A weight statistic and partial order on products of $m$-cycles

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**Article history:**
Received 20 March 2013
Received in revised form 26 September 2013
Accepted 2 October 2013
Available online 18 October 2013

**Keywords:**
Symmetric group
$m$-cycles
Bruhat order
EL-shellable posets

**Abstract**

R. S. Deodhar and M. K. Srinivasan defined a weight statistic on the set of involutions in the symmetric group and proved several results about the properties of this weight. These results include a recursion for a weight generating function, that the weight provides a grading for the set of fixed-point free involutions under a partial order related to the Bruhat partial order, and that this graded poset is EL-shellable and its order complex triangulates a ball. We extend the definition of weight to products of disjoint $m$-cycles in the symmetric group, and we generalize all of the results of Deodhar and Srinivasan just mentioned to the case of any $m \geq 2$.

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1. Introduction

Let $m$, $n$ be integers, with $2 \leq m \leq n$, let $S_n$ be the symmetric group on $[n] = \{1, 2, \ldots, n\}$, and let $\delta \in S_n$ be a product of disjoint $m$-cycles. In particular, if $m$ is prime, then $\delta$ is just an element of order $m$ in $S_n$ (or the identity, if it is the product of zero $m$-cycles). Writing $\delta$ in cycle notation, suppose we have $\delta$ is a product of $k$ disjoint $m$-cycles (so $mk \leq n$), so that

$$\delta = (a_{1,1} a_{1,2} \cdots a_{1,m})(a_{2,1} a_{2,2} \cdots a_{2,m}) \cdots (a_{k,1} a_{k,2} \cdots a_{k,m}).$$

(1.1)

Further, suppose that for each $i = 1, 2, \ldots, k$, $a_{i,1} < a_{i,j}$ for $j = 2, 3, \ldots, m$, and $a_{i,1} < a_{2,1} < \cdots < a_{k,1}$, and then we say that $\delta$ is in standard form. Let $f^{(m)}(n)$ denote the collection of all products of disjoint $m$-cycles in $S_n$, and let $f^{(m)}(n, k)$ denote the collection of all products of $k$ disjoint $m$-cycles in $S_n$, so that $\delta \in f^{(m)}(n, k)$ in (1.1). Given $\delta \in f^{(m)}(n, k)$ in standard form as in (1.1), define $\text{span}(\delta)$ as

$$\text{span}(\delta) = \sum_{i=1}^{k} \sum_{j=2}^{m} (a_{i,j} - a_{i,1} - 1).$$

For example, suppose $\delta \in f^{(3)}(9, 3)$, where $\delta = (1 6 9)(2 7 4)(3 5 8)$. Then

$$\text{span}(\delta) = (5 - 1) + (8 - 1) + (5 - 1) + (2 - 1) + (2 - 1) + (5 - 1) = 21.$$

Given an $m$-cycle $(a_1 a_2 \ldots a_m) \in S_n$ in standard form, draw its arc diagram by drawing, along a line containing points labeled from $[n]$, an arc for each pair $(a_i, a_j)$, $j = 2, \ldots, m$, where the arc $(a_1, a_j)$ is drawn under $(a_1, a_j)$ when $l > j$. When $a_j > a_i$ and $j > l$, then the arcs $(a_i, a_j)$ and $(a_1, a_j)$ intersect in the arc diagram, which we call an internal crossing of the $m$-cycle. That is, the number of internal crossings of the $m$-cycle $(a_1 a_2 \ldots a_m)$ is equal to the number of pairs $(a_i, a_j)$ from the sequence $a_2, a_3, \ldots, a_m$, which satisfy $i < j$ and $a_i < a_j$, which we call ascents of this sequence (which are also known as non-inversions). In Figs. 1 and 2, for example, we show the arc diagrams for the 5-cycles $(1 4 5 3 2)$ and $(1 2 4 5 3)$, with the internal crossings circled.

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0012-365X/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.disc.2013.10.002
If $\delta \in J_f^{(m)}(n, k)$, the arc diagram for $\delta$ is drawn by drawing the arc diagram for each of the $k$ disjoint $m$-cycles of $\delta$. There may be intersections of the arcs from different $m$-cycles of $\delta$, which we call external crossings of $\delta$. Let $C_{in}(\delta)$ denote the total number of internal crossings in the arc diagram of $\delta$, and $C_{ex}(\delta)$ denote the total number of external crossings in the arc diagram of $\delta$. The crossing number of $\delta$, $C(\delta)$, is then defined as $C(\delta) = C_{ex}(\delta) + C_{in}(\delta)$. In Fig. 3, we show the arc diagram for $\delta = (169)(274)(358) \in J_f^{(3)}(9, 3)$, with internal crossings in circles and external crossings in boxes.

Now, given any $\delta \in J_f^{(m)}(n, k)$, we define the weight of $\delta$, which we denote $wt_m(\delta)$, as

$$wt_m(\delta) = \text{span}(\delta) - C(\delta) = \text{span}(\delta) - C_{ex}(\delta) - C_{in}(\delta).$$

So, for $\delta = (169)(274)(358)$, since $\text{span}(\delta) = 21$ and $C(\delta) = 7$ from Fig. 3, then $wt_3(\delta) = 14$. We note that for $m = 2$, our definition of weight coincides precisely with that from [5], since if $\delta \in S_n$ is an involution, then $C_{in}(\delta) = 0$.

Given integers $n, k \geq 0, m \geq 2$, with $mk \leq n$, define the weight generating function, denoted $J_f^{(m)}(n, k)$, to be the following polynomial in an indeterminate $q$:

$$J_f^{(m)}(n, k) = \sum_{\delta \in J_f^{(m)}(n, k)} q^{wt_m(\delta)}.$$

In Section 2, we prove a recursive relation on the weight generating function in terms of $n$ and $k$ (Theorem 2.1), and we use it to compute an exact formula (Corollary 2.1) for $J_f^{(m)}(mn, n)$, the weight generating function for the set of products of disjoint $m$-cycles in $S_{mn}$ which are fixed-point free. We use the notation $F^{(m)}(mn) = J_f^{(m)}(mn, n)$ for the set of fixed-point free products of disjoint $m$-cycles in $S_{mn}$.

