# ON THE CHARACTERISTIC MAP OF FINITE UNITARY GROUPS

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## 1. Introduction

In his seminal work [9], Green described a remarkable connection between the class functions of the finite general linear group $\mathrm{GL}(n, \mathbb{F}_q)$ and a generalization of the ring of symmetric functions of the symmetric group $S_n$. In particular, Green defines a map, called the characteristic map, that takes irreducible characters to Schur-like symmetric functions, and recovers the character table of $\mathrm{GL}(n, \mathbb{F}_q)$ as the transition matrix between these Schur functions and Hall-Littlewood polynomials [19, Chapter IV]. Thus, we can use the combinatorics of the symmetric group $S_n$ to understand the representation theory of $\mathrm{GL}(n, \mathbb{F}_q)$. Some of the implications of this approach include an indexing of irreducible characters and conjugacy classes of $\mathrm{GL}(n, \mathbb{F}_q)$ by multi-partitions and a formula for the degrees of the irreducible characters in terms of these partitions.

This paper describes the parallel story for the finite unitary group $\mathrm{U}(n, \mathbb{F}_{q^2})$ by collecting known results for this group and examining some applications of the unitary characteristic map. Inspired by Green, Ennola [5, 6] used results of Wall [22] to construct the appropriate ring of symmetric functions and characteristic map. Ennola was able to prove that the analogous Schur-like functions correspond to an orthonormal basis for the class functions, and conjectured that they corresponded to the irreducible characters. He theorized that the representation theory of $\mathrm{U}(n, \mathbb{F}_{q^2})$ should be deduced from the representation theory of $\mathrm{GL}(n, \mathbb{F}_q)$ by substituting "$-q$" for every occurrence of "$q$". The general phenomenon of obtaining a polynomial invariant in $q$ for $\mathrm{U}(n, \mathbb{F}_{q^2})$ by this substitution has come to be known as "Ennola duality".

Roughly a decade after Ennola made his conjecture, Deligne and Lusztig [3] constructed a family of virtual characters, called Deligne-Lusztig characters, to study the representation theory of arbitrary finite reductive groups. Lusztig and Srinivasan [18] then computed an explicit decomposition of the irreducible characters of $\mathrm{U}(n, \mathbb{F}_{q^2})$ in terms of Deligne-Lusztig characters. Kawanaka [15] used this composition to demonstrate that Ennola duality applies to Green functions, thereby improving results of Hotta and Springer [12] and finally proving Ennola's conjecture.

This paper begins by describing some of the combinatorics and group theory associated with the finite unitary groups. Section 2 defines the finite unitary groups, outlines the combinatorics of multi-partitions, and gives a description of some of the key subgroups. Section 3 analyzes the conjugacy classes of $\mathrm{U}(n,\mathbb{F}_{q^2})$ and the Jordan decomposition of these conjugacy classes.

Section 4 outlines the statement and development of the Ennola conjecture from two perspectives. Both points of view define a map from a ring of symmetric functions to the character ring $C$ of $\mathrm{U}(n,\mathbb{F}_{q^2})$. However, the first uses the multiplication for $C$ as defined by Ennola, and the second uses Deligne-Lusztig induction as the multiplicative structure of $C$. This multiplicative structure on the graded ring of characters of the unitary group was studied by Digne and Michel in [4], where the focus is that this multiplication induces a Hopf algebra structure. This structure theorem in [4] is equivalent to our Corollary 4.1, although our approach focuses on the explicit map between characters and symmetric functions.

The main results are

**I.** (Theorem 4.2) The Deligne-Lusztig characters correspond to power-sum symmetric functions via the characteristic map of Ennola.

**II.** (Corollary 4.2) The multiplicative structure that Ennola defined on $C$ is Deligne-Lusztig induction.

Section 5 computes the degrees of the irreducible characters, and uses this result to evaluate various sums of character degrees (see [19, IV.6, Example 5] for the $\mathrm{GL}(n,\mathbb{F}_q)$ analogue of this method). The main results are

**III.** (Theorem 5.1) An irreducible $\chi^{\boldsymbol{\lambda}}$ character of $\mathrm{U}(m,\mathbb{F}_{q^2})$ corresponds to

$$(-1)^{\lfloor m/2\rfloor + n(\boldsymbol{\lambda})} s_{\boldsymbol{\lambda}} \quad \text{and} \quad \chi^{\boldsymbol{\lambda}}(1) = q^{n(\boldsymbol{\lambda}')} \frac{\displaystyle\prod_{1\le i\le m} (q^i - (-1)^i)}{\displaystyle\prod_{\Box\in\boldsymbol{\lambda}} (q^{\mathbf{h}(\Box)} - (-1)^{\mathbf{h}(\Box)})},$$

where $s_{\boldsymbol{\lambda}}$ is a Schur-like function, and both $n(\boldsymbol{\lambda})$ and $\mathbf{h}(\Box)$ are combinatorial statistics on the multi-partition $\boldsymbol{\lambda}$.

**IV.** (Corollary 5.2) If $\mathcal{P}_n^{\Theta}$ indexes the irreducible characters $\chi^{\boldsymbol{\lambda}}$ of $\mathrm{U}(n,\mathbb{F}_{q^2})$, then

$$\sum_{\boldsymbol{\lambda}\in\mathcal{P}_n^{\Theta}} \chi^{\boldsymbol{\lambda}}(1) = |\{g\in \mathrm{U}(n,\mathbb{F}_{q^2}) \mid g \text{ symmetric}\}|.$$

Section 6 uses results by Ohmori [21] and Henderson [11] to adapt a model for the general linear group, found by Klyachko [17] and Inglis and Saxl [13], to the finite unitary group. The main result is

**V.** (Theorem 6.2) Let $U_m = \mathrm{U}(m,\mathbb{F}_{q^2})$, where $q$ is odd, and let $\Gamma_m$ be the Gelfand-Graev character of $U_m$, $\mathbf{1}$ be the trivial character of the finite symplectic group $Sp_{2r} = \mathrm{Sp}(2r,\mathbb{F}_q)$, and $R_L^G$ be the Deligne-Lusztig induction functor. Then

$$\sum_{0\le 2r\le m} R^{U_m}_{U_{m-2r}\oplus U_{2r}}\big(\Gamma_{m-2r}\otimes \mathrm{Ind}^{U_{2r}}_{Sp_{2r}}(\mathbf{1})\big) = \sum_{\boldsymbol{\lambda}\in\mathcal{P}_m^{\Theta}} \chi^{\boldsymbol{\lambda}}.$$

That is, in the theorem of Klyachko, one may replace parabolic induction by Deligne-Lusztig induction to obtain a theorem for the unitary group.

These results give considerable combinatorial control over the representation theory of the finite unitary group, and there are certainly more applications to these results than what we present in this paper. Furthermore, this characteristic map gives some insight as to how a characteristic map might look in general type, using the invariant rings of other Weyl groups.

## 2. Preliminaries

2.1. **The unitary group and its underlying field $\mathbb{K}$.** Let $\mathbb{K} = \bar{\mathbb{F}}_q$ be the algebraic closure of the finite field with $q$ elements and let $\mathbb{K}_m = \mathbb{F}_{q^m}$ denote the finite subfield with $q^m$ elements. Let $\mathrm{GL}(n, \mathbb{K})$ denote the general linear group over $\mathbb{K}$, and define Frobenius maps

$$\begin{array}{rccc} F: & \mathrm{GL}(n,\mathbb{K}) & \longrightarrow & \mathrm{GL}(n,\mathbb{K}) \\ & (a_{ij}) & \mapsto & (a_{ji}^q)^{-1}, \end{array} \quad \text{and} \quad \begin{array}{rccc} F': & \mathrm{GL}(n,\mathbb{K}) & \longrightarrow & \mathrm{GL}(n,\mathbb{K}) \\ & (a_{ij}) & \mapsto & (a_{n-j,n-i}^q)^{-1}. \end{array} \tag{2.1}$$

Then the unitary group $U_n = \mathrm{U}(n, \mathbb{K}_2)$ is given by

$$U_n = \mathrm{GL}(n,\mathbb{K})^F = \{a \in \mathrm{GL}(n,\mathbb{K}) \mid F(a) = a\} \tag{2.2}$$

$$\cong \mathrm{GL}(n,\mathbb{K})^{F'} = \{a \in \mathrm{GL}(n,\mathbb{K}) \mid F'(a) = a\}. \tag{2.3}$$

In fact, it follows from the Lang-Steinberg theorem (see, for example, [2]) that there exists $y \in \mathrm{GL}(n, \mathbb{K}_2)$ such that $\mathrm{GL}(n,\mathbb{K})^F = y\mathrm{GL}(n,\mathbb{K})^{F'} y^{-1}$.

We define the multiplicative groups $T_m$ as

$$T_m = \mathrm{GL}(1,\mathbb{K})^{F^m} = \{x \in \mathbb{K} \mid x^{q^m-(-1)^m} = 1\}.$$

Note that $T_m \cong \mathbb{K}_m^\times$ only if $m$ is even. We identify $\mathbb{K}^\times$ with the inverse limit $\varprojlim T_m$ with respect to the norm maps

$$\begin{array}{rccc} N_{mr}: & T_m & \longrightarrow & T_r \\ & x & \mapsto & xx^{-q}\cdots x^{(-q)^{m/r-1}} \end{array}, \quad \text{where } m, r \in \mathbb{Z}_{\geq 1} \text{ with } r \mid m.$$

If $T_m^*$ is the group of characters of $T_m$, then the direct limit

$$\mathbb{K}^* = \varinjlim T_m^*$$

gives the group of characters of $\mathbb{K}^\times$. Let

$$\Theta = \{F\text{-orbits of } \mathbb{K}^*\}.$$

A polynomial $f(t) \in \mathbb{K}_2[t]$ is *$F$-irreducible* if there exists an $F$-orbit $\{x, x^{-q}, \ldots, x^{(-q)^d}\}$ of $\mathbb{K}^\times$ such that

$$f(t) = (t - x)(t - x^{-q}) \cdots (t - x^{(-q)^d}).$$

Let

$$\Phi = \{f \in \mathbb{K}_2[t] \mid f \text{ is } F\text{-irreducible}\} \overset{1-1}{\longleftrightarrow} \{F\text{-orbits of } \mathbb{K}^\times\}. \tag{2.4}$$

The set $\Phi$ has an alternate description, as given in [5]. For $f = t^d + a_1 t^{d-1} + \cdots + a_d \in \mathbb{K}_2[t]$ with $a_d \neq 0$, let

$$\tilde{f} = t^d + a_d^{-q}(a_{d-1}^q t^{d-1} + \cdots + a_1^q t + 1),$$

which has the effect of applying $F$ to the roots of $f$ in $\mathbb{K}$. Then a polynomial $f \in \mathbb{K}_2[t]$ is $F$-irreducible if and only if either

(a) $f$ is irreducible in $\mathbb{K}_2[t]$ and $\tilde{f} = f$ ($d$ must be odd in this case), or
(b) $f(t) = h(t)\tilde{h}(t)$ where $h$ is irreducible in $\mathbb{K}_2[t]$ and $\tilde{h}(t) \neq h(t)$.

