Abstract. Let $G = U(2m, \mathbb{F}_{q^2})$ be the finite unitary group, with $q$ the power of an odd prime $p$. We prove that the number of irreducible complex characters of $G$ with degree not divisible by $p$ and with Frobenius-Schur indicator $-1$ is $q^{m-1}$.

2010 AMS Subject Classification: 20C33

1. Introduction

Let $U(n, \mathbb{F}_q)$ denote the finite unitary group defined over the finite field $\mathbb{F}_q$ with $q$ elements, where $q$ is a power of the prime $p$. A semisimple character of $U(n, \mathbb{F}_q)$ is an irreducible complex character with degree prime to $p$. If an irreducible complex character $\chi$ of a finite group $G$ is real-valued, then $\chi$ is called symplectic if its associated complex representation cannot be defined over the real numbers (otherwise, the real-valued character $\chi$ is called orthogonal; see Section 2.1). In [8], it was conjectured that when $q$ is odd, the group $U(2m, \mathbb{F}_{q^2})$ has exactly $q^{m-1}$ semisimple symplectic characters. The main result of this paper is the proof of this conjecture.

In Section 2, we give some preliminary definitions and results, including Theorem 2.1, which says that the Frobenius-Schur indicator of a real-valued semisimple character of $U(2m, \mathbb{F}_{q^2})$ is given by the value of the central character evaluated at the generator of the center of the group. In Section 3, we give an explicit description of the semisimple conjugacy classes of the finite unitary groups, along with their centralizers and associated linear characters. In Section 4, we give the parameterization of the irreducible characters of the finite unitary groups in terms of Deligne-Lusztig induction and Jordan decomposition of characters. In Proposition 4.1 and Lemma 4.2, we establish a bijection between real semisimple classes and real semisimple characters of the finite unitary groups. Along with Section 3.4, this gives a bijection between real semisimple characters of $U(n, \mathbb{F}_{q^2})$ and the self-dual polynomials in $\mathbb{F}_q[x]$ of degree $n$. Finally, the main result is proven in Section 5. After a key lemma, it is shown that the semisimple symplectic characters of $U(2m, \mathbb{F}_{q^2})$, $q$ odd, correspond to the self-dual polynomials in $\mathbb{F}_q[x]$ of degree $2m$ with constant term $-1$, and the proof of Theorem 5.1 is completed by counting these polynomials.
The context of the main result of this paper, and the reason why it was originally conjectured, are as follows. Let \( \tau \) denote the transpose-inverse automorphism of the group \( U_n = U(n, \mathbb{F}_{q^2}) \), so that \( \tau(g) = Tg^{-1} \) for any \( g \in U(n, \mathbb{F}_{q^2}) \), and let \( U_n(\tau) \) denote the split extension of the group \( U_n \) by the order 2 automorphism \( \tau \). Then, the irreducible characters of \( U_n \) which extend to irreducible characters of \( U_n(\tau) \) are exactly the \( \tau \)-invariant characters, which are the real-valued characters of \( U_n \). If \( \chi \) is an irreducible real-valued character of \( U_n \), then by [12, Corollary 5.3], an irreducible character of \( U_n(\tau) \) which extends \( \chi \) is real-valued exactly when \( \chi \) is orthogonal, and takes non-real values whenever \( \chi \) is symplectic.

As proven in [8], when \( n \) is odd, there exists an element \( x\tau \) of \( U_n(\tau) \) such that \( (x\tau)^2 = u \), where \( u \) is regular unipotent in \( U_n \). Part of the main result of [8] is that if \( \psi \) is an irreducible character of \( U_n(\tau) \) which extends a semisimple real-valued character of \( U_n \), and \( n \) is odd, then \( \psi(x\tau) = \pm 1 \). In fact, in the case that \( n \) is odd, every real-valued semisimple character of \( U_n \) is orthogonal (see Section 2.3).

When \( n \) is even, in [8] it is shown that although \( U_n(\tau) \) has no elements which square to regular unipotent elements of \( U_n \), there does exist an element \( y\tau \) such that \( (y\tau)^2 = -u \), where \( u \) is regular unipotent in \( U_n \). In [8, Theorem 6.4], it is proven that if \( n = 2m \) is even and \( q \) is odd, and \( \psi \) is an irreducible character of \( U_n(\tau) \) which is the extension of a semisimple symplectic character of \( U_n \), then \( \chi(y\tau) \) is a nonzero purely imaginary number, where \( (y\tau)^2 = -u \) and \( u \) is regular unipotent in \( U_n \). As a consequence, the element \( y\tau \) is not a real element of \( U_n(\tau) \). It is then conjectured that \( \chi(y\tau) \) is equal to \( \pm \sqrt{-q} \), and that there are exactly \( q^{m-1} \) semisimple symplectic characters of \( U(2m, \mathbb{F}_{q^2}) \). Here, we prove the second part of this conjecture, in Theorem 5.1 below. Counting the number of semisimple symplectic characters has significance in proving the first part of this conjecture, as it could be used in a calculation similar to that in the proof of [8, Theorem 6.3].

As discussed in the Introduction of [8], there is a more general conjecture about the values of the characters of \( U_n(\tau) \) which are extended from irreducible real-valued characters of \( U_n \), and their correspondence with characters extended from \( \text{GL}(n, \mathbb{F}_q) \). In particular, let \( G_n = \text{GL}(n, \mathbb{F}_q) \), and let \( G_n(\tau) \) be the split extension of \( G_n \) by the transpose-inverse automorphism. When \( n = 2m \) is even and \( q \) is odd, we expect there to be exactly \( q^{m-1} \) semisimple real-valued characters of \( G_n \) with the property that the extensions of these characters to \( G_n(\tau) \) take the values \( \pm \sqrt{q} \) on elements \( w\tau \), where \( (w\tau)^2 \) is the negative of a regular unipotent element of \( G_n \).