In Section 3, we introduce the Bruhat order (sometimes called the strong Bruhat order) on the symmetric group $S_n$, which makes $S_n$ a graded poset with grading precisely given by the number of inversions of a permutation. We consider a specific subset $E(n_m)$ of a permutation group $S(n_m)$ (identified with $S_{mn}$), and the set $E(n_m)$ is in bijection with the set $F^{(m)}(mn)$ of fixed-point free products of disjoint $m$-cycles in $S_{mn}$. In Proposition 3.1, we show that an explicit bijection $\phi$ defined between these two sets maps the weight $wt_m$ of a permutation in $F^{(m)}(mn)$ to the number of inversions of the permutation in $E(n_m)$.

Now let $\delta, \pi \in F^{(m)}(mn)$, and suppose that

$$\delta = (a_{1,1} a_{1,2} \cdots a_{1,m})(a_{2,1} a_{2,2} \cdots a_{2,m}) \cdots (a_{n,1} a_{n,2} \cdots a_{n,m})$$

is in standard form. Then $\pi$ is obtained from $\delta$ by an interchange if one of the following holds:

(i) There is some $i, 1 \leq i \leq n$, and some $j, l, 2 \leq j, l \leq m$, such that the standard form of $\pi$ is obtained by interchanging $a_{i,j}$ and $a_{i,l}$ in $\delta$.

(ii) There are some $i, j, 1 \leq i < j \leq n$, and some $l, 2 \leq l \leq m$, such that the standard form of $\pi$ is obtained by interchanging $a_{i,l}$ and $a_{j,l}$ in $\delta$.

(iii) There are some $i, j, l, h, 2 \leq l, h \leq m, l \neq h, 1 \leq i < j \leq n$, such that the standard form of $\pi$ is obtained by interchanging $a_{i,j}$ and $a_{j,h}$ in $\delta$.
If $\pi$ is obtained from $\delta$ by an interchange, then such an interchange is weight increasing if $w_m(\delta) < w_m(\pi)$. We define a relation on the set $F(m)(mn)$ as follows. If $\delta, \sigma \in F(m)(mn)$, then we write $\delta \preceq \sigma$ if $\sigma$ is obtained by $\delta$ by a sequence of zero or more weight increasing interchanges. In Proposition 3.2, we show that the bijection $\phi : F(m)(mn) \rightarrow E(n_m)$, which we show in Proposition 3.1 carries $w_m$ to the number of inversions, is an isomorphism of posets, where $F(m)(mn)$ is given the partial order $\preceq$ just defined, and $E(n_m)$ is given the Bruhat order. This is not quite enough to conclude that $F(m)(mn)$ is a graded poset with rank given by $w_m$, since it is not apparent that the sub-poset $E(n_m)$ of $S(n_m)$ is graded by the number of inversions. This requires the notion of EL-labelings, which we introduce in Section 4.

After defining EL-labelings and EL-shellable graded posets in Section 4, and the EL-labeling defined on $S_n$ with respect to the Bruhat order, we show that $E(n_m)$ inherits this EL-labeling from $S(n_m)$ in Proposition 4.1. It follows that $E(n_m)$ is an EL-shellable graded poset. We immediately obtain our next main result, Theorem 4.1, which states that $(F(m)(mn), \preceq)$ is an EL-shellable graded poset, of rank $(mn-1)!/((m-1)!m(n-2)!)$, with grading given by $w_m$, and with explicit rank generating function found in Corollary 2.1. Finally, we introduce the notion of the order complex of a poset, and we show in Theorem 4.2 that the order complex of $F(m)(mn)$, less its maximal and minimal elements, triangulates a ball of dimension $(m-1)!/(m(n-2)! - 2$.

All of the results mentioned above are generalizations of results obtained by Deodhar and Srinivasan [5]. We adapt our arguments from those given by Deodhar and Srinivasan, and throughout we specify precisely which results and arguments from [5] are being generalized. We order material somewhat differently than in [5], partially to stress which parts of this paper require more work than the case $m = 2$. That is, the paper is ordered roughly in order from results requiring more argument than in [5] to results requiring essentially the same. The main idea in this paper is finding the “right” generalization of weight for products of disjoint $m$-cycles, for any $m \geq 2$. The fact that our definition fits the bill reveals itself in the proofs of Theorem 2.1 and Proposition 3.1, which require a more intricate proof than their $m = 2$ counterparts. Proposition 3.2 then requires more cases to check than the case $m = 2$. The arguments given in Section 4 then go through nearly the same as in the $m = 2$ case, with only cosmetic changes.

Other than their results which inspired this paper, Deodhar and Srinivasan also showed [6] that their weight defined on involutions in $S_n$ is a specialization of the weight defined by W. P. Johnson [9,8] on set partitions. So it is reasonable to expect that our weight function on products of disjoint $m$-cycles should be a specialization of a more general weight than Johnson’s on set partitions. It would be interesting to have such a generalization and to use it to find combinatorial applications which generalize those found by Johnson.

Can, Cherniavsky, and Twelbeck [3] have studied the Bruhat order on fixed-point free involutions, and in particular have shown that the poset of fixed-point free involutions studied by Deodhar and Srinivasan is a sub-poset of the fixed-point free involutions under the Bruhat order [3, Theorem 10]. It seems to be a worthwhile question to understand the (very likely more complicated) relationship between the poset $F(m)(mn)$ we study here and the Bruhat poset of fixed-point free products of disjoint $m$-cycles in $S_{mn}$ when $m > 2$.

Finally, we point out that after the original version of this paper was written, many of its results were further generalized by Can and Cherniavsky [2]. We make remarks on these results at the end of Section 3.

2. Recursion for the generating function

As in the introduction, let $j(m)(n, k)$ denote the set of permutations in $S_n$ which are the product of $k$ disjoint $m$-cycles. A counting argument gives the recursive relation

$$|j(m)(n + 1, k)| = |j(m)(n, k)| + n(n - 1) \cdots (n - m + 1)j(m)(n - m + 1, k - 1).$$

The main result of this section is a refinement of this recursion in terms of the weight function $w_m$ defined in the introduction. In particular, Theorem 2.1 gives a recursive relation for the generating function $j_q(m)(n, k) = \sum_{k \in j(m)(n, k)} q^{w_m(\delta)}$, giving a $q$-analog of the recursion for $|j(m)(n, k)|$ above. This generalizes [5, Proposition 2.1].

