



Semisimple symplectic characters of finite unitary groups

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ABSTRACT

Let $G = U(2m, \mathbb{F}_{q^2})$ be the finite unitary group, with q the power of an odd prime p . We prove that the number of irreducible complex characters of G with degree not divisible by p and with Frobenius–Schur indicator -1 is q^{m-1} .

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1. Introduction

Let $U(n, \mathbb{F}_{q^2})$ denote the finite unitary group defined over the finite field \mathbb{F}_q , where q is a power of the prime p . A *semisimple* character of $U(n, \mathbb{F}_{q^2})$ is an irreducible complex character with degree prime to p . If an irreducible complex character χ of a finite group G is real-valued, then χ is called *symplectic* if its associated complex representation cannot be defined over the real numbers (otherwise, the real-valued character χ is called *orthogonal*). In [6], it was conjectured that when q is odd, the group $U(2m, \mathbb{F}_{q^2})$ has exactly q^{m-1} semisimple symplectic characters. The main result of this paper is the proof of this conjecture.

In Section 2, we describe the semisimple characters of a connected reductive group over a finite field, and give some general results. In Proposition 2.2, under certain conditions we establish a bijection between real-valued semisimple characters of a finite reductive group G and real semisimple classes of a dual group G^* . In Section 3, we give an explicit description of the semisimple conjugacy

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classes of the finite unitary groups, and a bijection between the real semisimple classes of $U(n, \mathbb{F}_{q^2})$ and self-dual polynomials in $\mathbb{F}_q[x]$ of degree n . Together with Proposition 2.2, this gives a bijection between real semisimple characters of $U(n, \mathbb{F}_{q^2})$ and the self-dual polynomials in $\mathbb{F}_q[x]$ of degree n . In Section 4, we start by stating Theorem 4.1, which says that the Frobenius–Schur indicator of a real-valued semisimple character of $U(2m, \mathbb{F}_{q^2})$ is given by the value of the corresponding central character evaluated at a generator of the center of the group. After a few key lemmas needed to evaluate central characters, we show that the semisimple symplectic characters of $U(2m, \mathbb{F}_{q^2})$, q odd, correspond to the self-dual polynomials in $\mathbb{F}_q[x]$ of degree $2m$ with constant term -1 , and the proof of the main result, Theorem 4.2, is completed by counting these polynomials.

For the larger context of the main result of this paper and why it was originally conjectured, we refer the reader to the introduction of [6].

2. Semisimple characters of finite reductive groups

Let \mathbf{G} be a connected, reductive algebraic group defined over \mathbb{F}_q , $F : \mathbf{G} \rightarrow \mathbf{G}$ a Frobenius morphism and $G = \mathbf{G}^F$ the finite reductive group of F -fixed points of \mathbf{G} . The main tool in describing the set of irreducible characters $\text{Irr}(\mathbf{G}^F)$ of \mathbf{G}^F is Deligne–Lusztig induction, as first given in [3], and presented in [1,2,4]. If \mathbf{L} is an F -stable Levi subgroup of \mathbf{G} with $L = \mathbf{L}^F$, let ψ be a character of L . We then have the Deligne–Lusztig induced virtual character $R_L^G(\psi)$ of ψ to G . If $T = \mathbf{T}^F$ where \mathbf{T} is an F -stable maximal torus of \mathbf{G} , then a *unipotent* character of G is an irreducible character of G which is a constituent of $R_T^G(\mathbf{1})$, where $\mathbf{1}$ is the trivial character of T . We let ε_G denote $(-1)^{r(\mathbf{G})}$, where $r(\mathbf{G})$ is the \mathbb{F}_q -rank of \mathbf{G} .

