TOPOLOGICAL GROUPS

MATH 519

The purpose of these notes is to give a mostly self-contained topological background for the study of the representations of locally compact totally disconnected groups, as in [BZ] or [B, Chapter 4]. These notes have been adapted mostly from the material in the classical text [MZ, Chapters 1 and 2], and from [RV, Chapter 1]. An excellent resource for basic point-set topology is [M].

1. Basic examples and properties

A *topological group* G is a group which is also a topological space such that the multiplication map $(g, h) \mapsto gh$ from $G \times G$ to G, and the inverse map $g \mapsto g^{-1}$ from G to G, are both continuous. Similarly, we can define *topological rings* and *topological fields*.

Example 1. Any group given the discrete topology, or the indiscrete topology, is a topological group.

Example 2. $\mathbb{R}$ under addition, and $\mathbb{R}^\times$ or $\mathbb{C}^\times$ under multiplication are topological groups. $\mathbb{R}$ and $\mathbb{C}$ are topological fields.

Example 3. Let R be a topological ring. Then $\mathrm{GL}(n, R)$ is a topological group, and $M_n(R)$ is a topological ring, both given the subspace topology in R^{n^2}.

If G is a topological group, and $t \in G$, then the maps $g \mapsto tg$ and $g \mapsto gt$ are homeomorphisms, and the inverse map is a homeomorphism. Thus, if $U \subset G$, we have

$$U \text{ is open} \iff tU \text{ is open} \iff Ut \text{ is open} \iff U^{-1} \text{ is open.}$$

A topological space X is called *homogeneous* if given any two points $x, y \in X$, there is a homeomorphism $f : X \to X$ such that $f(x) = y$. A homogeneous space thus looks topologically the same near every point. Any topological group G is homogeneous, since given $x, y \in G$, the map $t \mapsto yx^{-1}t$ is a homeomorphism from G to G which maps x to y.

If X is a topological space, $x \in X$, a *neighborhood* of x is a subset U of X such that x is contained in the interior of U. That is, U is not necessarily open, but there is an open set $W \subset X$ containing x such that $W \subset U$.

If G is a group, and S and T are subsets of G, we let ST and S^{-1} denote

$$ST = \{st \mid s \in S, t \in T\} \quad \text{and} \quad S^{-1} = \{s^{-1} \mid s \in S\}.$$

The subset S is called *symmetric* if $S^{-1} = S$. We will let 1 denote the identity element of a group unless otherwise stated. The following result, although innocent enough looking, will be the most often used in all of the results which follow.

Proposition 1.1. *Let G be a topological group. Every neighborhood U of 1 contains an open symmetric neighborhood V of 1 such that $VV \subset U$.*

Proof. Let U' be the interior of U. Consider the multiplication map $\mu : U' \times U' \to G$. Since μ is continuous, then $\mu^{-1}(U')$ is open and contains $(1,1)$. So, there are open sets $V_1, V_2 \subset U$ such that $(1,1) \in V_1 \times V_2$, and $V_1V_2 \subset U$. If we let $V_3 = V_1 \cap V_2$, then $V_3V_3 \subset U$ and V_3 is an open neighborhood of 1. Finally, let $V = V_3 \cap V_3^{-1}$, which is open, contains 1, is symmetric, and satisfies $VV \subset U$. □

Proposition 1.2. *If G is a topological group, then every open subgroup of G is also closed.*

Proof. Let H be an open subgroup of G. Then any coset xH is also open. So,

$$Y = \bigcup_{x \in G \setminus H} xH$$

is also open. From elementary group theory, $H = G \setminus Y$, and so H is closed. □

Proposition 1.3. *If G is a topological group, and if K_1 and K_2 are compact subsets of G, then K_1K_2 is compact.*

Proof. The set $K_1 \times K_2$ is compact in $G \times G$, and multiplication is continuous. Since the continuous image of a compact set is compact, K_1K_2 is compact. □

If X is a topological space, and A is a subset of X, recall that the *closure* of A, denoted $\overline{A}$, is the intersection of all closed subsets containing A. A necessary and sufficient condition for x to be an element of $\overline{A}$ is for every open neighborhood U of x, $U \cap A$ is nonempty, which may be seen as follows. If $x \notin \overline{A}$, then there is a closed set F which contains A, but $x \notin F$. Then $U = X \setminus F$ is an open neighborhood of x such that $U \cap A = \varnothing$. Conversely, if U is an open neighborhood of x such that $U \cap A = \varnothing$, then $X \setminus U$ is a closed set containing A which does not contain x, so $x \notin \overline{A}$.