Before giving the result, we clarify the following notation. For the indeterminate $q$, define $[n]_q = q^{n-1} + q^{n-2} + \cdots + q + 1$ for any $n \geq 1$. If $\delta \in j(m)(n, k)$ and the $m$-cycle $(a_1, a_2, \ldots, a_m)$ is one of the cycles in $\delta$ in standard form, then we write $(a_1, a_2, \ldots, a_m) | \delta$.

**Theorem 2.1.** The following recursion holds for the generating function $j_q(m)(n, k)$:

$$j_q(m)(n + 1, k) = j_q(m)(n, k) + [n]_q[n - 1]_q \cdots [n - m + 2]_q j_q(m)(n - m + 1, k - 1).$$

**Proof.** We begin by obtaining a bijection

$$\Theta : j(m)(n + 1, k) \rightarrow j(m)(n, k) \cup ([n] \times [n - 1] \times \cdots \times [n - m + 2] \times j(m)(n - m + 1, k - 1)).$$

For any $\delta \in j(m)(n + 1, k)$, if $(1 a_1, a_2 a_3 \cdots a_m)^+ \not\delta$, that is, if $a_{i+1} \not= 1$ when $\delta$ is in standard form, then label each $a_{i,j}$ of $\delta$ by $a_{i,j} - 1$ for all $1 \leq i \leq k$, $1 \leq j \leq m$ and define the resulting element to be $\Theta(\delta) \in j(m)(n, k)$.

Now consider the case that $(1 a_1, a_2 a_3 \cdots a_m) \not\delta$, so $a_{i,1} = 1$ in $\delta$ in standard form. Delete the $m$-cycle $(1 a_1, a_2 a_3 \cdots a_m)$ from $\delta$ and name the resulting element $\delta$. Now, in the set $[n + 1]/\{1, a_1, a_2, a_3, \ldots, a_m\}$, relabel the elements
of this set, in increasing order, by $1, 2, \ldots, n - m + 1$. Apply this relabeling to the entries of $\tilde{\delta}$ and name the resulting element $\delta' \in f^{(m)}(n - m + 1, k - 1)$. For each $a_{1,j}, 2 \leq l \leq m$, define

$$f(a_{1,l}) = \text{the number of } a_{1,j} \text{ such that } 1 < j < l \text{ and } a_{1,j} < a_{1,l}.$$ 

That is, $f(a_{1,l})$ is equal to the number of arcs in the arc diagram of the cycle $(a_{1,1} \ldots a_{1,m})$ which are above the arc from 1 to $a_{1,j}$ and which intersect with the arc between $a_{1,j} = 1$ and $a_{1,l}$. So, $\sum_{j=2}^{m} f(a_{1,j})$ is the total number of internal crossings of this cycle, which is also the number of ascents in the sequence $a_{1,2}, \ldots, a_{1,m}$. Note that we have $a_{1,l} > f(a_{1,l}) + 1$, and if $a_{1,l} > n - l + 2$, say $a_{1,l} = n - l + 2 + j$ for some $j \geq 1$, then it follows that $f(a_{1,l}) \geq j - 1$. That is, $a_{1,l} - 1 - f(a_{1,l}) \in [n - l + 2]$. With these definitions, define $\Theta(\delta)$ in the case that $(a_{1,2} a_{1,3} \ldots a_{1,m})/\delta$ by

$$\Theta(\delta) = (a_{1,2} - 1, a_{1,3} - 1 - f(a_{1,3}), \ldots, a_{1,m} - 1 - f(a_{1,m}), \delta').$$

To see that $\Theta$ is indeed a bijection, it suffices to show that it is injective. If $\Theta(\delta_1) = \Theta(\delta_2) \in f^{(m)}(n, k)$, it follows immediately that $\delta_1 = \delta_2$, so suppose $\Theta(\delta_1) = \Theta(\delta_2) \not\in f^{(m)}(n, k)$. So,

$$\Theta(\delta_1) = (a_{1,2} - 1, a_{1,3} - 1 - f(a_{1,3}), \ldots, a_{1,m} - 1 - f(a_{1,m}), \delta_1')$$

$$= (b_{1,2} - 1, b_{1,3} - 1 - f(b_{1,3}), \ldots, b_{1,m} - 1 - f(b_{1,m}), \delta_2').$$

Then $a_{1,2} = b_{1,2}$ and $a_{1,3} - f(a_{1,3}) = b_{1,3} - f(b_{1,3})$. If $f(a_{1,3}) \neq f(b_{1,3})$, say $f(a_{1,3}) > f(b_{1,3})$, then we must have $f(a_{1,3}) = 1$ and $f(b_{1,3}) = 0$, so $a_{1,3} - b_{1,3} = 1$. This is impossible, since we must also have $b_{1,2} < b_{1,2} = a_{1,2} < a_{1,3}$. Thus $f(a_{1,3}) = f(b_{1,3})$ and $a_{1,3} = b_{1,3}$. By induction, suppose $j < m$ and for each $l \leq j$ we have $a_{1,l} = b_{1,l}$ and so $f(a_{1,l}) = f(b_{1,l})$. If $f(a_{1,j+1}) \neq f(b_{1,j+1})$, say $f(a_{1,j+1}) = f(b_{1,j+1}) + k, k > 0$, then $a_{1,j+1} = b_{1,j+1} + k$ since $a_{1,j+1} - f(a_{1,j+1}) = b_{1,j+1} - f(b_{1,j+1})$.

However, we then have

$$k = \text{the number of } a_{1,l} = b_{1,l} \text{ such that } 2 \leq l \leq j \text{ and } b_{1,j+1} < a_{1,l} < a_{1,j+1},$$

which is impossible. Thus $a_{1,l} = b_{1,l}$ for each $l$, and since $\delta_1' = \delta_2'$, we have $\delta_1 = \delta_2$.