As in [2, Chapter 4], [4, 13.10], [1, 8.2], there is a dual group \mathbf{G}^* to \mathbf{G} , with a dual Frobenius map F^* . More precisely, fix an injective homomorphism $\kappa : \overline{\mathbb{F}}_q^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$, where $\overline{\mathbb{Q}}_\ell$ is the field of ℓ -adic numbers and $\ell \neq p$. The groups \mathbf{G} and \mathbf{G}^* are in duality with respect to an F -stable maximal torus \mathbf{T} of \mathbf{G} contained in an F -stable Borel subgroup of \mathbf{G} and an F^* -stable maximal torus \mathbf{T}^* contained in an F^* -stable Borel subgroup of \mathbf{G}^* . We also say $G = \mathbf{G}^F$ and $G^* = \mathbf{G}^{*F^*}$ are in duality. Then there is a well-defined isomorphism ([1, 8.14], [2, Proposition 3.3.6])

$$(\mathbf{T}^*)^{F^*} \cong \text{Irr}(\mathbf{T}^F). \tag{2.1}$$

This isomorphism can also be defined for other pairs of tori in \mathbf{G} and \mathbf{G}^* , in particular for pairs in duality in the sense of [2, Proposition 4.3.1]. Any maximal torus in \mathbf{G}^F is of the form $(a\mathbf{T}a^{-1})^F$ for some $a \in \mathbf{G}$ and is isomorphic to \mathbf{T}^{wF} , for some $w \in W(\mathbf{T})$. We say that $(a\mathbf{T}a^{-1})^F$ is of type w and write it as $\mathbf{T}_w^F = T_w$. Then we also have a maximal torus of \mathbf{G}^{*F^*} isomorphic to $(\mathbf{T}^*)^{F^*w^*}$, where $w \rightarrow w^*$ is an anti-isomorphism between $W(\mathbf{T})$ and $W(\mathbf{T}^*)$ (see [4, pp. 104–106]). The tori \mathbf{T}^{wF} and $(\mathbf{T}^*)^{F^*w^*}$ are then in duality.

The isomorphism (2.1) then gives rise to a homomorphism

$$Z(\mathbf{G}^*)^{F^*} \rightarrow \text{Lin}(\mathbf{G}^F), \quad z \mapsto \hat{z}, \tag{2.2}$$

where $\text{Lin}(\mathbf{G}^F)$ is the group of linear characters of \mathbf{G}^F (see [1, 8.19] and [4, Proposition 13.30]). These characters play a key role in our main theorem.

From here, we suppose that \mathbf{G} has connected center.

Definition 2.1. (See [2, p. 280].) An irreducible character of G is *semisimple* if its average value on regular unipotent elements of G is nonzero.

If p is a *good prime* for G (see [2]), then the semisimple characters of G are precisely those with degree not divisible by p .

Definition 2.2. (See [2, p. 281].) An irreducible character of G is *regular* if it is a constituent of the Gelfand–Graev character of G .

The regular characters of G are in natural one-to-one correspondence with the semisimple characters through Curtis duality (see [2, 8.3.7]).

Since the center of G is connected, then the centralizer $C_{G^*}(s)$ of any semisimple element $s \in G^*$ is connected [1, Theorem 13.14]. The set $\text{Irr}(G)$ of irreducible characters of G is in one-to-one correspondence with G^* -conjugacy classes of pairs (s, ψ) , where s is a semisimple element of G^* , and ψ is a unipotent character of $C_{G^*}(s)$ [1, Theorem 15.8]. This is the Jordan decomposition of characters. If furthermore $C_{G^*}(s)$ is a Levi subgroup of G^* with a dual Levi subgroup L of G , then the character of G corresponding to the pair (s, ψ) is of the form $\varepsilon_G \varepsilon_L R_L^G(\hat{s}\psi)$ [1, Proposition 15.10].

We now describe the semisimple characters in this context. Since such a description in the precise form that we require is hard to find in the literature, we give it in the following lemma.

Lemma 2.1.

(i) *The semisimple characters of G are, up to sign, of the form*

$$\frac{1}{|W(s)|} \sum_{w \in W(s)} R_{T_w}^G(\hat{s}), \tag{2.3}$$

where s is a semisimple element of G^* , $W(s)$ is the Weyl group of $C_{G^*}(s)$, and \hat{s} is the character of each T_w corresponding to s .