In [7], Gow proves that the rational Schur index of any irreducible character of \( U(n, \mathbb{F}_{q^2}) \) is at most 2. It is not known in general which characters of the finite unitary group have Schur index 1 or 2, although partial results are obtained in [7, 10, 11]. By the Brauer-Speiser Theorem, any symplectic character of a finite group has rational Schur index 2, and so the main result of this paper can also be used to give a lower bound for the number of
irreducible characters of $U(2m, \mathbb{F}_{q^2})$, $q$ odd, with Schur index 2.

Acknowledgements. The second author thanks Fernando Rodriguez-Villegas for helpful communication. The second author was supported in part by NSF grant DMS-0854849.

2. Preliminaries

2.1. Frobenius-Schur indicators. Let $G$ be a finite group, and let $(\pi, V)$ be an irreducible complex representation of $G$ with character $\chi$. Define the Frobenius-Schur indicator of $\chi$ (or of $\pi$), denoted $\varepsilon(\chi)$ (or $\varepsilon(\pi)$), to be

$$\varepsilon(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

By classical results of Frobenius and Schur, $\varepsilon(\chi)$ always takes the value 0, 1, or $-1$, and $\varepsilon(\chi) = 0$ if and only if $\chi$ is not a real-valued character. Furthermore, if $\chi$ is real-valued, then $\varepsilon(\chi) = 1$ if and only if $(\pi, V)$ is a real representation. That is, $\varepsilon(\chi) = 1$ if and only if there exists a basis for $V$ such that the matrix representation corresponding to $\pi$ with respect to this basis has image in the invertible matrices defined over $\mathbb{R}$.

If $\chi$ is a real-valued irreducible character of $G$, then $\chi$ is called an orthogonal character of $G$ if $\varepsilon(\chi) = 1$, and $\chi$ is called a symplectic character of $G$ if $\varepsilon(\chi) = -1$.

2.2. The finite unitary group. Let $q$ be a power of a prime $p$, let $\mathbb{F}_q$ denote a finite field with $q$ elements, and let $\overline{\mathbb{F}}_q$ be a fixed algebraic closure of $\mathbb{F}_q$. Now let $G_n = \text{GL}(n, \overline{\mathbb{F}}_q)$ be the group of invertible $n$-by-$n$ matrices with entries in $\overline{\mathbb{F}}_q$. For $g = (a_{ij}) \in G_n$, let $^T g$ denote the transpose of $g$, so $^T g = (a_{ji})$. Now, define the map $F$ on $G_n$ by

$$F : g = (a_{ij}) \mapsto ^T (a_{ij}^q)^{-1} = (a_{ji}^q)^{-1}.$$ 

The finite unitary group, denoted $U(n, \mathbb{F}_{q^2})$, is defined to be the set of $F$-fixed points of $G_n$, so

$$U(n, \mathbb{F}_{q^2}) = \{g \in \text{GL}(n, \overline{\mathbb{F}}_q) \mid F(g) = g\}.$$ 

We will also denote $U(n, \mathbb{F}_{q^2})$ by $U_n$.

2.3. Semisimple and regular characters. If $G$ is a finite group of Lie type defined over $\mathbb{F}_q$, and $q$ is the power of the prime $p$, where $p$ is a good prime for $G$ (see [2]), then a character of $G$ is called semisimple when its degree is not divisible by $p$. In the case that $G = U(n, \mathbb{F}_{q^2})$, $p$ is always a good prime for $G$ (since the group is type $A$).

Recall that the Gelfand-Graev character of a finite group of Lie type $G$, is obtained by inducing a non-degenerate linear character of the Sylow $p$-subgroup of $G$ up to $G$ (see [2, Section 8.1]). An irreducible character of $G$ is regular if it is a component of the Gelfand-Graev character. Recall that the
semisimple and regular characters of a finite group of Lie type are in natural one-to-one correspondence through the duality functor (see [2, Section 8.3]).

The following result, proven in [13], gives the Frobenius-Schur indicators of the real-valued regular and semisimple characters of the finite unitary groups. We note that Theorem 2.1(1) below also follows from [11, Theorem 7(ii)].

**Theorem 2.1.** Let $\chi$ be a real-valued semisimple or regular character of $U(n, \mathbb{F}_{q^2})$. Then:

1. If $n$ is odd or $q$ is even, then $\varepsilon(\chi) = 1$.
2. If $n$ is even and $q$ is odd, then $\varepsilon(\chi) = \omega_{\chi}(\beta I)$, where $\beta$ is a multiplicative generator for the center of $U(n, \mathbb{F}_{q^2})$, and $\omega_{\chi}$ is the central character corresponding to $\chi$.

So, $U(n, \mathbb{F}_{q^2})$ only has semisimple (or regular) symplectic characters when $n$ is even and $q$ is odd. In the main result, we count the number of semisimple symplectic characters of $U(n, \mathbb{F}_{q^2})$, which is the same as the number of regular symplectic characters of $U(n, \mathbb{F}_{q^2})$. In fact, as proven in [13, Corollary 3.1], if $G$ is a finite group of Lie type defined over $\mathbb{F}_q$, which comes from a connected reductive group over $\overline{\mathbb{F}}_q$ with connected center, then the number of semisimple symplectic (respectively, orthogonal) characters of $G$ is equal to the number of regular symplectic (respectively, orthogonal) characters of $G$.