Proposition 1.4. *If G is a topological group, and H is a subgroup of G, then the topological closure of H, $\overline{H}$, is a subgroup of G.*

Proof. Let $g, h \in \overline{H}$. Let U be an open neighborhood of the product gh. Let $\mu : G \times G \to G$ denote the multiplication map, which is continuous, so $\mu^{-1}(U)$ is open in $G \times G$, and contains (g,h). So, there are open neighborhoods V_1 of g and V_2 of h such that $V_1 \times V_2 \subset \mu^{-1}(U)$. Since $g, h \in \overline{H}$, then there are points $x \in V_1 \cap H \neq \varnothing$ and $y \in V_2 \cap H \neq \varnothing$. Since $x, y \in H$, we have $xy \in H$, and since $(x,y) \in \mu^{-1}(U)$, then $xy \in U$. Thus, $xy \in U \cap H \neq \varnothing$, and since U was an arbitrary open neighborhood of gh, then we have $gh \in \overline{H}$. Now let $\iota : G \to G$ denote the inverse map, and let W be an open neighborhood of h^{-1}. Then $\iota^{-1}(W) = W^{-1}$ is open and contains h, so there is a point $z \in H \cap W^{-1} \neq \varnothing$. Then we have $z^{-1} \in H \cap W \neq \varnothing$, and as before this implies $h^{-1} \in \overline{H}$. □

Remark. Note that in the last part of the proof of Proposition 1.4, we have shown that the closure of a symmetric neighborhood of 1 is again symmetric.

Lemma 1.1. *Let G be a topological group, F a closed subset of G, and K a compact subset of G, such that $F \cap K = \varnothing$. Then there is an open neighborhood V of 1 such that $F \cap VK = \varnothing$ (and an open neighborhood V' of 1 such that $F \cap KV' = \varnothing$).*

Proof. Let $x \in K$, so $x \in G \setminus F$, and $G \setminus F$ is open. So, $(G \setminus F)x^{-1}$ is an open neighborhood of 1. By Proposition 1.1, there is an open neighborhood W_x of 1 such that $W_x W_x \subset (G \setminus F)x^{-1}$. Now, $K \subset \cup_{x \in K} W_x x$, and K is compact, so there exists a finite number of points $x_1, \ldots, x_n \in K$, such that $K \subset \cup_{i=1}^n W_i x_i$, where we write $W_i = W_{x_i}$. Now let

$$V = \bigcap_{i=1}^{n} W_i.$$

For any $x \in K$, $x \in W_i x_i$ for some i. Now we have

$$Vx \subset W_i x \subset W_i W_i x_i \subset G \setminus F.$$

In other words, $F \cap Vx = \varnothing$. Since this is true for any $x \in K$, we now have $F \cap VK = \varnothing$. □

Remark. Note that from Proposition 1.1, the neighborhood V in Lemma 1.1 may be taken to be symmetric.

Proposition 1.5. *Let G be a topological group, K a compact subset of G, and F a closed subset of G. Then FK and KF are closed subsets of G.*

Proof. If $FK = G$, we are done, so let $y \in G \setminus FK$. This means $F \cap yK^{-1} = \varnothing$. Since K is compact, yK^{-1} is compact. By Lemma 1.1, there is an open neighborhood V of 1 such that $F \cap VyK^{-1} = \varnothing$, or $FK \cap Vy = \varnothing$. Since Vy is an open neighborhood of y contained in $G \setminus FK$, we have FK is closed. □

2. Separation properties and functions

A topological space X is said to be T_1 if for any two distinct points $x, y \in X$, there is an open set U in X such that $x \in U$, but $y \notin U$. This is equivalent to one-point sets being closed. If G is a topological group, then G being T_1 is equivalent to $\{1\}$ being a closed set in G, by homogeneity.

A topological space X is said to be *Hausdorff* (or T_2) if given any two distinct points $x, y \in X$, there are open sets $U, V \subset X$, $x \in U$, $y \in V$, such that $U \cap V = \varnothing$. Recall the following basic properties of Hausdorff spaces.

Exercise 1. If X is a Hausdorff space, then every compact subset of X is closed.