(ii) *Suppose $C_{G^*}(s)$ is a Levi subgroup of G^* with a dual subgroup L of G . Then $R_L^G(\hat{s})$ is equal to (2.3) and $\varepsilon_G \varepsilon_L R_L^G(\hat{s})$ is a semisimple character of G .*

Proof. (i) We first note that the T_w where $w \in W(s)$ are the tori of G dual to the maximal tori of $C_{G^*}(s)$, up to conjugacy, and hence the characters \hat{s} of each T_w are defined. From [4, 14.40] we see that the regular characters of G are of the form

$$\frac{1}{|W(s)|} \sum_{w \in W(s)} \varepsilon_G \varepsilon_{T_w} R_{T_w}^G(\hat{s}).$$

Taking the Curtis dual, we get a formula for a semisimple character, up to sign, in the form

$$\frac{1}{|W(s)|} \sum_{w \in W(s)} R_{T_w}^G(\hat{s}),$$

since the dual of $\varepsilon_G \varepsilon_{T_w} R_{T_w}^G$ is $R_{T_w}^G$ [4, p. 138].

(ii) We now assume $C_{G^*}(s)$ is a Levi subgroup of G^* with a dual subgroup L of G , which will be the case if the order of s is divisible only by good primes [1, Proposition 13.16]. Then the T_w are precisely the maximal tori of L up to conjugacy, and the linear character \hat{s} of L restricts to the characters \hat{s} of each of the T_w in (2.3). From a formula for the trivial character of L [2, 7.4.2] and hence for \hat{s} we get $\hat{s} = \frac{1}{|W(s)|} \sum_{w \in W(s)} R_{T_w}^L(\hat{s})$. By applying R_L^G to this formula we see that the semisimple characters of G are of the form $R_L^G(\hat{s})$ up to sign. Finally the sign is given by $\varepsilon_G \varepsilon_L$ [4, 12.17]. \square

Recall that an element of a group is *real* if it is conjugate to its inverse in the group. A real semisimple class is thus a semisimple class (s) with the property that $(s^{-1}) = (s)$. The next proposition shows that the Lusztig map R_L^G behaves nicely on real semisimple classes.

Proposition 2.1. *Let $G = G^F$ as above and $L = L^F$ a Levi subgroup of G , with a dual Levi subgroup L^* of G^* . Let $s \in Z(L^*)$ and let \hat{s} be the corresponding linear character of L . Let $g \in G$. Then $R_L^G(\hat{s})(g^{-1}) = R_L^G(\widehat{s^{-1}})(g)$.*

Proof. We denote by G_u the subset of unipotent elements of G . Let $g = tu = ut$ be the Jordan decomposition of g , with t semisimple and u unipotent in G . The Deligne–Lusztig character formula [4, 12.2] implies

$$R_L^G(\hat{s})(g) = \frac{1}{|L||C_G(t)|} \sum_{\{h \in G | t \in {}^hL\}} |C_{hL}(t)| \sum_{v \in C_{hL}(t)_u} Q_{C_{hL}(t)}^{C_G(t)}(u, v^{-1}) {}^h\hat{s}(tv),$$

which implies

$$R_L^G(\hat{s})(g^{-1}) = \frac{1}{|L||C_G(t)|} \sum_{\{h \in G | t \in {}^hL\}} |C_{hL}(t)| \sum_{v \in C_{hL}(t)_u} Q_{C_{hL}(t)}^{C_G(t)}(u^{-1}, v^{-1}) {}^h\hat{s}(t^{-1}v),$$

and

$$R_L^G(\hat{s}^{-1})(g) = \frac{1}{|L||C_G(t)|} \sum_{\{h \in G | t \in {}^hL\}} |C_{hL}(t)| \sum_{v \in C_{hL}(t)_u} Q_{C_{hL}(t)}^{C_G(t)}(u, v^{-1}) {}^h\hat{s}^{-1}(tv),$$

yielding

$$R_L^G(\hat{s}^{-1})(g) = \frac{1}{|L||C_G(t)|} \sum_{\{h \in G | t \in {}^hL\}} |C_{hL}(t)| \sum_{v \in C_{hL}(t)_u} Q_{C_{hL}(t)}^{C_G(t)}(u, v) {}^h\hat{s}^{-1}(tv^{-1}).$$