Let $\delta \in f^{(m)}(n + 1, k)$. If $(1 a_{1,2} a_{1,3} \ldots a_{1,m}) \not\subseteq \delta$, it follows that $\text{wt}_{m}(\delta) = \text{wt}_{m}(\Theta(\delta))$. If $(1 a_{1,2} a_{1,3} \ldots a_{1,m})/\delta$, we claim that

$$\text{wt}_{m}(\delta) = \text{wt}_{m}(\delta') + a_{1,2} - 2 + a_{1,3} - 2 + \cdots + a_{1,m} - 2 - f(a_{1,1}).$$

By considering the bijection $\Theta$ just constructed, along with the powers of $q$ which occur on both sides of the desired recursion, one sees that proving this claim finishes the proof.

Consider the arc diagram for $\delta$, and define $A_k$ to be the number of arcs of $\delta$ without 1 as an endpoint which cross exactly $k$ arcs of $\delta$ which do have endpoint 1. It follows that we have

$$C_{ex}(\delta) = C_{ex}(\tilde{\delta}) + \sum_{j=1}^{m-1} jA_j = C_{ex}(\delta') + \sum_{j=1}^{m-1} jA_j.$$

We also have

$$\text{span}(\delta) = \text{span}(\tilde{\delta}) + a_{1,2} - 2 + a_{1,3} - 2 + \cdots + a_{1,m} - 2$$

$$= \text{span}(\delta') + a_{1,2} - 2 + a_{1,3} - 2 + \cdots + a_{1,m} - 2 + \sum_{j=1}^{m-1} jA_j,$$

since each arc counted by an $A_j$ corresponds to an element of $[n + 1]$ underneath an arc of $\tilde{\delta}$, which is removed in the relabeling process when constructing $\delta'$. From the definition of the function $f$, we also have

$$C_{in}(\delta) = C_{in}(\tilde{\delta}) + f(a_{1,1}) + \cdots + f(a_{1,m}) = C_{in}(\delta') + f(a_{1,1}) + \cdots + f(a_{1,m}).$$

Therefore, we have

$$\text{wt}_{m}(\delta) = \text{span}(\delta) - C_{ex}(\delta) - C_{in}(\delta)$$

$$= \text{span}(\delta') + a_{1,2} - 2 + a_{1,3} - 2 + \cdots + a_{1,m} - 2 + \sum_{j=1}^{m-1} jA_j$$

$$= \left( C_{ex}(\delta') + \sum_{j=1}^{m-1} jA_j \right) - (C_{in}(\delta') + f(a_{1,1}) + \cdots + f(a_{1,m}))$$

$$= \text{wt}_{m}(\delta') + a_{1,2} - 2 + a_{1,3} - 2 - f(a_{1,1}) + \cdots + a_{1,m} - 2 - f(a_{1,m}),$$

giving the claim. $\square$
Using Theorem 2.1, we may calculate a precise formula for the weight generating function in the fixed-point free case. For \( n > 0 \), define \( [n]_q^! = [n]_q[n - 1]_q \cdots [1]_q \), and define \( [0]_q^! = 1 \). The following generalizes [5, Proposition 2.3].

**Corollary 2.1.** For any \( n \geq 0 \), we have

\[
j_q^{(m)}(mn, n) = \sum_{\delta \in F^{(m)}(mn)} q^{wt_{\delta}} = \frac{[mn]_q^!}{[mn]_q[m(n - 1)]_q \cdots [m]_q^!}.
\]

**Proof.** Since \( j_q^{(m)}(0, 0) = 1 \), and \( j_q^{(m)}(mn - 1, n) = 0 \) for any \( n \geq 1 \), the result follows from Theorem 2.1 and induction. \( \square \)

**Remark.** It is at this point in [5] when Deodhar and Srinivasan obtain an expansion of the \( q \)-binomial coefficient as a sum over involutions, in terms of the weight function and the regular binomial coefficients, with implications about the poset of subspaces of a finite vector space. This was the only result from [5] for which we were unable to obtain a meaningful generalization. It would be nice to have such a generalization and to understand the meaning of Theorem 2.1 in the context of finite vector spaces.

## 3. Bruhat order and the poset \( E(n_m) \)

Given any element \( \pi \) in the symmetric group \( S_n \), we may write \( \pi \) in permutation notation, as in \( \pi = \pi_1 \pi_2 \cdots \pi_n \). An inversion of \( \pi \) is a pair \((i, j) \in [n] \times [n]\) such that \( i < j \) and \( \pi_i > \pi_j \). Let \( i(\pi) \) denote the number of inversions of \( \pi \). If \( \pi' \in S_n \) such that \( \pi' \) is obtained from \( \pi \) by interchanging two \( \pi_i, \pi_j \) in permutation notation, and \( i(\pi) < i(\pi') \), then we say that \( \pi' \) is obtained by \( \pi \) by an inversion increasing interchange. For \( \pi, \sigma \in S_n \), define \( \pi \preceq \sigma \) if \( \pi \) can be obtained from \( \sigma \) by a sequence of zero or more inversion increasing interchanges. This partial order is the (strong) Bruhat order, and it makes \( S_n \) a graded poset with grading given by \( \iota \) and rank generating function \([n]_q^! \) [10, Chapter 3, Exercise 183(a)].

Now, given \( m \geq 2 \), extend the set \([n]\) by defining, for each \( l \in [n] \), elements \( l_1, l_2, \ldots, l_m \), which are ordered so that \( l_i < l_j \) when \( i < j \), and \( k < l \) for any \( i \) and \( j \). Let \([n]_m \) be the resulting linearly ordered set, that is,

\[
[n]_m = \{1 < 2 < \cdots < 1_m < 2_1 < \cdots < 2_m < \cdots < n_1 < \cdots < n_m\}.
\]

Then the symmetric group \( S(n_m) \) on \([n]_m \) may be identified with \( S_{mn} \), and \( S(n_m) \) is a graded poset under the Bruhat order.

