbf{N}_G(S_1)$. We therefore deduce that G has at least $k + 1$ real-valued irreducible characters, and this is contradiction. Thus we conclude that $|S|$ is bounded in terms of k, as desired. □

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