But ${}^h\hat{s}(t^{-1}v) = {}^h\hat{s}(t^{-1})$ and ${}^h\hat{s}^{-1}(tv) = {}^h\hat{s}(t^{-1})$ since \hat{s} is a linear character. Thus the lemma is true if $Q_{C_{hL}(t)}^{C_G(t)}(u^{-1}, v^{-1}) = Q_{C_{hL}(t)}^{C_G(t)}(u, v)$. This follows from the definition of the Green functions and the fact that their values are rational, in fact in \mathbb{Z} (see [4, 12.1]). \square

Under appropriate conditions, we may now define a bijection from semisimple classes of G^* to semisimple characters of G , which behaves nicely upon restriction to real semisimple classes.

Proposition 2.2. *Suppose that for any semisimple element s of G^* , $C_{G^*}(s)$ is a Levi subgroup of G^* with a dual subgroup L of G . Define a map Θ by $\Theta : (s) \mapsto \varepsilon_G \varepsilon_L R_L^G(\hat{s})$. Then Θ is a bijection from semisimple classes of G^* to semisimple characters G , and $\Theta(s)$ is a real-valued semisimple character of G if and only if (s) is a real semisimple class of G^* .*

Proof. First, it follows directly from Lemma 2.1 that Θ is a bijection. By Proposition 2.1, for any $g \in G$, $\Theta(s)(g^{-1}) = \Theta(s^{-1})(g)$, that is, $\overline{\Theta(s)} = \Theta(s^{-1})$. This implies that $\Theta(s)$ is real-valued if and only if $R_L^G(\hat{s}) = R_L^G(\widehat{s^{-1}})$, which occurs if and only if $(s) = (s^{-1})$ by the Jordan decomposition of characters, and hence if and only if (s) is a real class of G^* . \square

Finally, we state a result as in [1, (8.15)] regarding the values of linear characters of the form \hat{s} . Consider the fixed maximal torus \mathbf{T} and an F^* -stable maximal torus \mathbf{T}^* in duality with \mathbf{T} . Let $X(\mathbf{T}) = \text{Hom}(\mathbf{T}, \overline{\mathbb{F}}_q^\times)$ and $Y(\mathbf{T}) = \text{Hom}(\overline{\mathbb{F}}_q^\times, \mathbf{T})$ be as usual the groups of roots and coroots with pairing (\cdot, \cdot) as in [2, Section 1.9]. We make the identifications $X(\mathbf{T}^*) = Y(\mathbf{T})$, $Y(\mathbf{T}^*) = X(\mathbf{T})$ with \mathbf{T}^* as above. We choose d such that F^d acts as $t \rightarrow t^{q^d}$ on \mathbf{T} and \mathbf{T}^* . For any $s \in \mathbf{T}^{*F^*w^*}$, let $\lambda \in X(\mathbf{T}) = Y(\mathbf{T}^*)$ map to s by a norm map in the short exact sequence in [1, (8.11)] (with \mathbf{T} replaced by \mathbf{T}^*), and for any $t \in \mathbf{T}^{w^*F}$, let t correspond to $\eta \in Y(\mathbf{T}) = X(\mathbf{T}^*)$ in the same way. Thus $t = N_{F^d/w^*F}(\eta(\zeta))$, where N_{F^d/w^*F} is defined on \mathbf{T} by

$$N_{F^d/w^*F}(x) = \prod_{i=0}^{d-1} ({}^{w^*F})^i x,$$

and is defined on $Y(\mathbf{T})$ as

$$N_{F^d/wF}(\eta) = \sum_{i=0}^{d-1} (wF)^i \eta.$$

Given these $s \in \mathbf{T}^{*F^*w^*}$, $t \in \mathbf{T}^{wF}$, $\lambda \in X(\mathbf{T})$, and $\eta \in Y(\mathbf{T})$ we have

$$\hat{s}(t) = \kappa(\zeta^{\langle \lambda, N_{F^d/wF}(\eta) \rangle}), \tag{2.4}$$

which gives the more general version of [1, (8.15)] where F^* and F are replaced by $(wF)^* = F^*w^*$ and wF , respectively.