2.2. **Combinatorics of $\Phi$-partitions and $\Theta$-partitions.** Fix an ordering of $\Phi$ and $\Theta$, and let

$$\mathcal{P} = \{\text{partitions}\} \quad \text{and} \quad \mathcal{P}_n = \{\nu \in \mathcal{P} \mid |\nu| = n\}.$$

Let $\mathcal{X}$ be either $\Phi$ or $\Theta$. An $\mathcal{X}$-*partition* $\boldsymbol{\nu} = (\boldsymbol{\nu}(x_1), \boldsymbol{\nu}(x_2), \ldots)$ is a sequence of partitions indexed by $\mathcal{X}$. The *size* of an $\mathcal{X}$-partition $\boldsymbol{\nu}$ is

$$||\boldsymbol{\nu}|| = \sum_{x \in \mathcal{X}} |x||\boldsymbol{\nu}(x)|, \quad \text{where} \quad |x| = \begin{cases} |x| & \text{if } \mathcal{X} = \Theta, \\ d(x) & \text{if } \mathcal{X} = \Phi, \end{cases} \tag{2.5}$$

$|x|$ is the size of the orbit $x \in \Theta$, and $d(x)$ is the degree of the polynomial $x \in \Phi$. Let

$$\mathcal{P}_n^{\mathcal{X}} = \{\mathcal{X}\text{-partitions } \boldsymbol{\nu} \mid ||\boldsymbol{\nu}|| = n\}, \quad \text{and} \quad \mathcal{P}^{\mathcal{X}} = \bigcup_{n=1}^{\infty} \mathcal{P}_n^{\mathcal{X}}. \tag{2.6}$$

For $\boldsymbol{\nu} \in \mathcal{P}^{\mathcal{X}}$, let

$$n(\boldsymbol{\nu}) = \sum_{x \in \mathcal{X}} |x| n(\boldsymbol{\nu}(x)), \quad \text{where} \quad n(\nu) = \sum_{i=1}^{\ell(\nu)} (i-1)\nu_i. \tag{2.7}$$

The *conjugate* $\boldsymbol{\nu}'$ of $\boldsymbol{\nu}$ is the $\mathcal{X}$-partition $\boldsymbol{\nu}' = (\boldsymbol{\nu}(x_1)', \boldsymbol{\nu}(x_2)', \ldots)$, where $\nu'$ is the usual conjugate partition for $\nu \in \mathcal{P}$.

The *semisimple part* $\boldsymbol{\nu}_s$ of $\boldsymbol{\nu} = (\boldsymbol{\nu}(x_1), \boldsymbol{\nu}(x_2), \ldots) \in \mathcal{P}_n^{\mathcal{X}}$ is

$$\boldsymbol{\nu}_s = ((1^{|\boldsymbol{\nu}(x_1)|}), (1^{|\boldsymbol{\nu}(x_2)|}), \ldots) \in \mathcal{P}_n^{\mathcal{X}}, \tag{2.8}$$

and the *unipotent part* $\boldsymbol{\nu}_u$ of $\boldsymbol{\nu} \in \mathcal{P}_n^{\mathcal{X}}$ is given by

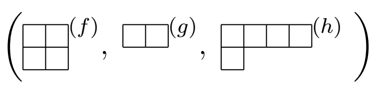
$$\boldsymbol{\nu}_u(\mathbb{1}) \quad \text{has parts} \quad \{|x|\boldsymbol{\nu}(x)_i \mid x \in \mathcal{X}, i = 1, \ldots, \ell(\boldsymbol{\nu}(x))\} \tag{2.9}$$

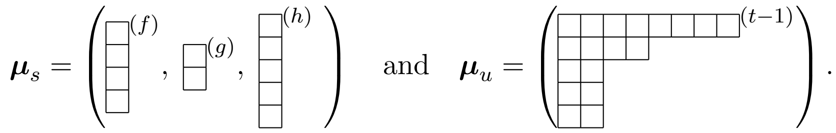
where

$$\mathbb{1} = \begin{cases} \{\mathbf{1}\} & \text{if } \mathcal{X} = \Theta, \\ t-1 & \text{if } \mathcal{X} = \Phi, \end{cases}$$

$\mathbf{1}$ is the trivial character in $\mathbb{K}^*$, and $\boldsymbol{\nu}_u(x) = \emptyset$ for $x \neq \mathbb{1}$.

**Example.** If $\boldsymbol{\mu} \in \mathcal{P}^{\Phi}$ is given by

$\boldsymbol{\mu} =$ , where $d(f) = 1$, $d(g) = d(h) = 2$, then $||\boldsymbol{\mu}|| = 1 \cdot 4 + 2 \cdot 2 + 2 \cdot 5 = 18$, $n(\boldsymbol{\mu}) = 1 \cdot 2 + 2 \cdot 0 + 2 \cdot 1 = 4$,



2.3. **Levi subgroups and maximal tori.** Let $\mathcal{X}$ be either $\Phi$ or $\Theta$ as in Section 2.2.

For $\boldsymbol{\nu} \in \mathcal{P}_n^{\mathcal{X}}$, let

$$L_{\boldsymbol{\nu}} = \bigoplus_{x \in \mathcal{X}_{\boldsymbol{\nu}}} L_{\boldsymbol{\nu}}(x), \quad \text{where} \quad \mathcal{X}_{\boldsymbol{\nu}} = \{x \in \mathcal{X} \mid \boldsymbol{\nu}(x) \neq \emptyset\}, \tag{2.10}$$

and for $x \in \mathcal{X}_{\boldsymbol{\nu}}$,

$$L_{\boldsymbol{\nu}}(x) = \begin{cases} \mathrm{U}(|\boldsymbol{\nu}(x)|, \mathbb{K}_{2|x|}) & \text{if } |x| \text{ is odd,} \\ \mathrm{GL}(|\boldsymbol{\nu}(x)|, \mathbb{K}_{|x|}) & \text{if } |x| \text{ is even.} \end{cases} \tag{2.11}$$

Then $L_{\boldsymbol{\nu}}$ is a Levi subgroup of $U_n = \mathrm{U}(n, \mathbb{K}_2)$ (though not uniquely determined by $\boldsymbol{\nu}$). The Weyl group

$$W_{\boldsymbol{\nu}} = \bigoplus_{x \in \mathcal{X}_{\boldsymbol{\nu}}} S_{|\boldsymbol{\nu}(x)|}, \tag{2.12}$$

of $L_{\boldsymbol{\nu}}$ has conjugacy classes indexed by

$$\mathcal{P}_s^{\boldsymbol{\nu}} = \{\boldsymbol{\gamma} \in \mathcal{P}^{\mathcal{X}} \mid \boldsymbol{\gamma}_s = \boldsymbol{\nu}_s\}, \tag{2.13}$$

and the size of the conjugacy class $c_{\boldsymbol{\gamma}}$ is

$$|c_{\boldsymbol{\gamma}}| = \frac{|W_{\boldsymbol{\gamma}}|}{z_{\boldsymbol{\gamma}}}, \quad \text{where} \quad z_{\boldsymbol{\gamma}} = \prod_{x \in \mathcal{X}} z_{\boldsymbol{\gamma}(x)} \quad \text{and} \quad z_\gamma = \prod_{i=1}^{\ell(\gamma)} i^{m_i} m_i!, \tag{2.14}$$

for $\gamma = (1^{m_1} 2^{m_2} \cdots) \in \mathcal{P}$.

For every $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathcal{P}_n$ there exists a maximal torus (unique up to isomorphism) $T_\nu$ of $U_n$ such that

$$T_\nu \cong T_{\nu_1} \times T_{\nu_2} \times \cdots \times T_{\nu_\ell}.$$

For every $\boldsymbol{\gamma} \in \mathcal{P}_s^{\boldsymbol{\mu}}$, there exists a maximal torus (unique up to isomorphism) $T_{\boldsymbol{\gamma}} \subseteq L_{\boldsymbol{\nu}}$ such that

$$T_{\boldsymbol{\gamma}} = \bigoplus_{x \in \mathcal{X}_{\boldsymbol{\nu}}} T_{\boldsymbol{\gamma}}(x), \quad \text{where} \quad T_{\boldsymbol{\gamma}}(x) \cong T_{|x|\boldsymbol{\gamma}(x)_1} \times \cdots \times T_{|x|\boldsymbol{\gamma}(x)_\ell}. \tag{2.15}$$

Note that as a maximal torus of $U_n$, the torus $T_{\boldsymbol{\gamma}} \cong T_{\boldsymbol{\gamma}_u(\mathbb{1})}$.

## 3. Conjugacy classes and Jordan decomposition

Let $U_n = \mathrm{U}(n, \mathbb{K}_2)$ as in (2.2). For $r \in \mathbb{Z}_{\geq 0}$, let

$$\psi_r(x) = \prod_{i=1}^{r}(1 - x^i). \tag{3.1}$$

In the following proposition, (i) is due to Ennola in [6], and the order of the centralizer in (ii) was obtained by Wall in [22] in a slightly different form, while the version in the proposition below appears in [6].

**Proposition 3.1** (Ennola, Wall)**.**
(a) *The conjugacy classes $c_{\boldsymbol{\mu}}$ of $U_n$ are indexed by $\boldsymbol{\mu} \in \mathcal{P}_n^{\Phi}$.*
(b) *Let $g \in c_{\boldsymbol{\mu}}$. The order $a_{\boldsymbol{\mu}}$ of the centralizer $g$ in $U_n$ is*

$$a_{\boldsymbol{\mu}} = (-1)^{||\boldsymbol{\mu}||} \prod_{f \in \Phi} a_{\boldsymbol{\mu}(f)}\big((-q)^{d(f)}\big), \text{ where } a_\mu(x) = x^{|\mu| + 2n(\mu)} \prod_j \psi_{m_j}(x^{-1}),$$

*for $\mu = (1^{m_1} 2^{m_2} 3^{m_3} \cdots) \in \mathcal{P}$.*

For $\boldsymbol{\mu} \in \mathcal{P}^{\Phi}$, let $L_{\boldsymbol{\mu}}$ be as in (2.10). Note that $|L_{\boldsymbol{\mu}}| = a_{\boldsymbol{\mu}_s}$.

**Lemma 3.1.** *Suppose $g \in c_{\boldsymbol{\mu}}$ with Jordan decomposition $g = su$. Then*

(a) *$s \in c_{\boldsymbol{\mu}_s}$ and $u \in c_{\boldsymbol{\mu}_u}$, where $\boldsymbol{\mu}_s$ and $\boldsymbol{\mu}_u$ are as in (2.8) and (2.9),*
(b) *the centralizer $C_{U_n}(s)$ of $s$ in $U_n$ is isomorphic to $L_{\boldsymbol{\mu}}$.*

*Proof.* (a) Suppose $f = t^d - a_{d-1}t^{d-1} - \cdots a_1 t - a_0 \in \Phi$ is irreducible (so $d$ is odd). Define

$$J(f) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{d-1} \end{pmatrix} \in \mathrm{GL}(d, \mathbb{K}_2). \tag{3.2}$$

For $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathcal{P}$, let

$$J_\nu(f) = J_{\nu_1}(f) \oplus \cdots \oplus J_{\nu_\ell}(f), \text{ where } J_m(f) = \begin{pmatrix} J(f) & \mathrm{Id}_d & & 0 \\ 0 & J(f) & \ddots & \\ \vdots & \ddots & \ddots & \mathrm{Id}_d \\ 0 & \cdots & 0 & J(f) \end{pmatrix}$$

is an $md \times md$ matrix and $\mathrm{Id}_d$ is the $d \times d$ identity matrix. Note that

$$J_m(f) = J_{(1^m)}(f) u_{(m)}(f), \quad \text{where} \quad u_{(m)}(f) = \begin{pmatrix} \mathrm{Id}_d & J(f)^{-1} & & 0 \\ & \mathrm{Id}_d & \ddots & \\ & & \ddots & J(f)^{-1} \\ 0 & & & \mathrm{Id}_d \end{pmatrix}$$

is the Jordan decomposition of $J_m(f)$ in $\mathrm{GL}(d, \mathbb{K}_2)$.

Similarly, if $f = h\tilde{h} \in \Phi$ (or $d(f)$ is even), then define

$$J(f) = J(h) \oplus J(\tilde{h}), \quad J_m(f) = J_m(h) \oplus J_m(\tilde{h}), \quad J_\nu(f) = J_\nu(h) \oplus J_\nu(\tilde{h}),$$

so that

$$J_m(f) = \big(J_{(1^m)}(h) \oplus J_{(1^m)}(\tilde{h})\big)\big(u_{(m)}(h) \oplus u_{(m)}(\tilde{h})\big)$$

is the Jordan decomposition of $J_m(f)$. If $g \in c_{\boldsymbol{\mu}}$, then $g$ is conjugate to

$$J_{\boldsymbol{\mu}} = \bigoplus_{f \in \Phi} J_{\boldsymbol{\mu}(f)}(f)$$

in $\mathrm{GL}(||\boldsymbol{\mu}||, \mathbb{K}_2)$. Claim (a) follows from the uniqueness of the Jordan decomposition.