Exercise 2. Let X be a topological space, and let $\Delta = \{(x, x) \,|\, x \in X\} \subset X \times X$ be the *diagonal* in $X \times X$. Then X is Hausdorff if and only if Δ is closed in $X \times X$.

Of course, if X is T_2, then X is T_1, but the converse does not hold in general. If G is a topological group however, the converse is true, which we now show.

Proposition 2.1. *Let G be a T_1 topological group. Then G is Hausdorff.*

Proof. Given distinct $g, h \in G$, take an open set U containing 1, such that $gh^{-1} \notin U$, which we may do since G is T_1. Applying Proposition 1.1, let V be an open symmetric neighborhood containing 1, such that $VV \subset U$. Now, Vg is open and contains g, and Vh is open and contains h. We must have $Vg \cap Vh = \varnothing$, otherwise there are $v_1, v_2 \in V$ such that $v_1 g = v_2 h$, which would mean

$$gh^{-1} = v_2 v_1^{-1} \in VV^{-1} = VV \subset U,$$

while gh^{-1} was chosen to be not an element of U. Thus G is Hausdorff. □

We can say even more than Proposition 2.1. A topological space X is called *regular* or T_3 if X is T_1, and for any point $x \in X$ and any closed subset $F \subset X$ such that $x \notin F$, there is an open set U containing x and an open set V containing F such that $U \cap V = \varnothing$. The space X is called *completely regular* or *Tychonoff* or $T_{3\frac{1}{2}}$ if it is T_1 and for any point $x \in X$ and any closed set $F \subset X$ such that $x \notin F$, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for every $y \in F$. Every space which is completely regular is also regular, since, for example, $f^{-1}([0,1/3))$ and $f^{-1}((2/3,1])$ are disjoint open sets in X containing x and F, respectively. We now see that any topological group which is T_1 is also completely regular, and thus regular.

Theorem 2.1. *Let G be a topological group, let 1_G denote the identity element in G, and let F be a closed subset of G such that $1_G \notin F$. Then there is a continuous function $f : G \to [0,1]$ such that $f(1_G) = 0$ and $f(y) = 1$ for every $y \in F$.*

Proof. See Problem Set 1. □

Corollary 2.1. *If G is a topological group which is T_1, then G is completely regular and thus regular.*

Proof. Let $x \in G$ and let F be a closed subset of G such that $x \notin F$. Then $x^{-1}F$ is a closed subset of G not containing 1_G, and from Theorem 2.1, there is a continuous function $f : G \to [0,1]$ such that $f(1_G) = 0$ and $f(y) = 1$ for $y \in x^{-1}F$. Now the function $h(g) = f(x^{-1}g)$ is the desired continuous function, and since G is also T_1, G is completely regular, and so is also regular. □

Let f be an $\mathbb{R}$-valued continuous function on a topological group G (we could also consider $\mathbb{C}$-valued functions). The *left* and *right translates* of f, written $L_h f$ and $R_h f$, respectively, are given by

$$L_h f(g) = f(h^{-1}g) \quad \text{and} \quad R_h f(g) = f(gh).$$

The function f is *left uniformly continuous* if for every $\varepsilon > 0$, there is a neighborhood V of 1 such that

$$h \in V \Longrightarrow ||L_h f - f||_\infty < \varepsilon,$$

where $||f||_\infty$ denotes the supremum norm. We may define a function to be *right uniformly continuous* similarly.

The *support* of a function f on a topological group G, written $\mathrm{supp}(f)$, is defined to be the topological closure of the set of points in G for which f is nonzero. That is,

$$\mathrm{supp}(f) = \overline{\{g \in G \mid f(g) \neq 0\}}.$$

We let $C_c(G)$ denote the set of continuous $\mathbb{R}$-valued functions on G with compact support. That is,

$$C_c(G) = \{f : G \to \mathbb{R} \mid f \text{ is continuous, } \mathrm{supp}(f) \text{ is compact}\}.$$

Proposition 2.2. *Let G be a topological group, and let $f \in C_c(G)$. Then f is left and right uniformly continuous.*