The following lemma will be used in the proof of the main theorem.

Lemma 2.2. *Let \mathbf{T} be any F -stable maximal torus of \mathbf{G} and \mathbf{T}^* an F^* -stable maximal torus of a dual group \mathbf{G}^* , dual to \mathbf{T} . Let $s \in \mathbf{T}^F$ and $t \in \mathbf{T}^{F^*}$ and let \hat{s} and \hat{t} be linear characters of \mathbf{T}^{*F^*} and \mathbf{T}^F as described above. Then $\hat{s}(t) = \hat{t}(s)$.*

Proof. We use the construction of the characters \hat{s} and \hat{t} given above. As before, choose d such that F^d acts as $t \rightarrow t^{q^d}$ on \mathbf{T} and \mathbf{T}^* . Let s correspond to $\lambda \in Y(\mathbf{T}^*) = X(\mathbf{T})$, and t correspond to $\eta \in Y(\mathbf{T})$, in the sense that, as in [1, (8.8)], $N_1(\lambda) = s$ and $N_1(\eta) = t$. Fix dual bases $\{\mu_1, \dots, \mu_n\}$ and $\{\nu_1, \dots, \nu_n\}$ of $Y(\mathbf{T})$ and $X(\mathbf{T})$, respectively, so that $\langle \mu_i, \nu_j \rangle = \delta_{ij}$. By (2.4), $\hat{s}(t) = \kappa(\zeta^{\langle \lambda, N_{F^d/wF}(\eta) \rangle})$ and $\hat{t}(s) = \kappa(\zeta^{\langle \eta, N_{F^d/F}(\lambda) \rangle})$, where $\zeta \in \bar{\mathbb{F}}_q^\times$ has order $q^d - 1$. Suppose that $\lambda = \sum_{i=1}^{d-1} m_i \mu_i$ and $\eta = \sum_{i=1}^{d-1} k_i \nu_i$ as elements of $Y(\mathbf{T}) = X(\mathbf{T}^*)$ and $Y(\mathbf{T}^*) = X(\mathbf{T})$. Now, $N_{F^d/wF}(\eta) = \sum_{i=1}^{d-1} F^{i-1} \eta$ and so we may calculate that

$$\langle \lambda, N_{F^d/wF}(\eta) \rangle = \sum_{i,j=1}^{d-1} k_i m_j (q^d)^{i+j-2} = \langle \eta, N_{F^d/F}(\lambda) \rangle.$$

Hence $\hat{s}(t) = \hat{t}(s)$, as desired. \square

3. Real semisimple classes of finite unitary groups

Let $\mathbf{G} = \text{GL}(n, \bar{\mathbb{F}}_q)$, and for $g = (a_{ij}) \in \mathbf{G}$, let ${}^T g$ denote the transpose of g , so ${}^T g = (a_{ji})$. Define the map F on \mathbf{G} by $F(a_{ij}) = (a_{ij}^q)^{-1}$. The finite unitary group, denoted $U(n, \mathbb{F}_{q^2})$, is defined as $U(n, \mathbb{F}_{q^2}) = \mathbf{G}^F$. In the rest of the paper we denote $U(n, \mathbb{F}_{q^2})$ by U_n .