(b) In $\mathrm{GL}(n, \mathbb{K}_2)$,

$$C_{\mathrm{GL}(n,\mathbb{K}_2)}(J_{(1^m)}(f)) \cong \begin{cases} \mathrm{GL}(m, \mathbb{K}_{2d(f)}) & \text{if } d(f) \text{ is odd,} \\ \mathrm{GL}(m, \mathbb{K}_{d(f)}) \oplus \mathrm{GL}(m, \mathbb{K}_{d(f)}) & \text{if } d(f) \text{ is even.} \end{cases}$$

If $d(f)$ is odd and $s = xJ_{(1^m)}(f)x^{-1}$ for some $x \in \mathrm{GL}(n, \mathbb{K}_2)$, then

$$(x\mathrm{GL}(m, \mathbb{K}_{2d(f)})x^{-1})^F = \mathrm{GL}(m, \mathbb{K}_{2d(f)})^{x \circ F} \cong \mathrm{GL}(m, \mathbb{K}_{2d(f)})^F = \mathrm{U}(m, \mathbb{K}_{2d(f)}).$$

Suppose $d(f)$ is even so that $f = h\tilde{h}$, and let $y \in \mathrm{GL}(n, \mathbb{K}_2)$ such that $\mathrm{GL}(n, \mathbb{K})^F = y\mathrm{GL}(n, \mathbb{K})^{F'}y^{-1}$ (see the comment after (2.2)). The element $J(f)$ is conjugate to

$$J(h) \oplus (F'(J(h))) \in \mathrm{GL}(n, \mathbb{K}_2)^{F'},$$

whose centralizer in $\mathrm{GL}(n, \mathbb{K}_2)^{F'}$ is

$$\{g \oplus F'(g) \mid g \in \mathrm{GL}(m, \mathbb{K}_{d(f)})\} \cong \mathrm{GL}(m, \mathbb{K}_{d(f)}). \quad \square$$

## 4. The Ennola Conjecture

4.1. **The characteristic map.** Let $X = \{X_1, X_2, \ldots\}$ be an infinite set of variables and let $\Lambda(X)$ be the graded $\mathbb{C}$-algebra of symmetric functions in the variables $\{X_1, X_2, \ldots\}$. For $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathcal{P}$, the *power-sum symmetric function* $p_\nu(X)$ is

$$p_\nu(X) = p_{\nu_1}(X)p_{\nu_2}(X)\cdots p_{\nu_\ell}(X), \quad \text{where} \quad p_m(X) = X_1^m + X_2^m + \cdots.$$

The irreducible characters $\omega^\lambda$ of $S_n$ are indexed by $\lambda \in \mathcal{P}_n$. Let $\omega^\lambda(\nu)$ be the value of $\omega^\lambda$ on a permutation with cycle type $\nu$. For $\lambda \in \mathcal{P}$, the *Schur function* $s_\lambda(X)$ is given by

$$s_\lambda(X) = \sum_{\nu \in \mathcal{P}_{|\lambda|}} \omega^\lambda(\nu) z_\nu^{-1} p_\nu(X), \quad \text{where} \quad z_\nu = \prod_{i \geq 1} i^{m_i} m_i! \tag{4.1}$$

is the order of the centralizer in $S_n$ of the conjugacy class corresponding to $\nu = (1^{m_1}2^{m_2}\cdots) \in \mathcal{P}$. Let $t \in \mathbb{C}$. For $\mu \in \mathcal{P}$, the *Hall-Littlewood symmetric function* $P_\mu(X;t)$ is given by

$$s_\lambda(X) = \sum_{\mu \in \mathcal{P}_{|\lambda|}} K_{\lambda\mu}(t) P_\mu(X;t), \tag{4.2}$$

where $K_{\lambda\mu}(t)$ is the Kostka-Foulkes polynomial (as in [19, III.6]). For $\nu, \mu \in \mathcal{P}_n$, the *classical Green function* $Q^{\mu}_{\nu}(t)$ is given by

$$p_\nu(X) = \sum_{\mu \in \mathcal{P}_{|\nu|}} Q^{\mu}_{\nu}(t^{-1}) t^{n(\mu)} P_\mu(X;t). \tag{4.3}$$

As a graded ring,

$$\begin{aligned} \Lambda(X) &= \mathbb{C}\text{-span}\{p_\nu(X) \mid \nu \in \mathcal{P}\} \\ &= \mathbb{C}\text{-span}\{s_\lambda(X) \mid \lambda \in \mathcal{P}\} \\ &= \mathbb{C}\text{-span}\{P_\mu(X;t) \mid \mu \in \mathcal{P}\}. \end{aligned}$$

For every $f \in \Phi$, fix a set of independent variables $X^{(f)} = \{X^{(f)}_1, X^{(f)}_2, \ldots\}$, and for any symmetric function $h$, we let $h(f) = h(X^{(f)})$ denote the symmetric function in the variables $X^{(f)}$. Let

$$\Lambda = \mathbb{C}\text{-span}\{P_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathcal{P}^{\Phi}\}, \quad \text{where} \quad P_{\boldsymbol{\mu}} = (-q)^{-n(\boldsymbol{\mu})} \prod_{f \in \Phi} P_{\boldsymbol{\mu}(f)}(f; (-q)^{-d(f)}).$$

Then

$$\Lambda = \bigoplus_{n \geq 0} \Lambda_n, \quad \text{where} \quad \Lambda_n = \mathbb{C}\text{-span}\{P_{\boldsymbol{\mu}} \mid \, ||\boldsymbol{\mu}|| = n\},$$

makes $\Lambda$ a graded $\mathbb{C}$-algebra. Define a Hermitian inner product on $\Lambda$ by

$$\langle P_{\boldsymbol{\mu}}, P_{\boldsymbol{\nu}} \rangle = a^{-1}_{\boldsymbol{\mu}} \delta_{\boldsymbol{\mu}\boldsymbol{\nu}}.$$

For each $\varphi \in \Theta$ let $Y^{(\varphi)} = \{Y^{(\varphi)}_1, Y^{(\varphi)}_2, \ldots\}$ be an infinite variable set, and for a symmetric function $h$, let $h(\varphi) = h(Y^{(\varphi)})$. Relate symmetric functions in the $X$ variables to symmetric functions in the $Y$ variables via the transform

$$p_n(\varphi) = (-1)^{n|\varphi|-1} \sum_{x \in T_{n|\varphi|}} \xi(x) p_{n|\varphi|/d(f_x)}(f_x), \tag{4.4}$$

where $\varphi \in \Theta$, $\xi \in \varphi$, and $f_x \in \Phi$ satisfies $f_x(x) = 0$.

Then

$$\Lambda = \mathbb{C}\text{-span}\{s_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}^{\Theta}\}, \quad \text{where} \quad s_{\boldsymbol{\lambda}} = \prod_{\varphi \in \Theta} s_{\boldsymbol{\lambda}(\varphi)}(\varphi). \tag{4.5}$$

Let $C_n$ denote the set of complex-valued class functions of the group $U_n$, and for $||\boldsymbol{\mu}|| = n$, let $\pi_{\boldsymbol{\mu}} : U_n \to \mathbb{C}$ be given by

$$\pi_{\boldsymbol{\mu}}(u) = \begin{cases} 1 & \text{if } u \in c_{\boldsymbol{\mu}}, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{where } u \in U_n.$$

Then the $\pi_{\boldsymbol{\mu}}$ form a $\mathbb{C}$-basis for $C_n$. By Proposition 3.1, the usual inner product on class functions of finite groups, $\langle \cdot, \cdot \rangle : C_n \times C_n \to \mathbb{C}$, satisfies

$$\langle \pi_{\boldsymbol{\mu}}, \pi_{\boldsymbol{\lambda}} \rangle = a^{-1}_{\boldsymbol{\mu}} \delta_{\boldsymbol{\mu}\boldsymbol{\lambda}}.$$

For $\alpha_i \in C_{n_i}$, Ennola [6] defined a product $\alpha_1 \star \alpha_2 \in C_{n_1+n_2}$, which takes the following value on the conjugacy class $c_{\boldsymbol{\lambda}}$:

$$\alpha_1 \star \alpha_2(c_{\boldsymbol{\lambda}}) = \sum_{||\boldsymbol{\mu}_i||=n_i} g^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}_1\boldsymbol{\mu}_2}\alpha_1(c_{\boldsymbol{\mu}_1})\alpha_2(c_{\boldsymbol{\mu}_2}),$$

where $g^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}_1\boldsymbol{\mu}_2}$ is the product of Hall polynomials (see [19, Chapter II])

$$g^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}_1\boldsymbol{\mu}_2} = \prod_{f\in\Phi} g^{\boldsymbol{\lambda}(f)}_{\boldsymbol{\mu}_1(f)\boldsymbol{\mu}_2(f)}((-q)^{d(f)}).$$

Extend the inner product to

$$C = \bigoplus_{n\geq 0} C_n,$$

by requiring the components $C_n$ and $C_m$ to be orthogonal for $n \neq m$. This gives $C$ a graded $\mathbb{C}$-algebra structure. The *characteristic map* is

$$\begin{array}{rccll} \mathrm{ch}: & C & \longrightarrow & \Lambda & \\ & \pi_{\boldsymbol{\mu}} & \mapsto & P_{\boldsymbol{\mu}}, & \quad \text{for } \boldsymbol{\mu} \in \mathcal{P}^{\Phi}. \end{array}$$

The following is implicit in [6], and a proof quickly follows from [19, III.3.6].

**Proposition 4.1.** *Let multiplication in the character ring $C$ of $U_n$ be given by $\star$. Then the characteristic map* $\mathrm{ch} : C \to \Lambda$ *is an isometric isomorphism of graded $\mathbb{C}$-algebras.*

Following the work of Green [9] on the general linear group, Ennola was able to obtain the following result. One may obtain a proof from the characteristic map point of view by following Macdonald [19, IV.4] on the general linear group case.

**Proposition 4.2** (Ennola)**.** *The set $\{s_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}^{\Theta}\}$ is an orthonormal basis for $\Lambda$.*

Now let $\chi^{\boldsymbol{\lambda}} \in C$ be class functions so that $\chi^{\boldsymbol{\lambda}}(1) > 0$ and $\mathrm{ch}(\chi^{\boldsymbol{\lambda}}) = \pm s_{\boldsymbol{\lambda}}$. Ennola conjectured that $\{\chi^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}^{\Theta}_n\}$ is the set of irreducible characters of $U_n$. He pointed out that if one could show that the product $\star$ takes virtual characters to virtual characters, then the conjecture would follow. There is no known direct proof of this fact, however. Significant progress on Ennola's conjecture was only made after the work of Deligne and Lusztig [3].

4.2. **Deligne-Lusztig Induction.** Let $T_\nu \cong T_{\nu_1} \times \cdots \times T_{\nu_\ell}$ be a maximal torus of $U_n$. If $t \in T_\nu$, then $t$ is conjugate to

$$J_{(1^{m_1})}(f_1) \oplus \cdots \oplus J_{(1^{m_\ell})}(f_\ell), \quad \text{where } f_i \in \Phi,\ m_i d(f_i) = \nu_i.$$

Define $\boldsymbol{\gamma}_t \in \mathcal{P}^{\Phi}$ by

$$\boldsymbol{\gamma}_t(f) \qquad \text{has parts} \qquad \{m_i \mid f_i = f\}. \tag{4.6}$$

Note that $(\boldsymbol{\gamma}_t)_u(t-1) = \nu$, but in general $t \notin c_{\boldsymbol{\gamma}_t}$.