Proof. We will prove that f is right uniformly continuous, as the proof for left uniformly continuous is exactly analogous. Let $K = \mathrm{supp}(f)$, and let $\varepsilon > 0$. Let $g \in G$, and let $B_{\varepsilon/3}(f(g))$ be the open ball of radius $\varepsilon/3$ in $\mathbb{R}$ centered at $f(g)$. Then $f^{-1}(B_{\varepsilon/3}(f(g)))$ is

an open neighborhood of g, call it W_g. Let $U_g = g^{-1}W_g$, which is an open neighborhood of 1, and if $h \in U_g$, then $gh \in W_g$. So, we have

$$h \in U_g \Longrightarrow |f(gh) - f(g)| < \varepsilon/3.$$

In other words, $f(g')$ is within $\varepsilon/3$ of $f(g)$ whenever $g^{-1}g' \in U_g$, or

$$g^{-1}g' \in U_g \Longrightarrow |f(g') - f(g)| < \varepsilon/3. \tag{2.1}$$

Applying Proposition 1.1, let V_g be an open symmetric neighborhood of 1 such that $V_gV_g \subset U_g$. K is compact, and

$$K \subset \bigcup_{g \in K} gV_g,$$

so we may take a finite number of $g \in K$, say $g_1, \ldots, g_n$, such that

$$K \subset \bigcup_{j=1}^{n} g_jV_{g_j}.$$

Let us write $V_j = V_{g_j}$ and $U_j = U_{g_j}$. Now let $V = \cap_{j=1}^{n} V_j$, which is an open symmetric neighborhood of 1. This will be the neighborhood which will give right uniform continuity.

Let $g \in K$, so that $g \in g_jV_j$ for some j, and let $h \in V$. Since $V_j \subset U_j$, we have $g_j^{-1}g \in U_j$. Since $h \in V_j$ and $V_jV_j \subset U_j$, we also have $g_j^{-1}gh \in U_j$. From (2.1) and our choice of U_j, we have, for any $h \in V$,

$$|f(g_j) - f(g)| < \varepsilon/3 \quad \text{and} \quad |f(gh) - f(g_j)| < \varepsilon/3.$$

The triangle inequality now gives

$$|f(gh) - f(g)| \leq |f(gh) - f(g_j)| + |f(g_j) - f(g)| < 2\varepsilon/3,$$

for any $h \in V$.

Now suppose $g \notin K$, $h \in V$, and that $f(gh) \neq 0$ (otherwise $|f(gh) - f(g)| = 0$). For some j, we have $gh \in g_jV_j$, so $g_j^{-1}(gh) \in V_j \subset U_j$, and by continuity $f(gh)$ is within $\varepsilon/3$ of $f(g_j)$. Now, $h^{-1} \in V_j^{-1} = V_j$, since V_j is symmetric, and so we have

$$g_j^{-1}g = g_j^{-1}ghh^{-1} \in V_jV_j \subset U_j.$$

By (2.1), $f(g_j)$ is within $\varepsilon/3$ of $f(g) = 0$. Finally, we have

$$|f(gh)| \leq |f(gh) - f(g_j)| + |f(g_j)| < 2\varepsilon/3.$$

Now, for any $g \in G$, $h \in V$, we have $|f(gh) - f(g)| < 2\varepsilon/3$. So,

$$h \in V \Longrightarrow ||R_hf - f||_\infty \leq 2\varepsilon/3 < \varepsilon,$$

and f is right uniformly continuous. □

3. Quotients

If X is a topological space, and $\sim$ is an equivalence relation on X, let $X/\sim$ denote the set of equivalence classes in X under $\sim$, and if $x \in X$, let $[x]$ denote the equivalence class of x under $\sim$. We may give the set $X/\sim$ the *quotient topology* as follows. Let

$$p : X \to X/\sim, \quad p(x) = [x],$$

be the natural projection map. Define $U \subset X/\sim$ to be open if and only if $p^{-1}(U)$ is open in X (forcing p to be continuous). Note that this implies that $F \subset X/\sim$ is closed if and only if $p^{-1}(F)$ is closed in X.

Now let G be a topological group, and H a subgroup of G. We can look at the collection G/H of left cosets of H in G (or the collection $H\backslash G$ of right cosets), which defines an equivalence relation on G. So, we can put the quotient topology on G/H as above. Recall that G/H is not a group under coset multiplication unless H is a normal subgroup of G.