Identifying $\text{GL}(1, \bar{\mathbb{F}}_q)$ with $\bar{\mathbb{F}}_q^\times$, consider the action of the map F on $\bar{\mathbb{F}}_q^\times$, where $F(\alpha) = \alpha^{-q}$. Then F^2 is the Frobenius endomorphism of $\bar{\mathbb{F}}_q^\times$ with fixed field \mathbb{F}_{q^2} . If Δ is an F^2 -orbit of $\bar{\mathbb{F}}_q^\times$, then so are

$$\Delta^{-q} = \{\alpha^{-q} \mid \alpha \in \Delta\}, \quad \Delta^{-1} = \{\alpha^{-1} \mid \alpha \in \Delta\}, \quad \text{and} \quad \Delta^q = \{\alpha^q \mid \alpha \in \Delta\}.$$

The elementary divisor theory in [5, Section 1] parameterizes semisimple conjugacy classes (s) of U_n by functions $v_{(s)}$ from F^2 -orbits Δ of $\bar{\mathbb{F}}_q^\times$ to non-negative integers such that

$$v_{(s)}(\Delta) = v_{(s)}(\Delta^{-q}) \quad \text{for all } \Delta, \quad \text{and} \quad \sum_{\Delta} v_{(s)}(\Delta) = n.$$

Here, (s) has characteristic polynomial

$$f_{(s)}(x) = \prod_{\Delta} c_{\Delta}(x)^{v_{(s)}(\Delta)},$$

where $c_{\Delta}(x) = \prod_{\alpha \in \Delta} (x - \alpha)$.

Let K be any field such that $\text{char}(K) \neq 2$, and fix an algebraic closure \bar{K} of K . A non-constant polynomial $g(x) \in K[x]$ is called a *self-dual polynomial* if it is monic, and has the property that any $\alpha \in \bar{K}$ is a root of $g(x)$ with multiplicity m if and only if α^{-1} is a root of $g(x)$ with multiplicity m .

Consider a semisimple class (s) of U_n . Then (s) is a real semisimple class if and only if, for all F^2 -orbits Δ ,

$$v_{(s)}(\Delta) = v_{(s)}(\Delta^{-1}) = v_{(s)}(\Delta^q) = v_{(s)}(\Delta^{-q}). \tag{3.1}$$

If (s) is real, then $f_{(s)}(x)$ is self-dual. Moreover, $f_{(s)}(x) \in \mathbb{F}_q[x]$ since $c_{\Delta}(x)c_{\Delta^q}(x) \in \mathbb{F}_q[x]$ when $\Delta \neq \Delta^q$. Thus $f_{(s)}(x)$ is a self-dual polynomial of degree n in $\mathbb{F}_q[x]$. Conversely, if $f(x)$ is a self-dual polynomial of degree n in $\mathbb{F}_q[x]$, then $f(x)$ factors in $\mathbb{F}_{q^2}[x]$ as a product of the $c_{\delta}(x)$'s since the $c_{\Delta}(x)$ are the monic irreducible polynomials in $\mathbb{F}_{q^2}[x]$. Since $f(x)$ is self-dual, $c_{\Delta}(x)$ and $c_{\Delta^{-1}}(x)$ have the same multiplicity in $f(x)$, and since $f(x) \in \mathbb{F}_q[x]$, $c_{\Delta}(x)$ and $c_{\Delta^{-1}}(x)$ have the same multiplicity in $f(x)$. Thus (3.1) holds and $f(x) = f_{(s)}(x)$ for a real semisimple conjugacy class (s) of U_n . Hence real semisimple conjugacy classes (s) of U_n are in bijection with self-dual polynomials $f_{(s)}(x)$ of degree n in $\mathbb{F}_q[x]$. Note that such self-dual $f_{(s)}(x)$ satisfy the following properties:

- (a) $f_{(s)}(0) = (-1)^{v_{(s)}(\{-1\})}$,
- (b) $f_{(s)}(x) = f_{(s)}(0)^{-1}x^n f_{(s)}(x^{-1})$,
- (c) $v_{(s)}(\{1\}) \equiv v_{(s)}(\{-1\}) \pmod{2}$ if n is even, since $\alpha \neq \alpha^{-1}$ for α a zero of $f_{(s)}(x)$ and $\alpha \neq \pm 1$.