**Example.** If $t \in T_{(4,4,2,2,1)}$ and $t$ is conjugate to

$$J_{\yng(1,1,1)}(t-1) \oplus J_{\yng(1,1)}(t^2+1) \oplus J_{\yng(1,1)}(t-1) \oplus J_{\yng(1)}(t^2+1) \oplus J_{\yng(1)}(t-1),$$

then

$$\boldsymbol{\gamma}_t = \left( \yng(4,2,1)^{(t-1)}, \yng(2,1)^{(t^2+1)} \right).$$

For $\boldsymbol{\mu} \in \mathcal{P}^\Phi$, let $L_{\boldsymbol{\mu}}$, $\boldsymbol{\gamma} \in \mathcal{P}_s^{\boldsymbol{\mu}}$ and $T_{\boldsymbol{\gamma}}$ be as in Section 2.3. Let $\theta$ be a character of $T_\nu$. The *Deligne-Lusztig character* $R_\nu(\theta) = R_{T_\nu}^{U_n}(\theta)$ is the virtual character of $U_n$ given by

$$\big(R_\nu(\theta)\big)(g) = \sum_{\substack{t \in T_\nu \\ \gamma_t \in \mathcal{P}_s^{\boldsymbol{\mu}}}} \theta(t) Q_{T_{\gamma_t}}^{L_{\boldsymbol{\mu}}}(u),$$

where $g \in c_{\boldsymbol{\mu}}$ has Jordan decomposition $g = su$ (thus, by Lemma 3.1 $C_{U_n}(s) \cong L_{\boldsymbol{\mu}}$), and $Q_{T_{\gamma_t}}^{L_{\boldsymbol{\mu}}}(u)$ is a Green function for the unitary group (see, for example, [2]).

It is proven by Lusztig and Srinivasan [18] that

$$\begin{aligned} C_n &= \{\text{class functions of } U_n\} \\ &= \mathbb{C}\text{-span}\{R_\nu(\theta) \mid \nu \in \mathcal{P}_n, \theta \in \mathrm{Hom}(T_\nu, \mathbb{C}^\times)\}, \end{aligned}$$

so we may define *Deligne-Lusztig induction* by

$$\text{(4.7)} \qquad \begin{array}{cccc} R_{U_m \oplus U_n}^{U_{m+n}}: & C_m \otimes C_n & \longrightarrow & C_{m+n} \\ & R_\alpha^{U_m}(\theta_\alpha) \otimes R_\beta^{U_n}(\theta_\beta) & \mapsto & R_{T_\alpha \oplus T_\beta}^{U_{m+n}}(\theta_\alpha \otimes \theta_\beta), \end{array}$$

for $\alpha \in \mathcal{P}_m$, $\beta \in \mathcal{P}_n$, $\theta_\alpha \in \mathrm{Hom}(T_\alpha, \mathbb{C})$, and $\theta_\beta \in \mathrm{Hom}(T_\beta, \mathbb{C})$.

Let $\Lambda$ and $C$ be as in Section 4.1, except we now give $C$ a graded $\mathbb{C}$-algebra structure using Deligne-Lusztig induction. That is, we define a multiplication $\circ$ on $C$ by

$$\chi \circ \eta = R_{U_m \oplus U_n}^{U_{m+n}}(\chi \otimes \eta), \quad \text{for } \chi \in C_m \text{ and } \eta \in C_n.$$

We recall the characteristic map defined in Section 4.1,

$$\begin{array}{cccc} \mathrm{ch}: & C & \longrightarrow & \Lambda \\ & \pi_{\boldsymbol{\mu}} & \mapsto & P_{\boldsymbol{\mu}} \qquad \text{for } \boldsymbol{\mu} \in \mathcal{P}^\Phi. \end{array}$$

As noted in Proposition 4.1, it is immediate that ch is an isometric isomorphism of vector spaces, but it is not yet clear if ch is also a ring homomorphism when $C$ has multiplication given by Deligne-Lusztig induction.

4.3. **The Ennola conjecture.** To prove the Ennola conjecture we require two further ingredients:

(1) Theorem 4.2 and Corollary 4.1 establish that this new characteristic map is also a ring isomorphism by using a key result by Kawanaka on Green functions to evaluate $\mathrm{ch}(R_\nu(\theta))$.

(2) Corollary 4.3 uses an explicit decomposition by Lusztig and Srinivasan of irreducible characters of $U_n$ into Deligne-Lusztig characters to complete the reproof of the Ennola conjecture.

To compute $\mathrm{ch}(R_\nu(\theta))$, we need to write the Green functions $Q_{T_\gamma}^{L_{\boldsymbol{\mu}}}(u)$ as polynomials in $q$. These Green functions turn out to be those of the general linear group, except with $q$ replaced by $-q$, which is the essence of Ennola's original idea. This fact was proven by Hotta and Springer [12] for the case that $p = \mathrm{char}(\mathbb{F}_q)$ is large compared to $n$, and was finally proven in full generality by Kawanaka [15].

**Theorem 4.1** (Hotta-Springer, Kawanaka)**.** *The Green functions for the unitary group are given by* $Q_{T_\gamma}^{L_{\boldsymbol{\mu}}}(u) = Q_{\boldsymbol{\gamma}}^{\boldsymbol{\mu}}(-q)$*, where*

$$Q_{\boldsymbol{\gamma}}^{\boldsymbol{\mu}}(-q) = \prod_{f\in\Phi_{\boldsymbol{\mu}}} Q_{\boldsymbol{\gamma}(f)}^{\boldsymbol{\mu}(f)}((-q)^{d(f)}),$$

*and* $Q_\gamma^\mu(q)$ *is the classical Green function as in (4.3).*

For $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathcal{P}$ and $\theta = \theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_\ell$ a character of $T_\nu$, define

$$p_{\boldsymbol{\mu}_{\nu\theta}} = \prod_{\varphi\in\Theta} p_{\boldsymbol{\mu}_{\nu\theta}(\varphi)}(\varphi), \quad \text{where} \quad \boldsymbol{\mu}_{\nu\theta}(\varphi) = (\nu_i/|\varphi| \mid \theta_i \in \varphi).$$

**Theorem 4.2.** *Let* $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathcal{P}$*,* $\theta = \theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_\ell$ *be a character of* $T_\nu$*, and* $\boldsymbol{\nu} = \boldsymbol{\mu}_{\nu\theta} \in \mathcal{P}^\Theta$*. Then*

$$\mathrm{ch}\big(R_\nu(\theta)\big) = (-1)^{||\boldsymbol{\nu}||-\ell(\boldsymbol{\nu})} p_{\boldsymbol{\nu}}.$$

*Proof.* By Theorem 4.1, and since $\mathrm{ch}(\pi_{\boldsymbol{\mu}}) = P_{\boldsymbol{\mu}}$, it suffices to show that the coefficient of $P_{\boldsymbol{\mu}}$ in the expansion of $(-1)^{|\boldsymbol{\nu}|-\ell(\boldsymbol{\nu})} p_{\boldsymbol{\nu}}$ is

$$\sum_{\substack{t\in T_\nu \\ \gamma_t\in\mathcal{P}_s^{\boldsymbol{\mu}}}} \theta_\nu(t) Q_{\boldsymbol{\gamma}_t}^{\boldsymbol{\mu}}(-q).$$

Since

$$p_{\boldsymbol{\nu}} = p_{\nu_1/|\varphi_1|}(\varphi_1)\cdots p_{\nu_\ell/|\varphi_\ell|}(\varphi_\ell), \qquad \text{where} \quad \theta_i \in \varphi_i,$$

the transform (4.4) implies

$$(-1)^{||\boldsymbol{\nu}||-\ell(\boldsymbol{\nu})} p_{\boldsymbol{\nu}} = \prod_{i=1}^{\ell} \sum_{t_i\in k_{\nu_i}^\times} \theta_i(t_i) p_{\nu_i/d_i}(f_i) = \sum_{t\in T_\nu} \theta(t) \bigg(\prod_{i=1}^{\ell} p_{\nu_i/d_i}(f_i)\bigg),$$

where $f_i \in \Phi$ satisfies $f_i(t_i) = 0$ and $d_i = d(f_i)$. By definition (4.6),

$$(-1)^{||\boldsymbol{\nu}||-\ell(\boldsymbol{\nu})} p_{\boldsymbol{\nu}} = \sum_{t\in T_\nu} \theta(t) \bigg(\prod_{f\in\Phi} p_{\boldsymbol{\gamma}_t(f)}(f)\bigg). \tag{4.8}$$

Change the basis from power-sums to Hall-Littlewood polynomials (4.3), and we obtain

$$
\begin{aligned}
&(-1)^{||\boldsymbol{\nu}||-\ell(\boldsymbol{\nu})} p_{\boldsymbol{\nu}} \\
&= \sum_{t \in T_\nu} \theta(t) \Big( \prod_{f \in \Phi} \sum_{\boldsymbol{\mu}(f) \in \mathcal{P}_{|\gamma_t(f)|}} Q^{\boldsymbol{\mu}(f)}_{\boldsymbol{\gamma}_t(f)}((-q)^{d(f)}) (-q)^{-d(f) n(\boldsymbol{\mu}(f))} P_{\boldsymbol{\mu}(f)}(f; (-q)^{-d(f)}) \Big) \\
&= \sum_{t \in T_\nu} \theta(t) \sum_{\substack{\boldsymbol{\mu} \in \mathcal{P}^\Phi \\ \gamma_t \in \mathcal{P}_s^{\boldsymbol{\mu}}}} Q^{\boldsymbol{\mu}}_{\boldsymbol{\gamma}_t}(-q) P_{\boldsymbol{\mu}} \\
&= \sum_{\boldsymbol{\mu} \in \mathcal{P}^\Phi} \Big( \sum_{\substack{t \in T_\nu \\ \gamma_t \in \mathcal{P}_s^{\boldsymbol{\mu}}}} \theta(t) Q^{\boldsymbol{\mu}}_{\boldsymbol{\gamma}_t}(-q) \Big) P_{\boldsymbol{\mu}},
\end{aligned}
$$

as desired. □

The following result immediately follows from the definition of the Deligne-Lusztig product (4.7) and Theorem 4.2. This result is equivalent to the Hopf algebra structure theorem obtained in [4].

**Corollary 4.1.** *Let multiplication in the character ring $C$ of $U_n$ be given by $\circ$. Then the characteristic map* $\mathrm{ch} : C \to \Lambda$ *is an isometric isomorphism of graded $\mathbb{C}$-algebras.*

An immediate consequence is that the graded multiplication that Ennola originally defined on $C$ is exactly Deligne-Lusztig induction, or

**Corollary 4.2.** *Let $\chi \in C_m$ and $\eta \in C_n$. Then*

$$\chi \circ \eta = \chi \star \eta.$$

We therefore have the advantage of using either product as convenience demands.

For $\boldsymbol{\lambda} \in \mathcal{P}^\Theta$, let $L_{\boldsymbol{\lambda}}$, $W_{\boldsymbol{\lambda}}$, and $T_{\boldsymbol{\gamma}}$, $\boldsymbol{\gamma} \in \mathcal{P}_s^{\boldsymbol{\lambda}}$, be as in Section 2.3.

Note that the combinatorics of $\boldsymbol{\gamma}$ almost specifies character $\theta_\gamma$ of $T_{\boldsymbol{\gamma}}$ in the sense that

$$\theta_{\boldsymbol{\gamma}}(T_{\boldsymbol{\gamma}}(\varphi)) = \theta_\varphi(T_{\boldsymbol{\gamma}}(\varphi)), \qquad \text{for some } \theta_\varphi \in \varphi.$$

In fact, we may define

$$R_{\boldsymbol{\gamma}} = R^{U_n}_{T_{\boldsymbol{\gamma}}}(\theta_{\boldsymbol{\gamma}}) = \mathrm{ch}^{-1}\big((-1)^{||\boldsymbol{\gamma}||-\ell(\boldsymbol{\gamma})} p_{\boldsymbol{\gamma}}\big), \tag{4.9}$$

where $\theta_{\boldsymbol{\gamma}}$ is any choice of the $\theta_\varphi$'s.