Proposition 3.1. *Let G be a topological group, and H a subgroup of G.*

(1) *G/H is a homogeneous space under translation by G.*
(2) *The map $p : G \to G/H$ is an open map.*
(3) *If H is compact, then $p : G \to G/H$ is a closed map.*
(4) *G/H is a Hausdorff space if and only if H is closed.*
(5) *H is open in G if and only if G/H is a discrete space. If G is compact, then H is open in G if and only if G/H is a finite and discrete space.*
(6) *If H is a normal subgroup of G, then G/H is a topological group.*
(7) *If H is the closure of the trivial subgroup, $H = \overline{\{1\}}$, then H is a normal subgroup of G and G/H is Hausdorff.*

Proof. (1): For $x \in G$, left translation by x on G/H gives a map $gH \mapsto xgH$. The inverse of this map is also a left translation, by x^{-1}, so to show this is a homeomorphism, we just need to show that it maps open sets to open sets, or is an open map. Let $U \subset G/H$ be open. By definition of the quotient topology, $p^{-1}(U) \subset G$ is open. It may be directly checked that we have $p^{-1}(xU) = xp^{-1}(U)$, which is also open. Since $p^{-1}(xU)$ is open, then xU is open by the definition of quotient topology, and so translation is an open map.

(2): Let $V \subset G$ be open. By the definition of quotient topology, $p(V) \subset G/H$ is open if and only if $p^{-1}(p(V)) \subset G$ is open. It may be checked that $p^{-1}(p(V)) = VH$. Since V is open, Vh is open for every $h \in H$. Since $VH = \cup_{h\in H} Vh$, VH is open, and so $p(V)$ is open.

(3): As in the proof of (2), we are reduced to showing that if $F \subset G$ is closed, then FH is closed. But H is compact, and so by Proposition 1.5, FH is closed.

(4): Suppose G/H is Hausdorff, so that it is T_1, and one point sets in G/H are closed. In particular, $\{H\}$ is closed in G/H. By the definition of the quotient topology, $\{H\} \subset G/H$ is closed if and only if $p^{-1}(\{H\}) = H \subset G$ is closed.

From Exercise 2, to show that G/H is Hausdorff, it is enough to show that the diagonal $\Delta = \{(gH, gH) \mid gH \in G/H\}$ is closed in $G/H \times G/H$. Through the natural map $f : (g_1H, g_2H) \mapsto (g_1, g_2)(H\times H)$, the space $G/H\times G/H$ is homeomorphic to $G\times G/H\times H$, and the image of the diagonal Δ under this map is $f(\Delta) = \{(g, g)(H \times H) \mid g \in G\}$. From the definition of the quotient topology, $f(\Delta)$ is closed if and only if

$$p^{-1}(f(\Delta)) = \{(g_1, g_2) \in G \times G \mid g_1 g_2^{-1} \in H\}$$

is closed. But this is the inverse image of H of the continuous map from $G \times G$ to G which maps (g_1, g_2) to $g_1 g_2^{-1}$. Since H is closed, then this set is closed as well.

(5): See Problem Set 1.

(6): Let T_g denote left multiplication by g, so that $T_g(x) = gx$, let ι and ι' denote the group inverse maps in G and G/H, respectively, and let $p : G \to G/H$ be the natural

projection map. Since for any $x \in G$ we have

$$(p \circ T_g)(x) = gxH = (gH)(xH) = (T_{p(g)} \circ p)(x) \quad \text{and} \quad (p \circ \iota)(x) = x^{-1}H = (\iota' \circ p)(x),$$

the following diagrams are commutative:

$$\begin{array}{ccc} G & \xrightarrow{T_g} & G \\ \downarrow p & & \downarrow p \\ G/H & \xrightarrow{T_{p(g)}} & G/H \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\iota} & G \\ \downarrow p & & \downarrow p \\ G/H & \xrightarrow{\iota'} & G/H \end{array}$$

Since p is an open map by (2), and T_g and ι are continuous, this means $T_{p(g)}$ and ι' must also be continuous, making G/H a topological group.

(7): By Proposition 1.4, $H = \overline{\{1\}}$ is a subgroup of G. H is then the minimal closed subgroup of G containing 1, while for any $x \in G$, xHx^{-1} is also a closed subgroup of G containing 1. Thus $H \subset xHx^{-1}$, and so $x^{-1}Hx \subset H$ for any $x \in G$, and H is a normal subgroup of G. Now, G/H is a topological group by (6), and G/H is Hausdorff by (4). □

4. Local compactness and connectedness

A topological space X is called *locally compact* if for every $x \in X$, there is a compact neighborhood $U \subset X$ of x. Before proving a few basic properties of locally compact spaces, recall the following, which we will need.

Exercise 3. If X is a compact space, then every closed subset of X is compact.