Define a subset \( E(n_m) \subset S(n_m) \) as follows. Let \( \pi \in S(n_m) \) be written in permutation form, \( \pi = \pi_1 \pi_2 \cdots \pi_{mn} \). Then \( \pi \in E(n_m) \) if and only if, in the permutation form of \( \pi, k_1 \) is to the left of \( l_1 \) whenever \( k < l \), and \( k_1 \) is to the left of \( k \) for any \( k \) and \( \pi_j \geq 2 \). For example, \( 1_1 1_2 2_2 1_3 \in E(2_3) \), while \( 1_1 1_2 3_2 2 \notin E(2_3) \). We now define a map \( \phi : F^{(m)}(mn) \to E(n_m) \) in the following way. Let \( \delta \in F^{(m)}(mn) \) be written in standard form,

\[
\delta = (a_{1_1} a_{1_2} \cdots a_{1_m})(a_{2_1} a_{2_2} \cdots a_{2_m}) \cdots (a_{n_1} a_{n_2} \cdots a_{n_m}),
\]

where \( a_{1_1} = 1 \). Starting with the identity in \( S_{mn} \) in permutation form, \( \cdots (mn) \), replace \( a_{ij} \) with \( l \) and define the resulting permutation in \( S(n_m) \) to be \( \phi(\delta) \). For example, if \( \delta = (1 \ 6 \ 9)(2 \ 7 \ 3)(4 \ 5 \ 8) \), then \( \phi(\delta) = 1_2 1_3 2_3 1_2 2_2 3_1 \). The fact that \( \phi(\delta) \in E(n_m) \) for any \( \delta \in F^{(m)}(mn) \) follows from the definitions of standard form and the set \( E(n_m) \). A direct counting argument gives \( |F^{(m)}(mn)| = |E(n_m)| \), and since \( \phi \) is injective by construction, then \( \phi \) is a bijection. Moreover, the map \( \phi \) carries the weight of \( \delta \) to the number of inversions of \( \phi(\delta) \), as we see next. We note that \( E(n_2) \) is exactly the set \( E(\bar{n}) \) defined by Deodhar and Srinivasan if we change each \( i_1 \) into \( i_2 \) and \( i_2 \) into \( i_1 \), our map \( \phi \) generalizes their bijection between fixed-point free involutions in \( S_{2n} \) and \( E(n) \), and the following is a generalization of [5, Proposition 3.3].

**Proposition 3.1.** For any \( \delta \in F^{(m)}(mn) \), we have \( wt_{\delta}(\delta) = \iota(\phi(\delta)) \).

**Proof.** The proof is by induction on \( n \). For the case \( n = 1 \), let \( \delta = (a_{1_1} a_{1_2} \cdots a_{1_m}) \in F^{(m)}(m) \) (where \( a_{1_1} = 1 \)). Then \( \delta = (a_{1_1} a_{1_2} \cdots a_{1_m}) \in F^{(m)}(m) \) and \( \delta = (a_{1_1} a_{1_2} \cdots a_{1_m}) \in F^{(m)}(m) \). The fact that \( \phi(\delta) \in E(n_2) \) for any \( \delta \in F^{(m)}(mn) \) follows from the definitions of standard form and the set \( E(n_2) \). A direct counting argument gives \( |F^{(m)}(mn)| = |E(n_m)| \), and since \( \phi \) is injective by construction, then \( \phi \) is a bijection. Moreover, the map \( \phi \) carries the weight of \( \delta \) to the number of inversions of \( \phi(\delta) \), as we see next. We note that \( E(n_2) \) is exactly the set \( E(\bar{n}) \) defined by Deodhar and Srinivasan if we change each \( i_1 \) into \( i_2 \) and \( i_2 \) into \( i_1 \), our map \( \phi \) generalizes their bijection between fixed-point free involutions in \( S_{2n} \) and \( E(n) \), and the following is a generalization of [5, Proposition 3.3].

Now consider some \( n > 1 \) under the assumption that the statement holds true for \( n - 1 \). Let \( \delta \in F^{(m)}(mn) \), where \( \delta \) in standard form is

\[
\delta = (a_{1_1} a_{1_2} \cdots a_{1_m}) \cdots (a_{n_1} a_{n_2} \cdots a_{n_m}),
\]

and let \( \pi = \phi(\delta) \in E(n_m) \). As in the proof of Theorem 2.1, form \( \delta' \in F^{(m)}(mn - 1) \) by deleting \( (a_{1_1} \cdots a_{1_m}) \) and relabeling (note that \( a_{1_1} = 1 \) necessarily here). Then, as we showed, we have

\[
wt_{\delta}(\delta) = wt_{\delta}(\delta') + a_{1_2} + a_{1_3} + \cdots + a_{1_m} - 2 - f(a_{1_3}) + \cdots + a_{1_m} - 2 - f(a_{1_m}),
\]

where \( f(a_{1_j}) \) is the number of \( a_{1_j} \) such that \( 1 < j < l \) and \( a_{1_j} < a_{1_i} \).
Now let $\pi' = \phi(\delta') \in E((n-1)_m)$, so we have $\iota(\pi') = \wt_m(\delta')$ by the induction hypothesis. Using the definitions of $\delta'$ and $\phi$, we obtain $\pi'$ from $\pi$ as follows. If $\pi = \pi_1 \pi_2 \cdots \pi_m$, then delete $1_1, 1_2, \ldots, 1_m$, and then replace each remaining $i_j$ with $(i-1)_j$. For example, if $\pi = 1_12_11_23_12_23_23_31_3$, then $\pi' = 1_11_21_21_22_2$. Then, every inversion of $\pi'$ corresponds to an inversion of $\pi$, and all other inversions of $\pi$ are the result of the positioning of $1_2, \ldots, 1_m$ in $\pi$. In particular, $1$ is in the $a_1$-th position of the string $\pi_1 \pi_2 \cdots \pi_m$, and $1$ forms an inversion with any element of this string to its left, except for any $1_j$ such that $j < I$. That is, if we define, for each $i \geq 2$,

$$g(1_i) = \text{the number of } 1_j \text{ such that } 1 < j < I \text{ and } \pi^{-1}(1_j) < \pi^{-1}(1_i),$$

then the number of inversions of $\pi$ which include $1_i$ is exactly $a_{1_i} - 2 - g(1_i)$. It follows from the definition of $\phi$ that we then have $g(1_i) = f(a_{1_i})$, so that we finally have

$$\iota(\pi) = \iota(\pi') + a_{1,2} - 2 + a_{1,3} - 2 - g(1_3) + \cdots + a_{1,m} - 2 - g(1_m) = \wt_m(\delta') + a_{1,2} - 2 + a_{1,3} - 2 - f(a_{1,3}) + \cdots + a_{1,m} - 2 - f(a_{1,m}) = \wt_m(\delta),$$

yielding the result. □

Now consider the partial order $\preceq$ on $E^m(mn)$ defined in Section 1. The following result is analogous to [5, Proposition 3.4].