For every $\boldsymbol{\lambda} \in \mathcal{P}^\Theta$ there exists a character $\omega^{\boldsymbol{\lambda}}$ of $W_{\boldsymbol{\lambda}}$ defined by

$$\omega^{\boldsymbol{\lambda}}(\boldsymbol{\gamma}) = \prod_{\varphi \in \Theta} \omega^{\boldsymbol{\lambda}(\varphi)}(\boldsymbol{\gamma}(\varphi)),$$

where $\omega^{\boldsymbol{\lambda}}(\boldsymbol{\gamma})$ is the value of $\omega^{\boldsymbol{\lambda}}$ on the conjugacy class $c_{\boldsymbol{\gamma}}$ corresponding to $\boldsymbol{\gamma} \in \mathcal{P}_s^\Theta$.

In [18], Lusztig and Srinivasan decomposed the irreducible characters of $U_n$ as linear combinations of Deligne-Lusztig characters, as follows.

**Theorem 4.3** (Lusztig-Srinivasan)**.** *Let* $\boldsymbol{\lambda} \in \mathcal{P}_n^\Theta$*. Then there exists* $\tau'(\boldsymbol{\lambda}) \in \mathbb{Z}_{\geq 0}$ *such that the class function*

$$R(\boldsymbol{\lambda}) = (-1)^{\tau'(\boldsymbol{\lambda}) + \lfloor n/2 \rfloor + \sum_{\varphi\in\Theta} |\boldsymbol{\lambda}(\varphi)| + |\varphi|\lceil |\boldsymbol{\lambda}(\varphi)|/2\rceil} \sum_{\boldsymbol{\gamma}\in\mathcal{P}_s^{\boldsymbol{\lambda}}} \frac{\omega^{\boldsymbol{\lambda}}(\boldsymbol{\gamma})}{z_{\boldsymbol{\gamma}}} R_{\boldsymbol{\gamma}}$$

*is an irreducible character of* $U_n$ *(*$z_{\boldsymbol{\gamma}}$ *is as in (2.14)).*

**Remark.** The sign

$$(-1)^{\lfloor n/2 \rfloor + \sum_{\varphi\in\Theta} |\boldsymbol{\lambda}(\varphi)| + |\varphi|\lceil |\boldsymbol{\lambda}(\varphi)|/2\rceil} = (-1)^{\mathbb{F}_q\text{-rank of } U_n \,+\, \mathbb{F}_q\text{-rank of } L_{\boldsymbol{\lambda}}},$$

and Theorem 5.1 will show that

$$\tau'(\boldsymbol{\lambda}) = n(\boldsymbol{\lambda}') + ||\boldsymbol{\lambda}|| - \sum_{\varphi\in\Theta} |\varphi|\lceil |\boldsymbol{\lambda}(\varphi)|/2\rceil = n(\boldsymbol{\lambda}') + \sum_{\varphi\in\Theta} |\varphi|\lfloor |\boldsymbol{\lambda}(\varphi)|/2\rfloor.$$

**Corollary 4.3** (Ennola Conjecture)**.** *For* $\boldsymbol{\lambda} \in \mathcal{P}^\Theta$*, there exists* $\tau(\boldsymbol{\lambda}) \in \mathbb{Z}_{\geq 0}$ *such that*

$$\left\{ \mathrm{ch}^{-1}\left( (-1)^{\tau(\boldsymbol{\lambda})} s_{\boldsymbol{\lambda}} \right) \;\middle|\; \boldsymbol{\lambda} \in \mathcal{P}_n^\Theta \right\}$$

*is the set of irreducible characters of* $U_n$*.*

*Proof.* By Theorem 4.3 and Theorem 4.2,

$$\begin{aligned}
\mathrm{ch}(R(\boldsymbol{\lambda})) &= (-1)^{\tau'(\boldsymbol{\lambda}) + \lfloor n/2 \rfloor + \sum_{\varphi\in\Theta} |\boldsymbol{\lambda}(\varphi)| + |\varphi|\lceil |\boldsymbol{\lambda}(\varphi)|/2\rceil} \sum_{\boldsymbol{\gamma}\in\mathcal{P}_s^{\boldsymbol{\lambda}}} \frac{\omega^{\boldsymbol{\lambda}}(\boldsymbol{\gamma})}{z_{\boldsymbol{\gamma}}} (-1)^{n-\ell(\boldsymbol{\gamma})} p_{\boldsymbol{\gamma}} \\
&= (-1)^{\tau'(\boldsymbol{\lambda}) + \lfloor n/2 \rfloor + n + \sum_{\varphi\in\Theta} |\varphi|\lceil |\boldsymbol{\lambda}(\varphi)|/2\rceil} \sum_{\boldsymbol{\gamma}\in\mathcal{P}_s^{\boldsymbol{\lambda}}} \frac{\omega^{\boldsymbol{\lambda}}(\boldsymbol{\gamma})}{z_{\boldsymbol{\gamma}}} (-1)^{\sum_{\varphi\in\Theta} |\boldsymbol{\lambda}(\varphi)| - \ell(\boldsymbol{\gamma})} p_{\boldsymbol{\gamma}}.
\end{aligned}$$

Note that the sign character $\omega^{\boldsymbol{\lambda}_s}$ of $W_{\boldsymbol{\lambda}}$ acts by

$$\omega^{\boldsymbol{\lambda}_s}(\boldsymbol{\gamma}) = (-1)^{\sum_{\varphi\in\Theta} |\boldsymbol{\gamma}(\varphi)| - \ell(\boldsymbol{\gamma})},$$

and that $\omega^{\boldsymbol{\lambda}} \otimes \omega^{\boldsymbol{\lambda}_s} = \omega^{\boldsymbol{\lambda}'}$, so since $\boldsymbol{\gamma} \in \mathcal{P}_s^{\boldsymbol{\lambda}}$,

$$\begin{aligned}
\mathrm{ch}(R(\boldsymbol{\lambda})) &= (-1)^{\tau'(\boldsymbol{\lambda}) + \lfloor n/2 \rfloor + n + \sum_{\varphi\in\Theta} |\varphi|\lceil |\boldsymbol{\lambda}(\varphi)|/2\rceil} \sum_{\boldsymbol{\gamma}\in\mathcal{P}_s^{\boldsymbol{\lambda}}} \frac{(\omega^{\boldsymbol{\lambda}} \otimes \omega^{\boldsymbol{\lambda}_s})(\boldsymbol{\gamma})}{z_{\boldsymbol{\gamma}}} p_{\boldsymbol{\gamma}} \\
&= (-1)^{\tau'(\boldsymbol{\lambda}) + \lfloor n/2 \rfloor + n + \sum_{\varphi\in\Theta} |\varphi|\lceil |\boldsymbol{\lambda}(\varphi)|/2\rceil} \sum_{\boldsymbol{\gamma}\in\mathcal{P}_s^{\boldsymbol{\lambda}}} \frac{\omega^{\boldsymbol{\lambda}'}(\boldsymbol{\gamma})}{z_{\boldsymbol{\gamma}}} p_{\boldsymbol{\gamma}},
\end{aligned}$$

and by applying (4.1) to a product over $\Theta$,

$$= (-1)^{\tau'(\boldsymbol{\lambda}) + \lfloor n/2 \rfloor + n + \sum_{\varphi\in\Theta} |\varphi|\lceil |\boldsymbol{\lambda}(\varphi)|/2\rceil} s_{\boldsymbol{\lambda}'}.$$ □

**Remark.** There are at least two natural ways to index the irreducible characters of $U_n$ by $\Theta$-partitions: Theorem 4.3 gives a natural indexing by $\Theta$-partitions, but Corollary 4.3 indicates that the conjugate choice is equally natural. Following Macdonald [19], we have chosen the latter indexing. However, several references, including Ennola [6], Ohmori [21], and Henderson [11], make use of the former.

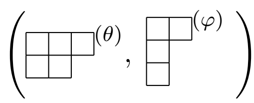
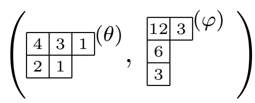
## 5. Characters degrees

**5.1. A formula for character degrees.** Let $\boldsymbol{\lambda} \in \mathcal{P}^{\Theta}$, and suppose $\square \in \boldsymbol{\lambda}$ is in position $(i,j)$ in $\boldsymbol{\lambda}(\varphi)$ for some $\varphi \in \Theta$. The *hook length* $\mathbf{h}(\square)$ of $\square$ is

$$\mathbf{h}(\square) = |\varphi| h(\square), \quad \text{where} \quad h(\square) = \boldsymbol{\lambda}(\varphi)_i - \boldsymbol{\lambda}(\varphi)'_j - i - j + 1,$$

is the usual hook length for partitions.

**Example.** For $|\theta| = 1$, $|\varphi| = 3$ and

$\boldsymbol{\lambda} =$  , the hook lengths are  .

Example 2 in [19, I.1] implies that

$$\sum_{\square \in \boldsymbol{\lambda}} \mathbf{h}(\square) = ||\boldsymbol{\lambda}|| + n(\boldsymbol{\lambda}) + n(\boldsymbol{\lambda}'). \tag{5.1}$$

For $\boldsymbol{\lambda} \in \mathcal{P}^{\Theta}$, let

$$\eta^{\boldsymbol{\lambda}} = \mathrm{ch}^{-1}(s_{\boldsymbol{\lambda}}).$$

**Theorem 5.1.** *Let $\boldsymbol{\lambda} \in \mathcal{P}^{\Theta}$ and let $1$ be the identity in $U_{||\boldsymbol{\lambda}||}$. Then*

$$\eta^{\boldsymbol{\lambda}}(1) = (-1)^{\tau(\boldsymbol{\lambda})} q^{n(\boldsymbol{\lambda}')} \frac{\prod_{1 \le i \le ||\boldsymbol{\lambda}||} (q^i - (-1)^i)}{\prod_{\square \in \boldsymbol{\lambda}} (q^{\mathbf{h}(\square)} - (-1)^{\mathbf{h}(\square)})},$$

*where $\tau(\boldsymbol{\lambda}) = ||\boldsymbol{\lambda}||(||\boldsymbol{\lambda}||+3)/2 + n(\boldsymbol{\lambda}) \equiv \lfloor ||\boldsymbol{\lambda}||/2 \rfloor + n(\boldsymbol{\lambda}) \pmod 2$. So for each $\boldsymbol{\lambda}$, we have $\chi^{\boldsymbol{\lambda}} = (-1)^{\tau(\boldsymbol{\lambda})} \eta^{\boldsymbol{\lambda}}$.*

*Proof.* We follow the computations in [19, IV.6]. Let $\eta^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}$ be the value of $\eta^{\boldsymbol{\lambda}}$ on the conjugacy class $c_{\boldsymbol{\mu}}$. Then

$$s_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu} \in \mathcal{P}^{\Phi}_{||\boldsymbol{\lambda}||}} \eta^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}} P_{\boldsymbol{\mu}},$$

implies $\eta^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}} = \langle s_{\boldsymbol{\lambda}}, a_{\boldsymbol{\mu}} P_{\boldsymbol{\mu}} \rangle$. Since the $s_{\boldsymbol{\lambda}}$ are orthonormal,

$$a_{\boldsymbol{\mu}} P_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\lambda}} \eta^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}} \bar{s}_{\boldsymbol{\lambda}} \qquad \text{where} \qquad \bar{s}_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu} \in \mathcal{P}^{\Phi}_{||\boldsymbol{\lambda}||}} \bar{\eta}^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}} P_{\boldsymbol{\mu}} \tag{5.2}$$

is obtained by taking the complex conjugates of the coefficients of the $P_{\boldsymbol{\mu}}$.