4. Proof of the main theorem

We now take $\mathbf{G} = \text{GL}(n, \bar{\mathbb{F}}_q)$ and $\mathbf{G}^F = G = U_n$ for the rest of this section. In this case we may take $\mathbf{G}^* = \mathbf{G}$, $F^* = F$ and $G = G^*$. We also have that the center of \mathbf{G} is connected, the prime p is always a good prime of G , and the centralizers of semisimple elements of \mathbf{G} are Levi subgroups. By [5, Proposition (1A)], the centralizer of a semisimple class (s) in U_n is isomorphic to a direct product of general linear groups and unitary groups over extensions of \mathbb{F}_q .

The irreducible characters of G were first described in [7], and the theory is given concisely in [5, Section 1], which we follow here. We can apply Lemma 2.1 to describe the semisimple characters of G as characters of the form $\varepsilon_G \varepsilon_L R_L^G(\hat{s})$ where s runs over a set of representatives of the semisimple classes of G and $L = C_G(s)$. However, in this case we can see this directly as follows. Any irreducible character of G is of the form $\varepsilon_G \varepsilon_L R_L^G(\hat{s}\psi)$ where ψ is a unipotent character of $L = C_G(s)$ and has degree $(G : L)_p \psi(1)$ [5, pp. 113, 116]. But $\psi(1)$ is divisible by p if and only if ψ is the trivial character of L (see [5, (1.15)]).

Recall that if H is a finite group, and π is an irreducible complex representation of H with character χ , then the *Frobenius–Schur indicator* of χ (or of π), denoted $\varepsilon(\chi)$ (or $\varepsilon(\pi)$) is defined to be $\varepsilon(\chi) = \frac{1}{|H|} \sum_{h \in H} \chi(h^2)$. By classical results of Frobenius and Schur, $\varepsilon(\chi)$ always takes the value 0, 1, or -1 , and $\varepsilon(\chi) = 0$ if and only if χ is not a real-valued character. Furthermore, if χ is real-valued, then $\varepsilon(\chi) = 1$ if and only if π is equivalent to a representation over \mathbb{R} . If χ is a real-valued irreducible character of H , then χ is called an *orthogonal character* of H if $\varepsilon(\chi) = 1$, and χ is called a *symplectic character* of H if $\varepsilon(\chi) = -1$.

The following result, proven in [10], gives the Frobenius–Schur indicators of the real-valued regular and semisimple characters of U_n . We note that Theorem 4.1(1) below also follows from [9, Theorem 7(ii)].

Theorem 4.1. Let χ be a real-valued semisimple or regular character of G .

- (1) If n is odd or q is even, then $\varepsilon(\chi) = 1$.
- (2) If n is even and q is odd, then $\varepsilon(\chi) = \omega_\chi(z)$, where z is a generator for the center of G , and ω_χ is the central character corresponding to χ .

Hence G has semisimple (or regular) symplectic characters only when n is even and q is odd. We remark that it follows from [10, Corollary 3.1] that the number of semisimple symplectic (respectively, orthogonal) characters of G is equal to the number of regular symplectic (respectively, orthogonal) characters of G .

In order to prove our main result, we also need to understand the central characters corresponding to the irreducible characters. The following is [5, Lemma (2H)].

Lemma 4.1. Let $\chi \in \text{Irr}(G)$ correspond to the G -conjugacy class of the pair (s, ψ) . Then for any elements $z \in Z(G)$, $g \in G$,

$$\chi(zg) = \hat{s}(z)\chi(g).$$

In particular, the central character ω_χ of χ is given by $\omega_\chi(z) = \hat{s}(z)$.

We now apply Lemma 2.2 to evaluate $\hat{s}(z)$, where z is a generator for the center of G .