### 2.4. Self-dual polynomials.

Let $K$ be any field such that $\text{char}(K) \neq 2$, and fix an algebraic closure $\overline{K}$ of $K$. A polynomial $g(x) \in K[x]$ is called a self-dual polynomial if it is monic, non-constant, has non-zero constant term, and has the property that for any $\alpha \in \overline{K}^\times$, $\alpha$ is a root of $g(x)$ with multiplicity $m$ if and only if $\alpha^{-1}$ is a root of $g(x)$ with multiplicity $m$. See [14, Section 1] for the basic results on self-dual polynomials which we now state. If $h(x) \in K[x]$ is a monic, non-constant polynomial with nonzero constant term, define $\overline{h}(x) \in K[x]$ by

$$
\overline{h}(x) = h(0)^{-1}x^dh(1/x), \quad \text{where} \quad d = \text{deg}(h(x)).
$$

Then $h(x)$ is a self-dual polynomial if and only if $h(x) = \overline{h}(x)$. Note that the only irreducible self-dual polynomials of odd degree are $x + 1$ and $x - 1$. Any self-dual polynomial $g(x) = \mathfrak{g}(x)$ may be written as a product

$$
g(x) = (x - 1)^s(x + 1)^t \prod_{i=1}^{k}(v_i(x)\pi_i(x))^{n_i} \prod_{j=1}^{l}r_j(x)^{m_j},
$$

where each $v_i(x), r_j(x) \in K[x]$ is irreducible over $K$, $v_i(x) \neq \pi_i(x)$, $r_j(x) = \mathfrak{r}_j(x)$, and $r_j(x) \neq x \pm 1$. Note also that any self-dual $g(x) \in K[x]$ has constant term $\pm 1$, and the constant term is 1 exactly when $s$ in the product (2.2) is even.
3. Semisimple classes

3.1. Description of semisimple classes. An element of GL(n, \(\overline{\mathbb{F}_q}\)) is semisimple if it is diagonalizable as a matrix over \(\overline{\mathbb{F}_q}\). A conjugacy class in the finite unitary group U(n, \(\mathbb{F}_q^2\)) is a semisimple class if each of its elements are semisimple as elements in GL(n, \(\overline{\mathbb{F}_q}\)).

Identifying GL(1, \(\overline{\mathbb{F}_q}\)) with \(\mathbb{F}_q^\times\), consider the action of the map F on \(\mathbb{F}_q^\times\), where F(\(\alpha\)) = \(\alpha^{-q}\). For any F-orbit \(\{\alpha, \alpha^{-q}, \ldots, \alpha^{(-q)^d}\}\) in \(\mathbb{F}_q^\times\), we have the polynomial

\[
(x - \alpha)(x - \alpha^{-q}) \cdots (x - \alpha^{(-q)^d}) \in \mathbb{F}_q[x].
\]

We call any such polynomial F-irreducible, and we denote by \(\mathcal{F}\) the collection of all F-irreducible polynomials in \(\mathbb{F}_q[x]\). The set \(\mathcal{F}\) has another useful description [5]. For any polynomial \(f(x) = x^d + c_1 x^{d-1} + \cdots + c_d \in \mathbb{F}_q[x]\) with \(c_d \neq 0\), define \(\tilde{f}(x)\) by

\[
\tilde{f}(x) = x^d + c_d^{-q} (c_{d-1} x^{d-1} + \cdots + c_1 x + 1),
\]

which has the effect of applying F to the roots of f in \(\overline{\mathbb{F}_q}\). Then, a polynomial \(f(x)\) is F-irreducible if and only if either

1. \(f\) is irreducible in \(\mathbb{F}_q[x]\) and \(\tilde{f} = f\), in which case \(d = \deg(f)\) must be odd, or
2. \(f(x) = h(x)\tilde{h}(x)\) for some irreducible polynomial h in \(\mathbb{F}_q[x]\) such that \(\tilde{h}(x) \neq h(x)\), in which case d must be even.

We let \(\mathcal{F}_1\) and \(\mathcal{F}_2\) denote the subsets of polynomials in \(\mathcal{F}\) with degree odd and even, respectively.

We now describe the semisimple classes of \(U_n = U(n, \mathbb{F}_q^2)\) (see [6, Sec. 1]). The semisimple classes of \(U_n\) are in one-to-one correspondence with an assignment of a non-negative integer \(m_f\) to each polynomial \(f \in \mathcal{F}\), such that \(\sum_{f \in \mathcal{F}} m_f \deg(f) = n\). Here, the finite product \(\prod_{f \in \mathcal{F}} f(x)^{m_f}\) is the characteristic polynomial of any element in the corresponding conjugacy class. Consider a semisimple element \(s \in U_n\), in the semisimple class \((s)\) described by the pairs \(\{(f, m_f) \mid f \in \mathcal{F}, m_f \geq 0\}\). For each \(f \in \mathcal{F}\) such that \(m_f > 0\), let \((f)\) denote the companion matrix of \(f\), and let \(m_f(f)\) denote the matrix direct sum of \(m_f\) copies of \((f)\), and denote \(s_f = m_f(f)\). Then the matrix direct sum over all \(f \in \mathcal{F}\) such that \(m_f > 0\), \(\prod_f s_f\), is a representative element of the semisimple conjugacy class \((s)\) in \(U_n\). We call \(\prod_f s_f\) the primary decomposition of \((s)\).

3.2. Centralizers. The centralizer in \(U_n\) of any element in the class \((s)\) may be described as follows (see [6, Prop. (1A)]). Given the primary decomposition \(\prod_f s_f\) of a semisimple conjugacy class \((s)\) of \(U(n, \mathbb{F}_q^2)\), the centralizer
\(L = C_{U_n}(s)\) of any element of \((s)\) is a direct product \(\prod_f L_f = \prod_f C(s)_f\) over all \(f \in \mathcal{F}\) such that \(m_f > 0\), where

1. \(L_f = C(s)_f \cong U(m_f, \mathbb{F}_{q^d})\) if \(f \in \mathcal{F}_1\), where \(d = \deg(f)\), and
2. \(L_f = C(s)_f \cong GL(m_f, \mathbb{F}_{q^d})\) if \(f \in \mathcal{F}_2\), where \(d = \deg(f)\).