If $c_{\boldsymbol{\mu}_1}$ is the conjugacy class corresponding to the identity element 1 of $U_m$, then $\boldsymbol{\mu}_1(t-1) = (1^m)$ and $\boldsymbol{\mu}_1(f) = 0$ for $f \neq t-1$. From the definition of $a_{\boldsymbol{\mu}}$, and the fact that $P_{(1^m)}(x;t) = e_m(x)$ (see [19, III.8]), we have

$$\begin{aligned} a_{\boldsymbol{\mu}_1} P_{\boldsymbol{\mu}_1} &= (-1)^m (-q)^{m+m(m-1)} \psi_m(-q^{-1})(-q)^{-m(m-1)/2} P_{(1^m)}(X^{(t-1)}; (-q)^{-1}) \\ &= (-1)^m (-q)^{m(m+1)/2} \psi_m((-q)^{-1}) P_{(1^m)}(X^{(t-1)}; (-q)^{-1}) \\ &= \psi_m(-q) e_m(t-1). \end{aligned}$$

Therefore, by (5.2),

$$\psi_m(-q) e_m(t-1) = \sum_{||\boldsymbol{\lambda}||=m} \eta^{\boldsymbol{\lambda}}(1) \bar{s}_{\boldsymbol{\lambda}}. \tag{5.3}$$

Let $\delta : \Lambda \to \mathbb{C}$ be the $\mathbb{C}$-algebra homomorphism defined by

$$\delta(p_m(f)) = \begin{cases} (-1)^{m-1}/(q^m - (-1)^m) & \text{if } f = t-1, \\ 0 & \text{if } f \neq t-1. \end{cases}$$

It follows from [19, I.4.3] and the argument from [19, p. 279] that

$$\log \left( \sum_{\boldsymbol{\lambda} \in \mathcal{P}^\Theta} s_{\boldsymbol{\lambda}} \otimes \bar{s}_{\boldsymbol{\lambda}} \right) = \sum_{n \geq 1} \frac{1}{n} \sum_{f \in \Phi} (q^{nd(f)} - (-1)^{nd(f)}) p_n(f) \otimes p_n(f). \tag{5.4}$$

Apply $\delta \otimes 1$ to both sides of Equation (5.4) to obtain

$$\log \Big( \sum_{\boldsymbol{\lambda} \in \mathcal{P}^\Theta} \delta(s_{\boldsymbol{\lambda}}) \bar{s}_{\boldsymbol{\lambda}} \Big) = \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} p_m(t-1) = \log \prod_i (1 + X_i^{(t-1)}).$$

By exponentiating and expanding the product to $e_m$'s, we have

$$\sum_{m \geq 0} e_m(t-1) = \sum_{\boldsymbol{\lambda}} \delta(s_{\boldsymbol{\lambda}}) \bar{s}_{\boldsymbol{\lambda}},$$

which gives

$$e_m(t-1) = \sum_{||\boldsymbol{\lambda}||=m} \delta(s_{\boldsymbol{\lambda}}) \bar{s}_{\boldsymbol{\lambda}}.$$

Compare coefficients with (5.3) to deduce

$$\eta^{\boldsymbol{\lambda}}(1) = \psi_m(-q) \delta(s_{\boldsymbol{\lambda}}), \quad \text{where} \quad \delta(s_{\boldsymbol{\lambda}}) = \prod_{\varphi \in \Theta} \delta(s_{\boldsymbol{\lambda}(\varphi)}(\varphi)). \tag{5.5}$$

By the definitions of $\delta$ and $p_m(\varphi)$, $\varphi \in \Theta$,

$$\delta(p_m(\varphi)) = (-1)^{|\varphi| m} \big( (-q)^{|\varphi| m} - 1 \big)^{-1} = (-1)^{|\varphi| m} \sum_{i \geq 1} (-q)^{-i|\varphi| m},$$

so

$$\delta(p_m(Y_1^{(\varphi)}, Y_2^{(\varphi)}, \ldots)) = (-1)^{|\varphi| m} p_m((-q)^{-|\varphi|}, (-q)^{-2|\varphi|}, \ldots).$$

That is, applying $\delta$ to a homogeneous symmetric function in $Y^{(\varphi)}$ of degree $m$ replaces each $Y_i^{(\varphi)}$ by $(-q)^{-i|\varphi|}$ and multiplies by $(-1)^{|\varphi|m}$. In particular, for $\lambda \in \mathcal{P}$, $\varphi \in \Theta$,

$$\delta(s_\lambda(\varphi)) = (-1)^{|\lambda||\varphi|} s_\lambda\big((-q)^{-|\varphi|}, (-q)^{-2|\varphi|}, \dots\big). \tag{5.6}$$

Example 2 in [19, I.3] implies

$$\delta(s_\lambda(\varphi)) = (-1)^{|\varphi||\lambda|}(-q)^{-|\varphi|(|\lambda|+n(\lambda))} \prod_{\square\in\lambda}\big(1-(-q)^{-|\varphi|h(\square)}\big)^{-1}. \tag{5.7}$$

Combine (5.5) and (5.7) to get

$$\begin{aligned}
\delta(s_{\boldsymbol\lambda}) &= (-1)^{||\boldsymbol\lambda||}(-q)^{-||\boldsymbol\lambda||-n(\boldsymbol\lambda)} \prod_{\square\in\boldsymbol\lambda}\big(1-(-q)^{-\mathbf{h}(\square)}\big)^{-1}\\
&= (-1)^{||\boldsymbol\lambda||}(-q)^{-||\boldsymbol\lambda||-n(\boldsymbol\lambda)+\sum_{\square\in\boldsymbol\lambda}\mathbf{h}(\square)} \prod_{\square\in\boldsymbol\lambda}\big((-q)^{\mathbf{h}(\square)}-1\big)^{-1}\\
&= (-1)^{||\boldsymbol\lambda||}(-q)^{n(\boldsymbol\lambda')} \prod_{\square\in\boldsymbol\lambda}\big((-q)^{\mathbf{h}(\square)}-1\big)^{-1}, \qquad \text{(by (5.1))}\\
&= (-1)^{||\boldsymbol\lambda||+n(\boldsymbol\lambda')+\sum_{\square\in\boldsymbol\lambda}\mathbf{h}(\square)} q^{n(\boldsymbol\lambda')} \prod_{\square\in\boldsymbol\lambda}\big(q^{\mathbf{h}(\square)}-(-1)^{\mathbf{h}(\square)}\big)^{-1}.
\end{aligned}$$

Since

$$||\boldsymbol\lambda|| + n(\boldsymbol\lambda') + \sum_{\square\in\boldsymbol\lambda}\mathbf{h}(\square) = 2||\boldsymbol\lambda|| + 2n(\boldsymbol\lambda') + n(\boldsymbol\lambda) \equiv n(\boldsymbol\lambda) \pmod 2,$$

we have

$$\delta(s_{\boldsymbol\lambda}) = (-1)^{n(\boldsymbol\lambda)} q^{n(\boldsymbol\lambda')} \prod_{\square\in\boldsymbol\lambda}\big(q^{\mathbf{h}(\square)}-(-1)^{\mathbf{h}(\square)}\big)^{-1}.$$

Since

$$\psi_m(-q) = \prod_{i=1}^m (1-(-q)^i) = (-1)^{m+m(m+1)/2}\prod_{i=1}^m (q^i-(-1)^i),$$

we have

$$\eta^{\boldsymbol\lambda}(1) = (-1)^{\tau(\boldsymbol\lambda)} q^{n(\boldsymbol\lambda')} \prod_{1\le i\le ||\boldsymbol\lambda||}(q^i-(-1)^i)\prod_{\square\in\boldsymbol\lambda}(q^{\mathbf{h}(\square)}-(-1)^{\mathbf{h}(\square)})^{-1}. \quad \square$$

By the Littlewood-Richardson rule, for any $\mu,\nu\in\mathcal{P}$, we have

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda,$$

where $c^\lambda_{\mu\nu}$ is the number of tableaux $T$ of shape $\lambda-\mu$ and weight $\nu$ such that the word $w(T)$ is a lattice permutation [19, I.9]. So we have

$$s_{\boldsymbol\mu} s_{\boldsymbol\nu} = \sum_{\boldsymbol\lambda} c^{\boldsymbol\lambda}_{\boldsymbol\mu\boldsymbol\nu} s_{\boldsymbol\lambda} \quad \text{where} \quad c^{\boldsymbol\lambda}_{\boldsymbol\mu\boldsymbol\nu} = \prod_{\varphi\in\Theta} c^{\boldsymbol\lambda(\varphi)}_{\boldsymbol\mu(\varphi)\boldsymbol\nu(\varphi)}. \tag{5.8}$$

**Corollary 5.1.** *Let $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}^{\Theta}$. Then $\chi^{\boldsymbol{\mu}} \circ \chi^{\boldsymbol{\nu}}$ is a character if and only if every $\boldsymbol{\lambda} \in \mathcal{P}^{\Theta}$ such that $c^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}\boldsymbol{\nu}} > 0$ satisfies*

$$n(\boldsymbol{\mu}) + n(\boldsymbol{\nu}) \equiv n(\boldsymbol{\lambda}) + ||\boldsymbol{\mu}||\, ||\boldsymbol{\nu}|| \pmod 2$$

*Proof.* Since

$$\mathrm{ch}(\chi^{\boldsymbol{\mu}} \circ \chi^{\boldsymbol{\nu}}) = (-1)^{\tau(\boldsymbol{\mu})+\tau(\boldsymbol{\nu})} \sum_{\boldsymbol{\lambda}} c^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}\boldsymbol{\nu}} s_{\boldsymbol{\lambda}},$$

the class function $\chi^{\boldsymbol{\mu}} \circ \chi^{\boldsymbol{\nu}}$ is a character if and only if every $\boldsymbol{\lambda}$ such that $c^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}\boldsymbol{\nu}} > 0$ satisfies

$$\tau(\boldsymbol{\mu}) + \tau(\boldsymbol{\nu}) \equiv \tau(\boldsymbol{\lambda}) \pmod 2.$$

For each $\boldsymbol{\lambda}$ such that $c^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}\boldsymbol{\nu}} > 0$, we have, for every $\varphi \in \Theta$, a tableau of shape $\boldsymbol{\lambda}(\varphi) - \boldsymbol{\mu}(\varphi)$ and weight $\boldsymbol{\nu}(\varphi)$. In particular, we have $|\boldsymbol{\mu}(\varphi)| + |\boldsymbol{\nu}(\varphi)| = |\boldsymbol{\lambda}(\varphi)|$ for every $\varphi \in \Theta$, so that $||\boldsymbol{\mu}|| + ||\boldsymbol{\nu}|| = ||\boldsymbol{\lambda}||$. The result follows from the definition of $\tau$. □

It follows that the operation $\circ$ does not always take a pair of characters to a character. For example, let $\varphi_1$ be the orbit of the trivial character, and define $\boldsymbol{\mu} = \boldsymbol{\nu} = \big(\square^{(\varphi_1)}\big)$. Then one $\boldsymbol{\lambda}$ such that $c^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}\boldsymbol{\nu}} > 0$ is

$$\boldsymbol{\lambda} = \left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}^{(\varphi_1)}\right).$$

Then we have $n(\boldsymbol{\mu}) + n(\boldsymbol{\nu}) = 0$ while $n(\boldsymbol{\lambda}) + ||\boldsymbol{\mu}||\, ||\boldsymbol{\nu}|| = 1$, and so $\chi^{\boldsymbol{\mu}} \circ \chi^{\boldsymbol{\nu}}$ is not a character by Corollary 5.1.