Lemma 4.1. *Let X be a locally compact Hausdorff space. Then X is regular, and any neighborhood V of any point $x \in X$ contains a compact neighborhood K of x.*

Proof. Let $x \in X$, and let F be a closed subset in X such that $x \notin F$. Let K be a compact neighborhood of x. Since K is closed (by Exercise 1), then $M = K \cap F$ is closed, and so $M \subset K$ is compact (by Exercise 3). For every $y \in M$, choose an open neighborhood U_y of y, and an open neighborhood U_x^y of x such that $U_y \cap U_x^y = \varnothing$, which we can do since X is Hausdorff. Since M is compact, there are a finite number of points $y_1, \ldots, y_n \in M$ such that $M \subset \cup_{i=1}^n U_i$, where $U_i = U_{y_i}$. Writing $U_x^{y_i} = U_x^i$, we have

$$x \in W = \bigcap_{i=1}^n U_x^i, \quad \text{and} \quad W \cap \bigcup_{i=1}^n U_i = \varnothing,$$

where W is an open neighborhood of x. Now, if $K' = K \setminus (\cup_{i=1}^n U_i)$, then K' is a closed subset of K, and so is compact, and it is a neighborhood of x since $W \subset K'$. In particular, K' is disjoint from F. Now let $V = \text{int}(K')$ be the interior of K', which is an open neighborhood of x. Then $V' = X \setminus K'$ is an open set containing F, and $V \cap V' = \varnothing$. Thus X is regular.

For the second statement, let V be any neighborhood of $x \in X$, and let $V' = \text{int}(V)$ be the interior of V. Let C be a compact neighborhood of x, and let $U' = \text{int}(C)$. Then $U = U' \cap V$ is an open neighborhood of x. Now $X \setminus U$ is closed, and since we have shown that X is regular, then there are open sets V_1, V_2 such that $X \setminus U \subset V_1$, $x \in V_2$, and $V_1 \cap V_2 = \varnothing$. Now let $K = \overline{V_2}$. Since $V_2 \subset X \setminus V_1$, which is closed, then $K \subset X \setminus V_1$.

Then $K \subset U$ since $X \setminus U \subset V_1$, and so $K \subset C$. Since C is compact and K is closed, K is also compact. Now K is a compact neighborhood of x and $K \subset U \subset V$. □

We will also need the following statement.

Lemma 4.2. *Let X be a Hausdorff space, and A a locally compact subspace of X. If A is dense in X, that is if $\overline{A} = X$, then A is open in X.*

Proof. Let $x \in A$. Since A is locally compact, there is an open neighborhood U of x in A, such that the closure of U in A is compact (see [M, Lemma 8.2]). Let $\mathrm{cl}_A(U)$ denote the closure of U in A (as opposed to $\overline{U}$ being the closure of U in X). Since X is Hausdorff, and $\mathrm{cl}_A(U)$ is compact, then $\mathrm{cl}_A(U)$ is closed in X, by Exercise 1. Since U is open in A, then $U = V \cap A$ for some open set V of X. Since $\mathrm{cl}_A(U)$ is closed in X, then $V \setminus \mathrm{cl}_A(U)$ is open in X. Since $U \subset A$ and $U \subseteq \mathrm{cl}_A(U)$, then we have $A \cap (V \setminus \mathrm{cl}_A(U)) = \varnothing$. Since A is dense in X, then we must have $V \setminus \mathrm{cl}_A(U) = \varnothing$. This implies $V \subseteq \mathrm{cl}_A(U) \subseteq A$. But now $U = V \cap A = V$, and so U is an open set of X which is a neighborhood of x and contained in A. That is, A is open in X. □

A topological group G is called a *locally compact group* if it is a locally compact space and it is Hausdorff.

Proposition 4.1. *Let G be a Hausdorff topological group. Any subgroup H of G which is locally compact (in the subspace topology) is closed.*

Proof. Consider the closure $\overline{H}$ of H in G. First, by definition we have H is dense in $\overline{H}$. Since H is locally compact, then we have H is open in $\overline{H}$ by Lemma 4.2. By Proposition 1.4, we have $\overline{H}$ is a subgroup of G, and so H is an open subgroup of $\overline{H}$. By Proposition 1.2, H is also a closed subgroup of $\overline{H}$. Now H is closed in $\overline{H}$ and $\overline{H}$ is closed in G, and so H is closed in G. □