The center \(Z(L)\) of a centralizer \(L = C_{U_n}(s)\) is the direct product of the centers \(Z(L_f)\) of the factors in the description given above. Using the notation as in the beginning of this section, let

\[T_d = (\overline{\mathbb{F}}_q^\times)^{F_d} = \left\{ a \in \overline{\mathbb{F}}_q^\times \mid a^{q^d(-1)^d} = 1 \right\},\]

so that \(T_m = \mathbb{F}_q^{\times m}\) when \(m\) is even. Then, given a centralizer of a semisimple element in \(U_n\), say \(C_{U_n}(s) = L = \prod_f L_f\), we have

\[Z(L) = \prod_f Z(L_f) \cong \prod_f T_{d(f)},\]

where \(d(f) = \deg(f)\), and the product is taken only over those \(f \in \mathcal{F}\) such that \(m_f > 0\).

### 3.3. Linear characters

Consider the general situation where \(G\) is a reductive group over \(\overline{\mathbb{F}}_q\), defined over \(\mathbb{F}_q\) by a Frobenius map \(F\). Then as in [2, Chapter 4], there is a dual group \(G^*\) to \(G\), with a dual Frobenius map \(F^*\). For an \(F\)-stable maximal torus \(T\) of \(G\), and the dual maximal torus \(T^*\) of \(G^*\) which is \(F^*\)-stable, then up to a choice of roots of unity, there is a natural isomorphism

\[(T^*)^{F^*} \cong \text{Irr}(T^F),\]

given in [1, Section 8.2]. We now give details of the construction of this isomorphism which we will need for the main result.

Fix an injective homomorphism \(\kappa : \overline{\mathbb{F}}_q^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times\), where \(\overline{\mathbb{Q}}_\ell\) are the \(\ell\)-adic rationals and \(\ell \neq p\). For our fixed \(F\)-stable maximal torus \(T\), let \(X(T) = \text{Hom}(T, \overline{\mathbb{F}}_q^\times)\) and \(Y(T) = \text{Hom}(\overline{\mathbb{F}}_q^\times, T)\) be the groups of roots and coroots with pairing \(\langle \cdot, \cdot \rangle\) as in [2, Sec. 1.9], and make the identifications \(X(T^*) = Y(T), Y(T^*) = X(T)\) for the dual root system corresponding to \(G^*\) and \(F^*\) and torus \(T^*\) dual to \(T\). As in [2, Sec. 3.3], any maximal torus in \(G^F\) is of the form \(T^{wF}\) for some \(w \in W(T)\), where \(T^{wF} = (\alpha T \alpha^{-1})^F\) for some \(\alpha \in G\). Given such a maximal torus \(T^{wF}\), let \(d\) be an integer such that \(\alpha Ta^{-1}\) splits over \(\mathbb{F}_{q^d}\), and let \(\zeta \in \overline{\mathbb{F}}_q^\times\) be an element of order \(q^d - 1\). Then we also have the maximal torus \((T^*)^{w^{F*}}\) of \(G^{*F^*}\), where \(w^{F*} \in W(T^*)\) corresponding to \(w\) (see [1, p. 124]). For any \(s \in (T^*)^{w^{F*}}\), let \(s\) correspond to \(\lambda \in X(T) = Y(T^*)\) by a norm map as in the short exact sequence in [1, (8.11)] (with \(T\) replaced by \(T^*)\), and for any \(t \in T^{wF}\), let \(t\) correspond to \(\eta \in Y(T) = X(T^*)\) by a norm map in the same exact sequence. That is, let.
t = N_{F^d/wF}(\eta(\zeta))$, where $N_{F^d/wF}$ is defined on $T$ by

$$N_{F^d/wF}(x) = \prod_{i=0}^{d-1} (wF)^i x,$$

and is defined analogously on $Y(T)$. Given these $s \in (T^*)^{wF^*}$, $t \in T^{wF}$, $\lambda \in X(T)$, and $\eta \in Y(T)$, define

$$\hat{s}(t) = \kappa(\zeta^{(\lambda, N_{F^d/wF}(\eta))}),$$

which gives the more general version of (3.1), where $F^*$ and $F$ are replaced by $w^* F^*$ and $wF$, respectively.

By way of the isomorphism (3.1), there exists a homomorphism

$$Z(G^*)^{F^*} \to \text{Lin}(G^F), \quad z \mapsto \hat{z},$$

where $\text{Lin}(G^F)$ is the group of linear characters of $G^F$, and so $\hat{z}$ is a linear character of $G^F$. The homomorphism (3.3) is constructed by identifying the characters of the appropriate abelian quotient of $G^F$ with a subgroup of $\text{Irr}(T^F)$, and extending the linear character of the quotient to a character of $G^F$ (see [1, 8.19] and [4, Proposition 13.30]).

Returning to our specific case of the reductive group $GL(n, \bar{F}_q)$ where $F$ is the twisted Frobenius so that $GL(n, \bar{F}_q)^F = U(n, \bar{F}_q) = U_n$ is the finite unitary group, then we have $GL(n, \bar{F}_q)^* = GL(n, \bar{F}_q)$ and $F^* = F$. If $s$ is any semisimple element of $U_n$, then as in the previous section we have the centralizer $C_{U_n}(s)$ of $s$ in $U_n$ is isomorphic to a direct product of finite general linear and unitary groups, and in particular we have $C_{U_n}(s) = L = L^F$ is the set of $\bar{F}_q$-points of a rational Levi subgroup $L$ of $GL(n, \bar{F}_q)$. Then $L^* = L$ and $Z(L^*)^{F^*} = Z(L)^F = Z(L^F)$ (by [2, Proposition 3.6.8]), and so we have $Z(L^*)^{F^*} = Z(L)$. Since $L$ is itself a reductive group, then from the homomorphism in (3.3), we may associate the element $s \in Z(L)$ with a linear character $\hat{s}$ of $L$. Note that if $s \in U_n$ is a semisimple element with $L = C_{U_n}(s)$, then $s^{-1} \in L$ and the homomorphism in (3.3) yields

$$\hat{s}^{-1} = \overline{s^{-1}},$$

which we will use later.