5.2. **Character degree sums.** Let $d_r$ denote the number of $F$-orbits in $\Theta$ of size $r$. Since the $F$-action preserves $T^*_m$ and $|T^*_m| = q^m - (-1)^m$, we have

$$q^m - (-1)^m = \sum_{r|m} r d_r, \qquad \text{for } m \in \mathbb{Z}_{>0}. \tag{5.9}$$

**Theorem 5.2.** *The sum of the degrees of the complex irreducible characters of $U_m$ is given by*

$$\sum_{||\boldsymbol{\lambda}||=m} \chi^{\boldsymbol{\lambda}}(1) = (q+1)q^2(q^3+1)q^4(q^5+1)\cdots\Big(q^m + \frac{(1-(-1)^m)}{2}\Big).$$

*Proof.* Following a similar approach to [19, IV.6, Example 5], we consider the coefficient of $t^m$ in the series

$$S = \sum_{\boldsymbol{\lambda} \in \mathcal{P}^{\Theta}} (-1)^{n(\boldsymbol{\lambda}) + ||\boldsymbol{\lambda}||} \delta(s_{\boldsymbol{\lambda}}) t^{||\boldsymbol{\lambda}||},$$

where $n(\boldsymbol{\lambda})$ is as in (2.7). Note that $S = S_{\mathrm{o}} S_{\mathrm{e}}$, where

$$S_{\mathrm{o}} = \prod_{\substack{|\varphi| \\ \text{odd}}} \sum_{\lambda} (-1)^{n(\lambda)} \delta(s_\lambda(\varphi))(-t)^{|\lambda||\varphi|} \text{ and } S_{\mathrm{e}} = \prod_{\substack{|\varphi| \\ \text{even}}} \sum_{\lambda} \delta(s_\lambda(\varphi)) t^{|\lambda||\varphi|}.$$

Combine the following identity from Example 6 in [19, I.5],

$$\sum_{\lambda} (-1)^{n(\lambda)} s_\lambda = \prod_i (1-x_i)^{-1} \prod_{i<j} (1 + x_i x_j)^{-1}$$

with (5.6) to obtain

$$\sum_{\lambda\in\mathcal{P}}(-1)^{n(\lambda)}(-1)^{|\varphi||\lambda|}\delta(s_\lambda(\varphi))t^{|\lambda||\varphi|}$$
$$=\prod_{i\geq 1}\big(1-(t(-q)^{-i})^{|\varphi|}\big)^{-1}\prod_{1\leq i<j}\big(1+(t^2(-q)^{-i-j})^{|\varphi|}\big)^{-1}.$$

Taking the logarithm,

$$\begin{aligned}\log S_{\mathrm{o}} &= \sum_{m\text{ odd}} d_m\Big(\sum_{i\geq 1}\sum_{r\geq 1}\frac{(t(-q)^{-i})^{mr}}{r}+\sum_{1\leq i<j}\sum_{r\geq 1}\frac{\big(-(t^2(-q)^{-i-j})^m\big)^r}{r}\Big)\\ &= \sum_{m\text{ odd}} d_m\sum_{r\geq 1}\left(\frac{t^{mr}}{r}\sum_{i\geq 1}(-q)^{-imr}+(-1)^r\sum_{1\leq i<j}t^{2mr}(-q)^{-(i+j)mr}\right)\\ &= \sum_{m\text{ odd}} d_m\sum_{r\geq 1}\frac{t^{mr}}{r((-q)^{mr}-1)}\left(1+(-1)^r\sum_{i\geq 1}t^{mr}(-q)^{-2imr}\right)\\ &= \sum_{m\text{ odd}} d_m\sum_{r\geq 1}\frac{t^{mr}}{r(q^{mr}-(-1)^{mr})}\left((-1)^{mr}+\sum_{i\geq 1}t^{mr}q^{-2imr}\right),\end{aligned}$$

where $d_m$ is as in (5.9). Similarly for $|\varphi|$ even, by Example 4 in [19, I.5],

$$\sum_{\lambda}s_\lambda=\prod_i(1-x_i)^{-1}\prod_{i<j}(1-x_ix_j)^{-1}$$

so

$$\sum_{\lambda\in\mathcal{P}}\delta(s_\lambda(\varphi))t^{|\lambda||\varphi|}=\prod_{i\geq 1}\big(1-(tq^{-i})^{|\varphi|}\big)^{-1}\prod_{1\leq i<j}\big(1-(t^2q^{-i-j})^{|\varphi|}\big)^{-1}.$$

Taking logarithms,

$$\begin{aligned}\log S_{\mathrm{e}} &= \sum_{m\text{ even}} d_m\Big(\sum_{i\geq 1}\sum_{r\geq 1}\frac{(tq^{-i})^{mr}}{r}+\sum_{1\leq i<j}\sum_{r\geq 1}\frac{(t^2q^{-i-j})^{mr}}{r}\Big)\\ &= \sum_{m\text{ even}} d_m\sum_{r\geq 1}\frac{t^{mr}}{r(q^{mr}-(-1)^{mr})}\left((-1)^{mr}+\sum_{i\geq 1}t^{mr}q^{-2imr}\right).\end{aligned}$$

Let $N=mr$, and by (5.9),

$$\begin{aligned}\log S &= \log S_{\mathrm{o}}+\log S_{\mathrm{e}}\\ &= \sum_{N\geq 1}\Big(\frac{(-t)^N}{N}+\sum_{i\geq 1}\frac{t^{2N}q^{-2iN}}{N}\Big)\\ &= \log(1+t)^{-1}+\sum_{i\geq 1}\log(1-t^2q^{-2i})^{-1}.\end{aligned}$$

By exponentiating,

$$S=\frac{1}{1+t}\prod_{i\geq 1}\frac{1}{1-t^2q^{-2i}}=(1-t)\prod_{i\geq 0}\frac{1}{1-t^2q^{-2i}}=(1-t)\sum_{l\geq 0}\frac{t^{2l}}{\psi_l(q^{-2})},$$

where the last equality comes from Example 4 in [19, I.2]. Multiply the coefficient of $t^m$ by $(-1)^m|\psi_m(-q)| = (-1)^m \prod_{i=1}^m (q^i - (-1)^i)$ to obtain

$$\sum_{||\boldsymbol{\lambda}||=m} \chi^{\boldsymbol{\lambda}}(1) = (q+1)q^2(q^3+1)q^4(q^5+1)\cdots\big(q^m + \frac{(1-(-1)^m)}{2}\big). \qquad \square$$

Write $f_{U_m}(q) = \sum_{||\boldsymbol{\lambda}||=m} \chi^{\boldsymbol{\lambda}}(1)$. The polynomial $f_{G_m}(q)$ expressing the sum of the degrees of the complex irreducible characters of $G_m = \mathrm{GL}(m, \mathbb{F}_q)$ was computed in [7] for odd $q$ and in [17] and Example 6 of [19, IV.6] for general $q$. From these results we see that

$$f_{U_m}(q) = (-1)^{m(m+1)/2} f_{G_m}(-q),$$

giving another example of Ennola duality.

Gow [7] and Klyachko [17] proved that the sum of the degrees of the complex irreducible characters of $G_n$ is equal to the number of symmetric matrices in $G_n$. Gow accomplished this by considering the split extension of $G_n$ by the transpose-inverse automorphism,

$$G_n^+ = \langle G_n, \kappa \mid \kappa^2 = 1, \kappa g \kappa = {}^t g^{-1} \text{ for every } g \in G_n \rangle, \tag{5.10}$$

and showed that every complex irreducible representation of $G_n^+$ could be realized over the real numbers. Instead, we show directly that the sum of the degrees of the complex irreducible characters of $U_n$ is equal to the number of symmetric matrices in $U_n$. One can apply this result to determine reality properties of the characters of $U_n^+$, where

$$U_n^+ = \langle U_n, \kappa \mid \kappa^2 = 1, \kappa u \kappa = {}^t u^{-1} \text{ for every } u \in U_n \rangle. \tag{5.11}$$

**Corollary 5.2.** *The sum of the degrees of the complex irreducible characters of* $\mathrm{U}(n, \mathbb{F}_{q^2})$ *is equal to the number of symmetric matrices in* $\mathrm{U}(n, \mathbb{F}_{q^2})$.

*Proof.* Define $S_{U_n}$ to be

$$S_{U_n} = \{\text{symmetric matrices in } U_n\}.$$

Then $U_n$ acts on the set $S_{U_n}$ by the action

$$u \cdot s = {}^t u s u, \quad s \in S_{U_n},\ u \in U_n.$$

To find the number of elements in $S_{U_n}$, we find the stabilizers of the orbits under this action. Consider the same action of $G_n = \mathrm{GL}(n, \mathbb{F}_q)$ on its subset of symmetric matrices. That is, let

$$S_{G_n} = \{\text{symmetric matrices in } G_n\},$$

and let $G_n$ act on the set $S_{G_n}$ by the action

$$g \cdot s = {}^t g s g, \quad s \in S_{G_n},\ g \in G_n.$$

Gow [8] proved, using the Lang-Steinberg theorem, that there is a one-to-one correspondence between conjugacy classes of $G_n^+$ (5.10) and $U_n^+$ (5.11) of elements of the form $g\kappa$ and $u\kappa$, for $g \in G_n$ and $u \in U_n$. Moreover, Gow proved that this correspondence preserves orders of the elements, and the corresponding centralizers in $G_n$ and $U_n$ are isomorphic. The conjugacy classes of order 2 elements of this form correspond to the orbits of symmetric matrices in $G_n$ and $U_n$, and their centralizers in $G_n$ and $U_n$ to the stabilizers under the action described above. So to find the stabilizers of the orbits in the case for $U_n$, it is enough to do this for $G_n$.

If $q$ is odd, or when $q$ is even and $n$ is odd, the stabilizers of the orbits of $S_{G_n}$ are exactly the orthogonal groups (see, for example, [10, Chapters 9, 14] for a complete discussion and orders). When $q$ and $n$ are both even, there are two orbits, and one orbit consists of symmetric matrices which have at least one nonzero entry on the diagonal, the order of whose stabilizer is computed in [20]. The other orbit, consisting of symmetric matrices with zero diagonal, corresponds to the unique class of alternating forms, with stabilizer the symplectic group over $\mathbb{F}_q$. In each of these cases, it may be easily checked that the sum of the indices of these stabilizers as subgroups of $U_n$, which by Gow's result is the size of $S_{U_n}$, is exactly the polynomial obtained in Theorem 5.2. □

For a finite group $H$ with order 2 automorphism $\iota$, and irreducible complex representation $\pi$ of $H$, the *twisted Frobenius-Schur indicator* of $\pi$, denoted $\varepsilon_\iota(\pi)$, was originally defined in [16], and studied further in [1]. This indicator is a generalization of the classical Frobenius-Schur indicator $\varepsilon(\pi)$, which takes the value 1 if $\pi$ is a real representation, $-1$ if the character of $\pi$ is real-valued but $\pi$ is not real, and 0 if the character of $\pi$ is not real-valued. If $H = U_n$, and $\iota$ is the inverse-transpose automorphism, it follows from Corollary 5.2 and [1, Proposition 1] that $\varepsilon_\iota(\pi) = 1$ for every complex irreducible representation $\pi$ of $U_n$. Using this fact, along with the formula for the twisted Frobenius-Schur indicator established in [16], we obtain the following reality properties for the group $U_n^+$.

**Corollary 5.3.** *Let $U_n^+$ be the split extension of $U_n$ by the transpose-inverse involution, as in (5.11). Let $\pi$ be a complex irreducible representation of $U_n$. Then we have the following:*

(1) *If $\varepsilon(\pi) = 0$, then $\pi$ induces to an irreducible representation $\rho = \mathrm{Ind}_{U_n}^{U_n^+}(\pi)$ of $U_n^+$ such that $\varepsilon(\rho) = 1$.*
(2) *If $\varepsilon(\pi) = 1$, then $\pi$ extends to two irreducible representations $\pi'$ and $\pi''$ of $U_n^+$ such that $\varepsilon(\pi') = \varepsilon(\pi'') = 1$.*
(3) *If $\varepsilon(\pi) = -1$, then $\pi$ extends to two irreducible representations $\pi'$ and $\pi''$ of $U_n^+$ such that $\varepsilon(\pi') = \varepsilon(\pi'') = 0$.*

A $\Theta$-partition $\boldsymbol{\lambda}$ is *even* if every part of $\boldsymbol{\lambda}(\varphi)$ is even for every $\varphi \in \Theta$. Let $Sp_{2n} = \mathrm{Sp}(2n, \mathbb{F}_q)$ be the symplectic group over the finite field $\mathbb{F}_q$. The following was proven in [11].