Remark. In a previous draft of these notes, we followed the proof of Proposition 4.1 that is given in [RV, Proposition 1-6], which begins with the claim that if K is a compact neighborhood of 1 in H, so K is closed in H, then there is a closed neighborhood F of 1 in G such that $K = F \cap H$. However, this does not follow from point-set topology, as was pointed out to me by David Savitt, who showed me the following counterexample [DS]. The key is that we are assuming that 1 is in the interior of F, not simply that F is a closed set containing 1. In particular, if $X = \{1, 2, 3, 4\}$ with topology given by $\{\varnothing, \{4\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, X\}$, let $H = \{1, 2\}$, which has the discrete topology as a supspace of X. Then we have $K = \{1\}$ is a closed neighborhood of itself in H. However, the only closed neighborhood of 1 in X is X itself, whose intersection with H is not K. So, the previous proof given here was not complete. The proof for Proposition 4.1 now given above, using Lemma 4.2, was suggested to me by Santana Afton while he was an undergraduate at the College of William and Mary. The proof given of Lemma 4.2 above is adapted from a Mathematics Stack Exchange post by Eric Wofsey [EW].

We will need to apply the following technical lemma later.

Lemma 4.3. *Let G be a locally compact group, K a compact subset of G, and U an open neighborhood of 1 in G. Then there is a neighborhood V of 1 in G such that $x^{-1}Vx \subset U$ for every $x \in K$.*

Proof. Let $x \in K$. Then xUx^{-1} is an open neighborhood of 1. Since G is a locally compact group, there is a compact neighborhood V_x of 1 such that $V_x \subset xUx^{-1}$ by Lemma 4.1. Let $F = G \setminus U$, then $x^{-1}V_x x \cap F = \varnothing$. From Lemma 1.1, there is a neighborhood W'_x of 1, which may be chosen to be compact by Lemma 4.1, and symmetric by Proposition 1.1 and the remark after Proposition 1.4 (along with Exercise 3), such that

$$(x^{-1}V_x x)W'_x \cap F = \varnothing.$$

The fact that W'_x is symmetric implies that we also have

$$x^{-1}V_x x \cap FW'_x = \varnothing.$$

From Proposition 1.5, FW'_x is closed, since F is closed and W'_x is compact. So, again by Lemma 1.1 and Proposition 1.1, there is a symmetric neighborhood W''_x of 1 such that

$$W''_x(x^{-1}V_x x) \cap FW'_x = \varnothing. \tag{4.1}$$

Now let $W_x = W'_x \cap W''_x$, which is a symmetric neighborhood of 1. Then we must have

$$W_x(x^{-1}V_x x)W_x \cap F = \varnothing, \tag{4.2}$$

otherwise (4.1) would be violated (since W_x and W'_x are symmetric).

For each $x \in K$, let $U_x = \operatorname{int}(W_x)$ be the interior of W_x. Then, the collection of all xU_x, $x \in K$, constitutes an open cover of K, and so there is a finite number of points, say $x_1, x_2, \ldots, x_n$, such that, writing $W_{x_i} = W_i$, and $U_{x_i} = U_i$,

$$K \subset \bigcup_{i=1}^{n} x_i U_i \subset \bigcup_{i=1}^{n} x_i W_i.$$

Now let $V = \cap_{i=1}^{n} V_i$, where $V_i = V_{x_i}$. If $x \in K$, then $x \in x_i W_i$ for some i, and so $x^{-1} \in W_i x_i^{-1}$, since W_i is symmetric. Now, by (4.2), we have

$$x^{-1}Vx \subset W_i x_i^{-1} V_i x_i W_i \subset G \setminus F = U. \quad \square$$

A topological space X is *connected* if whenever $X = U \cup V$ where U and V are nonempty open sets, then $U \cap V \neq \varnothing$. That is, X is connected when X has no nonempty proper subsets which are both closed and open (or *clopen*). A maximal connected subset of X is called a *connected component* of X. The space X is *totally disconnected* if each one-point subset in X is its own connected component. Of course, every discrete space is totally disconnected. One familiar example of a totally disconnected space which is not discrete is the Cantor middle-thirds set.

Exercise 4. If $A \subset X$ is connected, then $\overline{A}$ is connected. That is, connected components are closed sets.

If G is a topological group, then G is totally disconnected if and only if $\{1\}$ is a connected component, by homogeneity. The connected component of 1 in G will be denoted G°, and its basic properties are as follows.