### 3.4 Real semisimple classes.

An element of a group is called real if it is conjugate to its inverse in the group, and a real conjugacy class is one whose elements are real. Thus, a semisimple class $(s)$ of $U_n$ is a real semisimple class if $(s) = (s^{-1})$. Here, we give a description of the real semisimple classes of $U_n$.

Consider a semisimple element $s \in U_n$, and recall from Section 3.1 that the semisimple class $(s)$ of $U_n$ may be parameterized by a collection of pairs $(f, m_f)$, such that $f \in F$ is an $F$-irreducible polynomial in $F_q[x]$, and $m_f$ is a non-negative integer such that $\sum_{f \in F} m_f \deg(f) = n$, and in particular, the finite product $\prod_{f \in F} f(x)^{m_f}$ is the characteristic polynomial of the element $s \in U_n$. If we assume that $(s)$ is a real semisimple class, then for eigenvalue
\(\alpha \in \overline{F}_q\) of the element \(s\), we must have \(\alpha^{-1}\) is an eigenvalue of \(s\) with the same multiplicity, and conversely this condition guarantees a real semisimple class. In particular, if \(f \in \mathcal{F}\) is the \(F\)-irreducible polynomial corresponding to the \(F\)-orbit \([\alpha]\) of \(\alpha\), so that \(m_f > 0\), then the \(F\)-irreducible polynomial \(\overline{f}\) corresponding to the \(F\)-orbit \([\alpha^{-1}]\) of \(\alpha^{-1}\) must satisfy \(m_{\overline{f}} = m_f\). If we let \(d\) be the smallest non-negative integer such that \(\alpha^{qd} = \alpha\), note that we have

\[
[\alpha] \cup [\alpha^{-1}] = \{\alpha, \ldots, \alpha^{q^{d-1}}\} \cup \{\alpha^{-1}, \ldots, \alpha^{-q^{d-1}}\}.
\]

It follows that

\[
\prod_{\gamma \in [\alpha] \cup [\alpha^{-1}]} (x - \gamma)
\]

is a self-dual polynomial in \(\mathbb{F}_q[x]\) which is of the type appearing in the factorization of any self-dual polynomial in (2.2), so that \(\prod_{f \in \mathcal{F}} f_{\mathcal{F}_{m_f}}\) is a self-dual polynomial in \(\mathbb{F}_q[x]\) of degree \(n\). Conversely, the factorization of self-dual polynomials in (2.2) and the observation in (3.5) that any self-dual polynomial in \(\mathbb{F}_q[x]\) of degree \(n\) is of the form \(\prod_{f \in \mathcal{F}} f_{\mathcal{F}_{m_f}}\) for a real semisimple class of \(U_n\) parameterized by the pairs \(\{(f, m_f)\}\). It follows that self-dual polynomials in \(\mathbb{F}_q[x]\) of degree \(n\) are in one-to-one correspondence with real semisimple classes of \(U_n\).

4. The characters of finite unitary groups

Let \(G\) be a connected reductive group over \(\overline{F}_q\), defined over \(\mathbb{F}_q\) by a Frobenius map \(F\). The main tool in describing the set of irreducible characters of \(G^F\), \(\text{Irr}(G^F)\), is Deligne-Lusztig induction, as first given in [3], and presented in [1, 2, 4]. If \(L\) is an \(F\)-rational Levi subgroup of \(G\), and \(L = L^F\), \(G = G^F\), and \(\psi\) is a character of \(L\), then we let \(R_G^\psi(\hat{1})\) denote the Deligne-Lusztig induction of \(\psi\) to \(G\). If \(T = T^F\) where \(T\) is a maximal \(F\)-stable torus of \(G\), then a \textit{unipotent} character of \(G\) is an irreducible character of \(G\) which appears as a constituent of \(R_G^\psi(\hat{1})\), where \(\hat{1}\) is the trivial character of \(T\). We let \(\varepsilon_G\) denote \((-1)^{r(G)}\), where \(r(G)\) is the \(\mathbb{F}_q\)-rank of \(G\).

We also need some general facts about regular and semisimple characters. If \(G\) is any connected reductive group defined over \(\mathbb{F}_q\) by a Frobenius map \(F\), then by [4, 14.40, 14.47], the regular characters of \(G^F\) are indexed by the semisimple classes \((s)\) of the dual group \(G^{\ast_{\mathbb{R}}}\). In this correspondence, the class \((s)\) corresponds to the character \(\chi(s)\) given by

\[
\chi(s) = \frac{1}{|W_\mathbb{F}(s)|} \sum_{w \in W_\mathbb{F}(s)} \varepsilon_{G^F} \varepsilon_{T_w^F} R_{T_w^{G^F}}^{G^{\ast_{\mathbb{R}}}}(s),
\]

where \(s\) is some element of \((s)\), \(W_\mathbb{F}(s)\) is the Weyl group of the connected component of the centralizer \(C_G(s)\), \(T_w^{G^{\ast_{\mathbb{R}}}}\) is a torus of \(G^{\ast_{\mathbb{R}}}\) of type \(w\) (see [4, p. 113]) containing \(s\), and \(R_{T_w^{G^F}}^{G^{\ast_{\mathbb{R}}}}(s) = R_{T_w^F}(s)\) by [4, 13.30]. If we apply
the duality functor to (4.1), then by [2, Proposition 9.3.4], we find that the following is a semisimple character, up to a sign, indexed by \((s, \psi)\), where we take \(T_w\) to be the torus dual to \(T^*\):

\[
\frac{1}{|W^\circ(s)|} \sum_{w \in W^\circ(s)} R^G_{T^*_w}(\hat{s}).
\]

We now turn our attention to the finite unitary group \(G^F = U_n = U(n, \mathbb{F}_q^2)\), so that \(G^* = G\) and \(F^* = F\). The irreducible characters of \(U_n\) were first described in [9], and the complete theory is given concisely in [6, Section 1], which we follow here. We now take \(G = U_n\) throughout.