Lemma 4.2. Let (s) be a real semisimple class of G and z a generator for $Z(G)$. Suppose that the semisimple class (s) has -1 as an eigenvalue with multiplicity k . Then $\hat{s}(z) = (-1)^k$.

Proof. Since s and z may be viewed as elements of some common maximal torus, then by Lemma 2.2, we have $\hat{s}(z) = \hat{z}(s)$, where we may consider \hat{z} to be a linear character of U_n . There are exactly $|Z(G)| = q + 1$ linear characters of G , which are given by powers of the determinant, composed with a fixed injective homomorphism from \mathbb{F}_q^\times to \mathbb{C}^\times . It follows from [1, 8.19] that in the case $G^F = U_n$, the map (2.2) from $Z(G)^F$ to the group of linear characters of G^F is an isomorphism, which implies that \hat{z} is the determinant map composed with the fixed injective homomorphism from \mathbb{F}_q^\times to \mathbb{C}^\times .

Since (s) is a real semisimple class of U_n , then from the description in Section 3, each eigenvalue of s in \mathbb{F}_q^\times occurs with the same multiplicity as its inverse, and so $\det(s) = (-1)^k$, where k is the multiplicity of -1 as an eigenvalue. Therefore $\hat{s}(z) = \hat{z}(s) = (-1)^k$. \square

Finally, we arrive at the main result.

Theorem 4.2. Let $n = 2m$, and let q be odd. The number of semisimple symplectic characters of U_n is q^{m-1} .

Proof. By Theorem 4.1, if $\chi = \Theta(s)$ is a real-valued semisimple character of $G = U_{2m}$, where q is odd, then $\varepsilon(\chi) = \omega_\chi(z)$, where z is a generator of $Z(G)$. By Lemma 4.1, $\varepsilon(\Theta(s)) = \hat{s}(z)$, and by Proposition 2.2, (s) is a real semisimple class of G . By Lemma 4.2, $\hat{s}(z) = (-1)^k$, where k is the multiplicity of -1 as an eigenvalue of any element of (s) . By Section 3, the characteristic polynomial $f_{(s)}$ corresponding to (s) is a self-dual polynomial in $\mathbb{F}_q[x]$ of degree $2m$. From observations (a) and (c) at the end of Section 3, the constant term of $f_{(s)}$ is $(-1)^k$, where k is the multiplicity of the factor $x + 1$. So, the semisimple characters with Frobenius–Schur indicator -1 are exactly those semisimple characters $\Theta(s)$ such that the characteristic polynomial $f_{(s)}$ of (s) has constant term -1 . That is, to count semisimple symplectic characters of G , we must count self-dual polynomials in $\mathbb{F}_q[x]$ of degree $2m$ with constant term -1 .

Suppose $f(x) \in \mathbb{F}_q[x]$, $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$, is a self-dual polynomial of degree $n = 2m$ with $a_0 = -1$. As in observation (b) at the end of Section 3, we have

$$f(x) = -x^n f(x^{-1}) = x^n - a_1x^{n-1} - \dots - a_{n-1}x - 1. \tag{4.1}$$

Since $n = 2m$ is even, (4.1) implies that we must have $a_m = -a_m$, and so $a_m = 0$. For $1 \leq i \leq m - 1$, we may let a_i be any of q elements of \mathbb{F}_q , and then each $a_{n-i} = -a_i$ is determined, giving a total of q^{m-1} polynomials. \square

We conclude by mentioning that our result on symplectic characters extends to regular characters. As proven in [6, Theorem 4.4], there are exactly $q^m + q^{m-1}$ real-valued semisimple characters of U_{2m} when q is odd. It follows from Theorem 4.2 that there are exactly q^m semisimple orthogonal characters of U_{2m} when q is odd. From our remark after Theorem 4.1 above, it follows that there are exactly q^{m-1} symplectic regular characters and q^m orthogonal regular characters of U_{2m} when q is odd.

Remark. Shortly before publication of this paper, the authors became aware that Proposition 2.2 is also implied by the stronger result [8, Lemma 9.1].

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