**Proposition 3.2.** The map $\phi : E^m(mn) \to E(n_m)$ is an order isomorphism, mapping the partial order $\preceq$ to the Bruhat order.

**Proof.** We first show that $\phi$ preserves order. Let $\delta \in E^m(mn)$, and let $\delta$ be in standard form as $\delta = (a_1,1 \cdots a_{1_m}) \cdots (a_n,1 \cdots a_{n,m})$. Suppose that $\tau \in E^m(mn)$ and $\tau$ is obtained by $\delta$ by an interchange. If $\tau$ in standard form is obtained from $\delta$ by exchanging $a_{i_1}$ and $a_{i_2}$, where $1 \leq i_1 \leq n$, and $2 \leq i_2 \leq m$, then $\phi(\tau)$ is obtained from $\phi(\delta)$ by exchanging $i_1$ with $i_2$. If $\tau$ is obtained from $\delta$ by exchanging $a_{i_1}$ and $a_{i_2}$, for some $1 \leq i_1 < j \leq n$ and $2 \leq i_2 \leq m$, then $\phi(\tau)$ is obtained from $\phi(\delta)$ by exchanging $i_1$ and $j$. If $\tau$ is obtained from $\delta$ by exchanging $a_{i_1}$ and $a_{i_2}$, where $I \neq i_1, i_2 \leq m$, and $1 \leq i_1 < j \leq n$, then $\phi(\tau)$ is obtained from $\phi(\delta)$ by exchanging $i_1$ and $j$. Now, if $\delta \preceq \gamma$ for some $\gamma \in E^m(mn)$, then $\phi(\tau)$ is obtained from $\delta$ by some number of such interchanges which are weight increasing. Since $\phi$ maps the weight to the number of inversions by Proposition 3.1, then $\phi(\delta)$ is obtained from $\phi(\delta)$ by some sequence of inversion increasing interchanges, that is, $\phi(\delta) \preceq \phi(\gamma)$.

Next we show that $\phi^{-1}$ is order preserving. To make notation a bit more flexible, we will identify the linearly ordered set $[n_m]$ with $\{1, 2, \ldots, mn\}$, when they appear as indices in $\pi \in S(n_m)$. That is, if $\pi \in S(n_m)$ with $\pi = \pi_1 \pi_2 \cdots \pi_m$, then we will also write $\pi = \pi_1 \pi_2 \cdots \pi_m$. Let $\pi, \sigma \in E^m(mn)$, and suppose $\pi < \sigma$, with $\iota(\sigma) = \iota(\pi) + 1$. Let $\pi = \pi_1 \pi_2 \cdots \pi_m$, and suppose $\pi$ is obtained from $\delta$ by exchanging $\pi_{i_1}$ and $\pi_{i_2}$, where $i_1 < i_2 \leq m$. In particular, $\pi_{i_1}$ is to the left of $\pi_{i_2}$.

We first claim that we must have $\pi_{i_1} = \pi_{i_2} \notin \{1_1, 2_1, \ldots, n_1\}$. If not, so $k_1 = k_2$, then we cannot have $\pi_{i_1} = h_1 \in \{1_1, 2_1, \ldots, n_1\}$, since $\pi_{i_1} < \pi_{i_2}$, and we must remain in $E(n_m)$ when exchanging $\pi_{i_1}$ and $\pi_{i_2}$. On the other hand, if $\pi_{i_1} = k_1$ and $\pi_{i_2} = h_1 \notin \{1_1, 2_1, \ldots, n_1\}$, then since $k_1 < h_1$, we have $k_1 < h_1$. Since $\pi \in E(n_m)$, then $k_1$ is to the left of $h_1$, which is to the left of $k_2$. Then we cannot change $\pi_{i_1} = k_1$ and $\pi_{i_2} = h_1$ and remain in $E(n_m)$. Thus $k_1 \notin \{1_1, 2_1, \ldots, n_1\}$.

Now assume $\pi_{i_1} = h_1 \notin \{1_1, 2_1, \ldots, n_1\}$. If $\pi_{i_1} = k_1$ is such that $k_1 = h_1$, then $h_1 < h_1$, so $t > 1$. Then exchanging $a_{k_1}$ and $a_{h_1}$ in $\phi^{-1}(\pi)$ yields $\phi^{-1}(\sigma)$ in standard form. If $k_1 \neq h_1$, then $t < h_1$ since $\pi_{i_1} = k_1 < h_1 = \pi_{i_2}$. In order to show that $\phi^{-1}(\sigma)$ is in standard form when exchanging $a_{k_1}$ and $a_{h_1}$ in $\phi^{-1}(\pi)$, we need to show that $a_{k_1} < a_{h_1}$ whenever $t > k_1$. Supposing there is a $r$ such that $t < r < h_1$ and $i = a_{k_1} < a_{h_1} = x < a_{h_1} = j$, we have $k_1$ to the left of $r_1$, to the left of $h_1$. Then we cannot change $\pi_{i_1} = k_1$ and $\pi_{i_2} = h_1$ and remain in $E(n_m)$. We now have that $\phi^{-1}(\sigma)$ is obtained in standard form by exchanging $a_{k_1}$ and $a_{h_1}$ in $\phi^{-1}(\pi)$ in all cases. □

Now, if we knew that the grading $\iota$ on $S(n_m)$ restricted to $E(n_m)$ makes $E(n_m)$ a graded poset, we could conclude that $E^m(mn)$ was a graded poset by the previous two results. We show that $E(n_m)$ is a graded poset in the next section by considering EL-labelings.

**Remark.** Consider now an arbitrary permutation $\omega \in S_n$, written in cycle form, including cycles of length 1,

$$\omega = (a_{1,1} \cdots a_{k_1,1})(a_{1,2} \cdots a_{k_2,2}) \cdots (a_{1,h} \cdots a_{k_h,h}),$$

such that $a_{1,1} < a_{1,2} < \cdots < a_{1,h}$ and $a_{i,j} < a_{i,j'}$ for every $1 \leq j < h$ and $2 \leq i \leq k$. Then $\sum_{i,j} k_{i,j} = n$, so $(k_1, \ldots, k_h)$ is a composition of $n$, and call $(k_1, \ldots, k_h)$ the composition type of $\omega$. One may consider the map $\Omega : S_n \to S_n$ defined by...
Theorem 4.1. The poset $E(n_m)$ is an EL-shelling graded poset, with grading and EL-labeling obtained by restriction from $S(n_m)$ under the Bruhat order.