The set \(\text{Irr}(G)\) of irreducible characters of \(G\) is in one-to-one correspondence with \(G\)-conjugacy classes of pairs \((s, \psi)\), where \(s\) is a semisimple element of \(G\), and \(\psi\) is a unipotent character of \(L = C_G(s)\). Specifically, a \(G\)-conjugacy class of pairs \((s, \psi)\) corresponds to the irreducible character

\[
\varepsilon_{G \varepsilon_L} R^G_{L_1}(\hat{s}\psi),
\]

where \(\hat{s}\) is the linear character corresponding to \(s\) as given in Section 3.3.

In particular, we have

\[
\varepsilon_{G \varepsilon_L} R^G_{L_1}(\hat{s_1}\psi_1) = \varepsilon_{G \varepsilon_L} R^G_{L_2}(\hat{s_2}\psi_2)
\]

if and only if \(s_1\) and \(s_2\) are in the same conjugacy class of \(G\), and the character \(\psi_2\) of \(L_2\) is a \(G\)-conjugate of the character \(\psi_1\) of \(L_1\). This is precisely the Jordan decomposition of characters for the group \(G\) [1, Proposition 15.10].

In order to prove our main result, we need to understand the central characters of the irreducible characters of the finite unitary group. The following is [6, Lemma (2H)].

**Lemma 4.1.** Let \(\chi \in \text{Irr}(G)\) correspond to the \(G\)-conjugacy class of the pair \((s, \psi)\). Then for any elements \(z \in Z(G), g \in G\),

\[
\chi(zg) = \hat{s}(z)\chi(g).
\]

In particular, the central character \(\omega_\chi\) of \(\chi\) is given by \(\omega_\chi(z) = \hat{s}(z)\).

Consider the irreducible character of \(G\) corresponding to the pair \((s, 1)\), where \(1\) is the trivial character of \(L = C_G(s)\), and denote this by \(\Theta(s) = \varepsilon_{G \varepsilon_L} R^G_L(\hat{s})\). We now see that \(\Theta\) gives us a useful bijection between semisimple classes and semisimple characters of \(G\).

**Proposition 4.1.** For a semisimple class \((s)\) of \(G\), define \(\Theta\) by \(\Theta : (s) \mapsto \varepsilon_{G \varepsilon_L} R^G_L(\hat{s})\). Then \(\Theta\) is a bijection from semisimple classes of \(G\) to semisimple characters of \(G\).

**Proof.** First, by the Jordan decomposition of characters, we have \(R^G_L(\hat{s}_1) = R^G_L(\hat{s}_2)\) if and only if \((s_1) = (s_2)\), and so \(\Theta\) is well-defined and injective.

For any semisimple element \(s\) of \(G\), let \(L = G_G(s), W(s)\) be the Weyl group of the \(\mathbb{F}_q\)-points of \(L\), and let \(T_w\) be a torus of type \(w \in W(s)\) in \(L\).
By [6, Lemma (2B)], we have
\begin{equation}
\hat{s} = \frac{1}{|W(s)|} \sum_{w \in W(s)} R^G_{\hat{L}}(\hat{s}).
\end{equation}
By applying $R^G_L$ to (4.3) and by transitivity of Deligne-Lusztig induction, we obtain
\begin{equation}
R^G_L(\hat{s}) = \frac{1}{|W(s)|} \sum_{w \in W(s)} R^G_L(\hat{s}),
\end{equation}
which is (4.2), a semisimple character of $G$ up to a sign. Thus $\varepsilon_G \hat{s} R^G_L(\hat{s})$ is a semisimple character of $G$, and all semisimple characters of $G$ are of this form. So $\Theta$ is a bijection.  \hfill \Box

The map $\Theta$ behaves nicely on real semisimple class, as we see now.

**Proposition 4.2.** If $\Theta$ is the bijection from Proposition 4.1, then $\Theta(s)$ is a real-valued semisimple character of $U_n$ if and only if $(s)$ is a real semisimple class of $U_n$.

**Proof.** We will show that for any $g \in G = U_n$, $\Theta(s)(g^{-1}) = \Theta(s^{-1})(g)$, that is, $\Theta(s) = \Theta(s^{-1})$. This is enough, since this implies that $\Theta(s)$ is real-valued if and only if $R^G_L(\hat{s}) = R^G_L(\hat{s}^{-1})$, which occurs if and only if $(s) = (s^{-1})$ by the Jordan decomposition of characters, so if and only if $(s)$ is a real class of $G$.

We apply the character formula for Deligne-Lusztig induction as given in [4, Proposition 12.2]. Let $g \in G$ with Jordan decomposition $g = tu$, where $t$ is semisimple and $u$ unipotent in $G$. Then we have
\begin{equation}
R^G_L(\hat{s})(g) = \frac{1}{|L||C_G(t)|} \sum_{h \in G|t \in hL} |C_hL(t)| \sum_{v \in C_hL(t)_u} Q^{C_G(t)}_{C_hL(t)}(u, v^{-1}) h \hat{s}(tv),
\end{equation}
where $hL = hLh^{-1}$, $C_hL(t)_u$ is the unipotent radical of $C_hL(t)$, $h \hat{s}(tv) = \hat{s}(h^{-1}tvh)$, and $Q^{C_G(t)}_{C_hL(t)}$ is the Green function defined on $C_G(t)_u \times C_hL(t)_u$ (see [4, Definition 12.1]).