**Theorem 5.3** (Henderson)**.** *Let $q$ be odd. The decomposition of* $\mathrm{Ind}_{Sp_{2n}}^{U_{2n}}(\mathbf{1})$ *into irreducibles is given by*

$$\mathrm{Ind}_{Sp_{2n}}^{U_{2n}}(\mathbf{1}) = \sum_{\substack{||\boldsymbol{\lambda}||=2n \\ \boldsymbol{\lambda}' \text{even}}} \chi^{\boldsymbol{\lambda}}.$$

So, for $q$ odd, Theorem 5.3 implies an identity for the sum of the degrees of characters appearing in the permutation character. We are able to calculate this degree sum for any $q$, suggesting that Theorem 5.3 should hold for $q$ even as well.

**Theorem 5.4.** *The sum of the degrees of the complex irreducible characters of $U_{2m}$ corresponding to $\boldsymbol{\lambda}$ such that $\boldsymbol{\lambda}'$ is even is given by*

$$\sum_{\substack{||\boldsymbol{\lambda}||=2m \\ \boldsymbol{\lambda}' \text{ even}}} \chi^{\boldsymbol{\lambda}}(1) = (q+1)q^2(q^3+1)\cdots q^{2m-2}(q^{2m-1}+1) = \frac{|\mathrm{U}(n, \mathbb{F}_{q^2})|}{|\mathrm{Sp}(2n, \mathbb{F}_q)|}.$$

*Proof.* To calculate this sum, we find the coefficient of $t^{2m}$ in the series

$$T = \sum_{\boldsymbol{\lambda}' \text{ even}} (-1)^{n(\boldsymbol{\lambda})+||\boldsymbol{\lambda}||} \delta(s_{\boldsymbol{\lambda}}) t^{||\boldsymbol{\lambda}||} = T_{\mathrm{o}} T_{\mathrm{e}},$$

where

$$T_{\mathrm{o}} = \prod_{\substack{|\varphi| \\ \text{odd}}} \sum_{\lambda' \text{ even}} (-1)^{n(\lambda)} \delta(s_\lambda(\varphi)) t^{|\lambda||\varphi|} \text{ and } T_{\mathrm{e}} = \prod_{\substack{|\varphi| \\ \text{even}}} \sum_{\lambda' \text{ even}} \delta(s_\lambda(\varphi)) t^{|\lambda||\varphi|}.$$

It follows from the computation in Example 6 of [19, I.5] that

$$\sum_{\lambda' \text{ even}} (-1)^{n(\lambda)} s_\lambda = \prod_{i<j} (1 + x_i x_j)^{-1},$$

and applying this yields

$$\sum_{\lambda' \text{ even}} (-1)^{n(\lambda)} \delta(s_\lambda(\varphi)) t^{|\lambda||\varphi|} = \prod_{1 \le i < j} \left(1 + \left(t^2(-q)^{-i-j}\right)^{|\varphi|}\right)^{-1}.$$

From Example 5(b) in [19, I.5], we have the identity

$$\sum_{\lambda' \text{ even}} s_\lambda = \prod_{i<j} (1 - x_i x_j)^{-1},$$

from which it follows, for $|\varphi|$ even, that

$$\sum_{\lambda' \text{ even}} \delta(s_\lambda(\varphi)) t^{|\lambda||\varphi|} = \prod_{1 \le i < j} \left(1 - \left(t^2 q^{-i-j}\right)^{|\varphi|}\right)^{-1}.$$

Proceeding similarly as in the proof of Theorem 5.2, we obtain

$$T = \prod_{i \ge 1} \frac{1}{1 - t^2 q^{-2i}} = (1 - t^2) \prod_{i \ge 0} \frac{1}{1 - t^2 q^{-2i}} = (1 - t^2) \sum_{l \ge 0} \frac{t^{2l}}{\psi_l(q^{-2})}.$$

Multiplying the coefficient of $t^{2m}$ by $|\psi_{2m}(-q)| = \prod_{i=1}^{2m}(q^i - (-1)^i)$, we obtain

$$\sum_{\substack{||\boldsymbol{\lambda}||=2m \\ \boldsymbol{\lambda}' \text{ even}}} \chi^{\boldsymbol{\lambda}}(1) = (q+1)q^2(q^3+1)\cdots q^{2m-2}(q^{2m-1}+1). \qquad \square$$

Write $g_{U_m}(q) = \sum_{||\boldsymbol{\lambda}||=2m, \boldsymbol{\lambda}' \text{ even}} \chi^{\boldsymbol{\lambda}}(1)$, and let $g_{G_m}(q)$ denote the corresponding sum for $G_m$. The polynomial $g_{G_m}(q)$ was calculated in Example 7 of [19, IV.6], and similar to the previous degree sum, we see that we have

$$g_{U_m}(q) = (-1)^m g_{G_m}(-q).$$

## 6. A Deligne-Lusztig model

A *model* of a finite group $G$ is a representation $\rho$, which is a direct sum of representations induced from one-dimensional representations of subgroups of $G$, such that every irreducible representation of $G$ appears as a component with multiplicity 1 in the decomposition of $\rho$.

Klyachko [17] and Inglis and Saxl [13] obtained a model for $\mathrm{GL}(n, \mathbb{F}_q)$, where the induced representations can be written as a Harish-Chandra product of Gelfand-Graev characters and the permutation character of the finite symplectic group.

In this section we show that the same result is true for the finite unitary group, except the Harish-Chandra product is replaced by Deligne-Lusztig induction. The result is therefore not a model for $\mathrm{U}(n, \mathbb{F}_{q^2})$ in the finite group character induction sense, but rather from the Deligne-Lusztig point of view.

Let $\Gamma_{(m)}$ be the Gelfand-Graev character of $\mathrm{U}(m, \mathbb{F}_{q^2})$ (see, for example, [2] for a general definition). For $\boldsymbol{\lambda} \in \mathcal{P}^\Theta$, define

$$\mathrm{ht}(\boldsymbol{\lambda}) = \max\{\ell(\boldsymbol{\lambda}(\varphi)) \mid \varphi \in \Theta\}.$$

It is well-known that the Gelfand-Graev character has a multiplicity free decomposition (see, for example, [2]), and Ohmori [21, Section 5.2] gives the following explicit decomposition. Alternatively, the theorem also follows by applying the characteristic map to [3, Theorem 10.7].

**Theorem 6.1.** *The decomposition of* $\Gamma_{(m)}$ *into irreducibles is given by*

$$\Gamma_{(m)} = \sum_{\substack{\boldsymbol{\lambda} \in \mathcal{P}_m^\Theta \\ \mathrm{ht}(\boldsymbol{\lambda})=1}} \chi^{\boldsymbol{\lambda}}.$$

For a partition $\lambda$, let $o(\lambda)$ denote the number of odd parts of $\lambda$, and for $\boldsymbol{\lambda} \in \mathcal{P}^\Theta$, let $o(\boldsymbol{\lambda}) = \sum_{\varphi \in \Theta} |\varphi| o(\boldsymbol{\lambda}(\varphi))$. The following gives the Deligne-Lusztig model for the finite unitary group.

**Theorem 6.2.** *Let $q$ be odd. For each $r$ such that* $0 \leq 2r \leq m$,

$$\Gamma_{m-2r} \circ \mathrm{Ind}_{Sp_{2r}}^{U_{2r}}(\mathbf{1}) = \sum_{o(\boldsymbol{\lambda}')=m-2r} \chi^{\boldsymbol{\lambda}}.$$

*Furthermore,*

$$\sum_{0\le 2r\le m} \Gamma_{m-2r} \circ \mathrm{Ind}_{Sp_{2r}}^{U_{2r}}(\mathbf{1}) = \sum_{||\boldsymbol{\lambda}||=m} \chi^{\boldsymbol{\lambda}}$$

*Proof.* Suppose $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}^{\Theta}$, such that $\mathrm{ht}(\boldsymbol{\mu}) = 1$ and $\boldsymbol{\nu}'$ is even. From (5.8), Corollary 4.3, and Pieri's formula [19, I.5.16],

$$\chi^{\boldsymbol{\mu}} \circ \chi^{\boldsymbol{\nu}} = (-1)^{\tau(\boldsymbol{\mu})+\tau(\boldsymbol{\nu})} \sum_{\boldsymbol{\lambda}} \chi^{\boldsymbol{\lambda}}, \tag{6.1}$$

where the sum is taken over all $\boldsymbol{\lambda}$ such that for every $\varphi \in \Theta$, $\boldsymbol{\lambda}(\varphi) - \boldsymbol{\nu}(\varphi)$ is a horizontal $|\boldsymbol{\mu}(\varphi)|$-strip.

We now use Corollary 5.1 to show that $\chi^{\boldsymbol{\mu}} \circ \chi^{\boldsymbol{\nu}}$ is a character. As $\boldsymbol{\lambda}(\varphi) - \boldsymbol{\nu}(\varphi)$ is a horizontal $|\boldsymbol{\mu}(\varphi)|$-strip, the part $\boldsymbol{\lambda}(\varphi)_i'$ is either $\boldsymbol{\nu}(\varphi)_i'$ or $\boldsymbol{\nu}(\varphi)_i' + 1$ for every $i = 1, 2, \ldots, \ell(\boldsymbol{\lambda}(\varphi))$. By assumption, $\boldsymbol{\nu}'$ is even, so $\boldsymbol{\nu}(\varphi)_i'$ is even for every $\varphi \in \Theta$, and so

$$\binom{\boldsymbol{\nu}(\varphi)_i' + 1}{2} = \boldsymbol{\nu}(\varphi)_i' + \binom{\boldsymbol{\nu}(\varphi)_i'}{2} \equiv \binom{\boldsymbol{\nu}(\varphi)_i'}{2} \pmod 2.$$

Thus, $n(\boldsymbol{\lambda}(\varphi)) = \sum_i \binom{\boldsymbol{\lambda}(\varphi)_i'}{2} \equiv n(\boldsymbol{\nu}(\varphi)) \pmod 2$. The assumption $\mathrm{ht}(\boldsymbol{\mu}) = 1$ implies $n(\boldsymbol{\mu}(\varphi)) = 0$, and since $||\boldsymbol{\nu}||$ is even,

$$n(\boldsymbol{\mu}) + n(\boldsymbol{\nu}) \equiv n(\boldsymbol{\lambda}) + ||\boldsymbol{\mu}||\,||\boldsymbol{\nu}|| \pmod 2.$$

By Corollary 5.1, $\chi^{\boldsymbol{\mu}} \circ \chi^{\boldsymbol{\nu}}$ is a character.

Use the decompositions of Theorem 5.3 and Theorem 6.1 in the product (6.1) to observe that the irreducible characters $\chi^{\boldsymbol{\lambda}}$ in the decomposition of $\Gamma_{m-2r} \circ \mathrm{Ind}_{Sp_{2r}}^{U_{2r}}(\mathbf{1})$ are indexed by $\boldsymbol{\lambda} \in \mathcal{P}_m^{\Theta}$ such that for every $\varphi$, $\boldsymbol{\lambda}(\varphi) - \boldsymbol{\nu}(\varphi)$ is a horizontal $|\boldsymbol{\mu}(\varphi)|$-strip, where $||\boldsymbol{\mu}|| = m - 2r$, for some $\boldsymbol{\nu}(\varphi)$ such that $\boldsymbol{\nu}(\varphi)'$ is even. Then the number of odd parts of $\boldsymbol{\lambda}(\varphi)'$ is exactly $|\boldsymbol{\mu}(\varphi)|$, and so the $\boldsymbol{\lambda}$ in the decomposition must satisfy $\sum_{\varphi\in\Theta} |\varphi| o(\boldsymbol{\lambda}(\varphi)') = ||\boldsymbol{\mu}|| = m - 2r$. □

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