Proof. By [5, Proposition 3.1], it is enough to show that $E(n_m)$ contains a maximal element under the Bruhat order, $E(n_m)$ contains the minimal element of $S(n_m)$, and for all $\pi, \rho \in E(n_m)$ with $\pi < \rho$, the unique unrefinable chain $c_{\pi, \rho}$ in $S(n_m)$ lies completely in $E(n_m)$.

Like in the proof of Proposition 3.2, we will identify the linearly ordered set $[n_m]$ with $\{1, 2, \ldots, m\}$, when they appear as indices in $\pi \in S(n_m)$, so $\pi = \pi_1 \pi_2 \cdots \pi_{n_m}$ will be written as $\pi = \pi_1 \pi_2 \cdots \pi_{n_m}$.

First, the element $1 \cdots 1_{m-1} \cdots 2_m \cdots n_1 \cdots n_{m-1} \cdots n_2 \cdots (n-1)_m \cdots (n-1)_{m-1} \cdots (n-1)_2 \cdots 1_m \cdots 1_2 \in E(n_m)$ will be both the minimal element of $S(n_m)$ and an element of $E(n_m)$. Next, consider the element

$$\zeta = 1_{2_1} \cdots 1_{n_1} n_{m-1} \cdots n_2 (n-1)_m (n-1)_{m-1} \cdots (n-1)_2 \cdots 1_m \cdots 1_2 \in E(n_m).$$

We claim that $\zeta$ is a maximal element of $E(n_m)$. Let $\pi = \pi_1 \cdots \pi_{n_m} = \pi_1 \pi_2 \cdots \pi_{n_m} \in E(n_m)$. If $\pi_1 \cdots \pi_i \neq 1_2 \cdots 1_{n_1}$, find the least $i \geq 2$ such that $\pi_1 \cdots \pi_{i-1} = 1_{i-1} (i-1)_i$, and then $\pi_i = \pi_i$, for some $i > 1$. Since $\pi \in E(n_m)$, then we must have $\pi_1 \pi_2 \cdots \pi_{i-1} \pi_{i+1} \cdots \pi_{n_m} \in 1_2 \cdots 1_{n_1} 2_2 \cdots 2_m \cdots (i-1)_2 \cdots (i-1)_m$. We may then make a sequence of inversion increasing interchanges, first $\pi_i$ with $\pi_{i+1}$, then $\pi_{i+2}$, until we have obtained $\pi_1 \pi_2 \cdots \pi_{i-1} \pi_i \pi_{i+1} \cdots \pi_{n_m} = 1_1 (i-1)_i 1_{i+1} \pi_{i+2} \cdots \pi_{n_m}$. By induction, we obtain from $\pi$ a permutation $\sigma$ of the form $\sigma = 1_2 \cdots 1_{m-1} \pi_{n-1} \cdots \pi_{n_m}$ by a sequence of inversion increasing interchanges, so that $\pi < \sigma$. Note that any such $\sigma$ in $E(n_m)$ is also an element of $S(n_m-1)$. It follows that we must have $\pi \geq \zeta$, since $n_m > n_{m-1} > \cdots > n_2 > \cdots > 1_2$, and $n_m n_{m-1} \cdots n_2 \cdots 1_2 \in S(n_m-1)$ (shifting each $j$ to $j-1$). Thus $\zeta$ is the maximal element of $E(n_m)$.

Now let $\pi, \rho \in E(n_m)$ such that $\pi < \rho$, and consider the unique unrefinable rising chain $c_{\pi, \rho}$ from $\pi$ to $\rho$ in $S(n_m)$. Let $l_i \in [n_m]$ be the least element such that $\pi^{-1}(l_i) \neq \rho^{-1}(l_i)$. Then [7, Remark 2] $\pi^{-1}(l_i) < \rho^{-1}(l_i)$. Suppose that $l_i \in \{1, 2, \ldots, n_1\}$. If $\pi^{-1}(l_i) = s$, then $\rho^{-1}(l_i) > s$, while for every $k_i < l_i$, $\pi^{-1}(k_i) = \rho^{-1}(k_i)$. Thus we must have $\rho_i > l_i$. But now, $l_i \in \{1, 2, \ldots, n_1\}$, $\rho_i > l_i$, and $\rho_i$ appears to the left of $l_i$. But this contradicts $\rho \in E(n_m)$. Thus, we must have $l_i > 1_2 \cdots 1_{m-1} 2_2 \cdots 2_m \cdots n_2 \cdots n_1$. Now let $l_i \in [n_m]$ be the least element such that $l_i \in \{1, 2, \ldots, n_1\}$. Let $t_i \in S(n_m)$ be the string $t_i > l_i$, $\pi^{-1}(t_i) \pi^{-1}(l_i) \leq \rho^{-1}(l_i)$. Now write $\pi = \alpha_1 [\alpha_2 \cdots \alpha_3]$, where $\alpha_1, \alpha_2, \alpha_3$ are strings of elements from $[n_m]$. Consider the element $\omega = \alpha_1 [\alpha_2 \cdots \alpha_3] \in S(n_m)$ obtained by exchanging $l_i$ and $t_i$ in $\pi$. Then by [7, Remark 5], $\omega$ is the element immediately after $\pi$ in the unique rising chain $c_{\pi, \rho}$. We claim $\omega \in E(n_m)$, which will be enough to show that $c_{\pi, \rho}$ is contained in $E(n_m)$, by induction. Suppose $i \geq 2$, so $t_i \notin \{1, 2, \ldots, n_1\}$. Then $t_i < l_i$ and $j \geq 2$, so $l_i < t_i < t_j$. Thus $t_j$ is not in the string $\alpha_3$ by how we have chosen $t_i$. This implies $\omega = \alpha_1 [\alpha_2 \cdots \alpha_3] \in E(n_m)$. Next suppose $i = 1$, so $t_i = t_1$. Then $l_i < (l+1)_{i+1} < (l+2)_{i+1} < \cdots < (t-1)_{i+1} < t_i$. Thus none of $(l+1)_1, (l+2)_1, \ldots, (t-1)_1$ are in the string $\alpha_2$. Since $\pi = \alpha_1 [\alpha_2 \cdots \alpha_3] \in E(n_m)$, then $l_i$ is in the string $\alpha_1$ and $t_i$ for $i \geq 2$ are in the string $\alpha_3$. Thus $\omega = \alpha_1 [\alpha_2 \cdots \alpha_3] \in E(n_m)$ again. So $c_{\pi, \rho}$ is contained in $E(n_m)$ as claimed. \(\square\)
Theorem 4.1. \((F(m)(mn), \preceq)\) is an EL-shellable graded poset, of rank \(\frac{(m-1)(mn-2)}{2}\), with grading given by \(\text{wt}_m\), and with rank generating function given by
\[
\sum_{\delta \in F(m)(mn)} q^{\text{wt}_m(\delta)} = \frac{[mn]_q!}{[mn][m(n-1)]_q \cdots [m]_q}.
\]