Now we apply (4.4) to compute $R^G_L(\hat{s}^{-1})(g)$. From (3.4), we have $\hat{s}^{-1} = \hat{s}^{-1}$, and so the only change from (4.4) will be that $h \hat{s}(tv)$ is replaced by $h \hat{s}^{-1}(tv) = h \hat{s}(t^{-1}v^{-1}) = h \hat{s}(t^{-1}v) = h \hat{s}(t^{-1}v)$, since $\hat{s}$ acts trivially on unipotent elements. That is, we have
\begin{equation}
R^G_L(\hat{s}^{-1})(g) = \frac{1}{|L||C_G(t)|} \sum_{h \in G|t \in hL} |C_hL(t)| \sum_{v \in C_hL(t)_u} Q^{C_G(t)}_{C_hL(t)}(u, v^{-1}) h \hat{s}(t^{-1}v).
\end{equation}
On the other hand, if we apply (4.4) to compute $R^G_L(\hat{s})(g^{-1})$, where $g^{-1} = t^{-1}u^{-1}$, then we replace $\hat{s}(tv)$ with $\hat{s}(t^{-1}v)$, and $u$ is replaced by $u^{-1}$ in the
evaluation of the Green function, while each centralizer remains the same since $C_G(t) = C_G(t^{-1})$ and $C_{hL}(t) = C_{hL}(t^{-1})$, and the index in the first sum remains since $t \in hL$ if and only if $t^{-1} \in hL$. That is, we have

\begin{equation}
R_{L}^G(\hat{s})(g^{-1}) = \frac{1}{|L||C_G(t)|} \sum_{t \in C_G(t) \in hL} |C_{hL}(t)| \sum_{v \in C_{hL}(t),u} Q_{C_{hL}(t)}^C(u^{-1}, v^{-1}) h(\hat{s}(t^{-1})v).
\end{equation}

The only difference between (4.6) and (4.5) is in the evaluation of the Green function. The value of the Green function $Q_{C_{hL}(t)}^C(u, v^{-1})$ is defined as $1/|C_{hL}(t)|$ times the trace of the action of $(u, v^{-1}) \in C_G(t) \times C_{hL}(t)$ on the cohomology of the corresponding Deligne-Lusztig variety. In particular, whenever $u_1, u_2 \in C_G(t)$ are conjugate in $C_G(t)$, then the invariance of the trace under conjugation gives $Q_{C_{hL}(t)}^C(u_1, v^{-1}) = Q_{C_{hL}(t)}^C(u_2, v^{-1})$. Since $u$ is a unipotent element in $C_G(t)$, which is a product of unitary and general linear groups, then $u$ is conjugate to $u^{-1}$ in $C_G(t)$. From this along with (4.5) and (4.6) it follows that $R_{L}^G(\hat{s}^{-1})(g) = R_{L}^G(\hat{s})(g^{-1})$, and thus $\Theta(s^{-1})(g) = \Theta(s)(g^{-1})$ as claimed. 

5. Proof of the main theorem

In order to apply the results from Section 4, we need the following result on the behavior of linear characters of the form $\hat{s}$ for semisimple $s \in U_n$.

**Lemma 5.1.** Let $s, t \in U_n$ be semisimple elements such that $st = ts$, and let $\hat{s}$ and $\hat{t}$ both be linear characters of $C_{U_n}(s)$, as described in Section 3.3. Then $\hat{s}(t) = \hat{t}(s)$.

**Proof.** Recall the construction of the characters $\hat{s}$ and $\hat{t}$ given in Section 3.3. We have $G = \text{GL}(n, \overline{F}_q)$, $U_n = G^F$, and we may identify $G^* = G$ and $F^* = F$. Fix the maximal torus $T = T^*$ to be the diagonal torus, where $T$ defines the roots $X(T)$ and coroots $Y(T)$ of $G$. Let $w \in W(T) \cong S_n$ be such that $T^w = Z(C_{U_n}(s))$ is a maximal torus of $U_n$ with $s, t \in T^w$. As in Section 3.2, $C_{U_n}(s)$ is a product of general linear and unitary groups defined over extensions of $F_q$, with $T^w$ the product of cyclic tori, and the element $w$ depends on the structure of $C_{U_n}(s)$. We need only to consider the more convenient case that $T^w = T_n = (F_q^\times)^{n}$, which corresponds to $w$ being the $n$-cycle $(1 2 \cdots n)$ in $S_n$, as the more general case of $w$ is proved in a completely parallel fashion.

Fix dual bases $\{\mu_1, \ldots, \mu_n\}$ and $\{\nu_1, \ldots, \nu_n\}$ of $Y(T)$ and $X(T)$, respectively, so that $\langle \mu_i, \nu_j \rangle = \delta_{ij}$, where we may take $\mu_1(a)$ to be the diagonal matrix with $a \in \overline{F}_q^\times$ in the $(i, i)$ position, with $1$'s in other diagonal positions, and $\nu_i$ to be the map which sends an element of $T$ to the entry in the $(i, i)$ position. Write the elements of these groups additively, and in our
In particular, the terms in this expression are of the form $\langle \lambda, N_{F^n/wF}(\eta) \rangle$ with $\sum_{i=1}^{n} m_i \nu_i$. Now, the values of $\hat{s}(t)$ and $\hat{t}(s)$ as characters of $CU_n(s)$ are the same as if the are considered characters of $T^{wF}$ by construction. Using the notation of Section 3.3, we have $\hat{s}(t) = \kappa(\zeta^{\lambda,N_{F^n/wF}(\eta)})$ and $\hat{t}(s) = \kappa(\zeta^{\eta,N_{F^n/wF}(\lambda)})$, where $\zeta \in \mathbb{F}_q^\times$ has order $q^n-1$. As elements of $Y(T)$ and $X(T)$, respectively, suppose that $\lambda = \sum_{i=1}^{n} m_i \mu_i$ and $\eta = \sum_{i=1}^{n} k_i \nu_i$. Now, $N_{F^n/wF}(\eta) = \sum_{i=1}^{n} (wF)^{i-1} \eta$ and $(wF)\nu_i = (-q)\nu_{i+1(\text{mod } n)}$, and so we may calculate that

$$\langle \lambda, N_{F^n/wF}(\eta) \rangle = \sum_{i=1}^{n} (k_1(-q)^{i-1} + k_2(-q)^i + \cdots + k_n(-q)^{i-1+n-1})m_i.$$ 

In particular, the terms in this expression are of the form $m_i k_j (-q)^{i+j-2}$, symmetric in $i$ and $j$. Since the value of $\langle \eta, N_{F^n/wF}(\lambda) \rangle$ is computed by exchanging the $m_i$’s and the $k_j$’s, it follows that we have $\langle \lambda, N_{F^n/wF}(\eta) \rangle = \langle \eta, N_{F^n/wF}(\lambda) \rangle$, and hence $\hat{s}(t) = \hat{t}(s)$, as desired. \hfill $\square$

We now apply Lemma 5.1 to evaluate $\hat{s}(z)$, where $z$ is a generator for the center of $U_n$.