Proof. Since \(E(n_m)\) is a graded EL-shellable poset by Proposition 4.1, and \(F(m)(mn)\) is isomorphic to \(E(n_m)\) as a poset by Proposition 3.2, then \(F(m)(mn)\) is a graded EL-shellable poset. Since the order isomorphism \(\phi\) maps the weight function \(\text{wt}_m\) of \(F(m)(mn)\) to the number of inversions \(\iota\) of an element of \(E(n_m)\) by Proposition 3.1, which is the grading for \(E(n_m)\) under the Bruhat order, then \(\text{wt}_m\) provides a grading for \(F(m)(mn)\) under the partial order \(\preceq\). Finally, the rank generating function is then given by
\[
\sum_{\delta \in F(m)(mn)} q^{\text{wt}_m(\delta)} = \frac{[mn]_q!}{[mn][m(n-1)]_q \cdots [m]_q},
\]
by Corollary 2.1, and one can compute directly that the degree of this polynomial is \(\frac{(m-1)(mn-2)}{2}\), which is thus the rank of the graded poset \((F(m)(mn), \preceq)\).

We now give some notation in order to state and prove our last result. Let \(P\) be a finite graded poset with minimal element \(\hat{0}\) and maximal element \(\hat{1}\), and let \(\mu_P\) be the Möbius function for \(P\). Define \(\overline{P} = P \setminus \{\hat{0}, \hat{1}\}\), and let \(\Delta(\overline{P})\) be the order complex of \(\overline{P}\). That is, \(\Delta(\overline{P})\) is the simplicial complex with faces given by chains in \(\overline{P}\), where a chain \(c\) consisting of \(n\) elements gives a face of dimension \(n - 1\). So, if the graded poset \(P\) has rank \(d\), then \(\Delta(\overline{P})\) has dimension \(d - 2\). We let \(|\Delta(\overline{P})|\) denote the topological space constructed from the complex \(\Delta(\overline{P})\) (see [10, Section 3.8]), and then \(\Delta(\overline{P})\) triangulates the space \(|\Delta(\overline{P})|\).

When \(P\) is a finite graded poset which admits an EL-labeling \(\lambda\), then the complex \(\Delta(\overline{P})\) is shellable [1], which is why \(P\) is then called EL-shellable. We do not define the notion of a shellable complex here, but it can be found in [4], for example.

We now need a lemma. The minimal and maximal elements \(\hat{0}\) and \(\hat{1}\) in \(E(n_m)\) are \(\hat{0} = 1 \cdot \cdots \cdot m\) and \(\hat{1} = 1 \cdot 2 \cdot \cdots \cdot m\). However, if \(m > 2\), there must be other edges in the chain between these. This implies that there will not be a descent at some point in this chain.

We may now give our last main result, which is a direct generalization of [5, Theorem 1.3(ii)], and the proof we give is essentially identical.

Theorem 4.2. The complex \(\Delta(F(m)(mn))\) triangulates a ball of dimension \(\frac{(m-1)(mn-2)}{2} - 2\).

Proof. We may equivalently prove the statement for \(E(n_m)\) in place of \(F(m)(mn)\), since these are isomorphic as EL-shellable graded posets. Let \(d = \frac{(m-1)(mn-2)}{2}\). Consider a chain \(c\) in \(E(n_m)\) of length one less than maximal, so that such a chain is of the form \(x_1 < \cdots < x_{i-1} < x_{i+1} < \cdots < x_{d}\) for some \(i\), where \(x_{i+1}\) does not cover \(x_{i-1}\) in \(E(n_m)\). It is known that the symmetric group under the Bruhat order is Eulerian [11], meaning that any rank 2 interval of \(S(n_m)\) contains exactly two elements apart from its endpoints. Thus, the chain \(c\) is contained in at most 2 chains of maximal length in \(E(n_m)\), since the elements \(x_{i-1}\) and \(x_{i+1}\) have only two elements between them in \(S(n_m)\). By [4, Proposition 1.2], it follows that \(\Delta(E(n_m))\) triangulates either a ball or a sphere of dimension \(d - 2\).

Now, by [10, Equation (3.54) and Theorem 3.14.2] and Lemma 4.1, it follows that \(\mu_{E(n_m)}(\hat{0}, \hat{1}) = 0\). For a simplicial complex \(\Delta\), let \(\chi(\Delta)\) denote its reduced Euler characteristic. By [10, Proposition 3.8.6], we have \(\mu_{E(n_m)}(\hat{0}, \hat{1}) = \chi(\Delta(E(n_m)))\), and so \(\chi(\Delta(E(n_m))) = 0\). Since the reduced Euler characteristic of a sphere is \(\pm 1\), while the reduced Euler characteristic of a ball is 0, we must have that \(\Delta(E(n_m))\) triangulates a ball of dimension \(d - 2\).

Acknowledgments

The authors thank Murali Srinivasan for helpful communication, and the anonymous referees for very helpful remarks, including bringing the relevant Refs. [2,3] to our attention. The first author was supported by a NOYCE Fellowship from the College of William and Mary, and the second author was supported by NSF grant DMS-0854849.
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