**Lemma 5.2.** Let $(s)$ be a real semisimple class of $U_n$, and let $z$ be a generator for $Z(U_n)$. Suppose that the semisimple class $(s)$ has $-1$ as an eigenvalue with multiplicity $k$. Then $\hat{s}(z) = (-1)^k$.

**Proof.** By Lemma 5.1, we have $\hat{s}(z) = \hat{z}(s)$, where we may consider $\hat{z}$ to be a linear character of $U_n$. There are exactly $|Z(U_n)| = q+1$ linear characters of $U_n$, which are given by powers of the determinant, composed with a fixed injective homomorphism from $\mathbb{F}_q^\times$ to $\mathbb{C}^\times$. It follows from [1, End of Sec. 8.3] that in the case $G^F = U_n$, the map (3.3) from $Z(G)^F$ to the group of linear characters of $G^F$ is an isomorphism, which implies that $\hat{z}$ is the determinant map composed with the fixed injective homomorphism from $\mathbb{F}_q^\times$ to $\mathbb{C}^\times$.

Since $(s)$ is a real semisimple class of $U_n$, then from the description in Section 3.4, each eigenvalue of $s$ in $\mathbb{F}_q^\times$ occurs with the same multiplicity as its inverse, and so $\det(s) = (-1)^k$, where $k$ is the multiplicity of $-1$ as an eigenvalue. Therefore $\hat{s}(z) = \hat{z}(s) = (-1)^k$. \hfill $\square$

Finally, we arrive at the main result.

**Theorem 5.1.** Let $n = 2m$, and let $q$ be odd. The number of semisimple symplectic characters of $U(n, \mathbb{F}_q)$ is $q^{m-1}$.

**Proof.** By Theorem 2.1, if $\chi = \Theta(s)$ is a real-valued semisimple character of $G = U_{2m}$, where $q$ is odd, then $\varepsilon(\chi) = \omega_\chi(z)$, where $z$ is a generator of $Z(G)$. By Lemma 4.1, $\varepsilon(\Theta(s)) = \hat{s}(z)$, and by Proposition 4.2, $(s)$ is a real semisimple class of $G$. By Lemma 5.2, $\hat{s}(z) = (-1)^k$, where $k$ is the multiplicity of $-1$ as an eigenvalue of any element of $(s)$. By Section 3.4, the characteristic polynomial corresponding to $(s)$ is a self-dual polynomial in $\mathbb{F}_q[x]$ of degree $2m$. By the factorization of self-dual polynomials in (2.2),
the constant term of a self-dual polynomial is the multiplicity of the factor $x - 1$. In the case that the degree is even, then since the factors other than $x - 1$ and $x + 1$ in (2.2) have total even degree, then the multiplicity of the factor $x - 1$ has the same parity as the multiplicity of $x + 1$. In particular, the constant term of the self-dual polynomial is $(-1)^k$, where $k$ is the multiplicity of the factor $x + 1$. So, the semisimple characters with Frobenius-Schur indicator $-1$ are exactly those semisimple characters $\Theta(s)$ such that the characteristic polynomial corresponding to $(s)$ has constant term $-1$. That is, to count semisimple symplectic characters of $G$, we must count self-dual polynomials in $\mathbb{F}_q[x]$ of degree $2m$ with constant term $-1$.

Suppose $g(x) \in \mathbb{F}_q[x]$, $g(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, is a self-dual polynomial of degree $n = 2m$, and we assume also that $a_0 = -1$. By (2.1), we also have

$$(5.1) \quad g(x) = -x^n g(x^{-1}) = x^n - a_1x^{n-1} - \cdots - a_{n-1}x - 1.$$ 

Since $n = 2m$ is even, (5.1) implies that we must have $a_m = -a_m$, and so $a_m = 0$. For $1 \leq i \leq m - 1$, we may let $a_i$ be any of $q$ elements of $\mathbb{F}_q$, and then each $a_{n-i} = -a_i$ is determined. This gives a total of $q^{m-1}$ polynomials. \qed

As proven in [8, Theorem 4.4], there are exactly $q^m + q^{m-1}$ real-valued semisimple characters of $U(2m, \mathbb{F}_{q^2})$ when $q$ is odd. It follows from Theorem 5.1 that there are exactly $q^m$ semisimple orthogonal characters of $U(2m, \mathbb{F}_{q^2})$ when $q$ is odd. As mentioned at the end of Section 2.3, it follows from [13, Corollary 3.1] that there are exactly $q^{m-1}$ symplectic regular characters and $q^m$ orthogonal regular characters of $U(2m, \mathbb{F}_{q^2})$ when $q$ is odd.

REFERENCES


DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE (MC 249), UNIVERSITY OF ILLINOIS AT CHICAGO, 851 SOUTH MORGAN STREET, CHICAGO, IL 60680-7045
E-mail address: srinivas@uic.edu

DEPARTMENT OF MATHEMATICS, COLLEGE OF WILLIAM AND MARY, P. O. BOX 8795, WILLIAMSBURG, VA 23187-8795
E-mail address: vinroot@math.wm.edu