Proposition 4.2. *If G is a topological group, then G° is a normal subgroup of G, the connected components of G are all of the form xG° for $x \in G$, and G/G° is a totally disconnected group.*

Proof. See Problem Set 1. $\square$

Finally, we turn to the study of spaces which are locally compact and totally disconnected. Before proving the main statements, we first need a few more preliminary lemmas.

A topological space X is called *normal* or T_4 if it is T_1 (one point sets are closed) and for any disjoint closed subsets E and F of X, there are open sets U and V such that $E \subset U$, $F \subset V$, and $U \cap V = \varnothing$.

Exercise 5. Every compact Hausdorff space is normal. Note that from Lemma 4.1, we already know compact Hausdorff spaces are regular.

Lemma 4.4. *Let X be a compact Hausdorff space, and let $x \in X$. Let $\mathcal{U}_x$ denote the collection of compact open neighborhoods of x. Then $\cap_{U \in \mathcal{U}_x} U$ is the connected component of x.*

Proof. Let $F = \cap_{U \in \mathcal{U}_x} U$, which is a nonempty closed set since X itself is a compact open neighborhood of x, and each $U \in \mathcal{U}_x$ is compact and thus closed (by Exercise 3). Suppose that V' and W' are closed and open subsets of F (in the subspace topology of F) such that

$$F = V' \cup W', \quad \text{and} \quad V' \cap W' = \varnothing.$$

Since F is closed, then V' and W' are closed subsets in X. Since X is normal by Exercise 5, then there are disjoint open sets V and W of X such that $V' \subset V$ and $W' \subset W$. To show F is connected, we must show that one of V' or W' is empty.

Now, $B = X \setminus (V \cup W)$ is closed, and thus compact, and does not intersect F. So, the sets $X \setminus U$, $U \in \mathcal{U}_x$, cover B, and are all open (and closed) since each U is compact (thus closed) and open. Since B is compact, there are a finite number of neighborhoods of x, $U_1, \ldots, U_n \in \mathcal{U}_x$, such that $B \subset \cup_{i=1}^n (X \setminus U_i)$. In other words, if we let $A = \cap_{i=1}^n U_i$, then $A \cap B = \varnothing$, $x \in A$, and A is compact and open. Now $A \subset X \setminus B = V \cup W$, and so

$$A = (A \cap V) \cup (A \cap W),$$

where $A \cap V$ and $A \cap W$ are disjoint open sets. Since A is closed, $A \cap V$ and $A \cap W$ are also both closed (and thus compact). So, x can only be an element of one of them, say $x \in A \cap V$, which means that $F \subset A \cap V$ (since $A \cap V$ is a compact open neighborhood of x), while $F \cap (A \cap W) = \varnothing$. This means we must have $F = V'$ and $W' = \varnothing$, so that F is connected.

Now let C be the connected component of x, so that $F \subset C$. Suppose that $F \neq C$, so that there is a point $y \in C \setminus F$. Then there must be a compact open neighborhood M of x such that $y \notin M$. Now $M \cap C$ is closed and open in C, while $(X \setminus M) \cap C$ contains y, contradicting the fact that C is connected. Thus F is the connected component of x. □

Lemma 4.5. *Let X be a compact Hausdorff space, let C be a connected component of X, and let F be a closed subset of X such that $F \cap C = \varnothing$. Then there is a compact open set V such that $C \subset V$ and $F \cap V = \varnothing$.*

Proof. We have F is compact (Exercise 3), and if $x \in C$, then $C = \cap_{U \in \mathcal{U}_x} U$, where $\mathcal{U}_x$ is the collection of compact open neighborhoods of x, by Lemma 4.4. The open sets $X \setminus U$, $U \in \mathcal{U}_x$ cover F, and so for a finite number of sets in $\mathcal{U}_x$, say $U_1, \ldots, U_n$, F is covered by $\cup_{i=1}^n (X \setminus U_i)$. If we let $V = \cap_{i=1}^n U_i$, we have $F \cap V = \varnothing$, and $C \subset V$, as desired. □

Theorem 4.1. *Let X be a Hausdorff space. Then X is locally compact and totally disconnected if and only if every neighborhood of every point $x \in X$ contains a compact open neighborhood of x.*

Proof. ($\Rightarrow$): Let $x \in X$, and let U be the interior of an arbitrary neighborhood of x. By Lemma 4.1, there is a compact neighborhood K of x contained in U. Now let V be an open neighborhood of x, $V \subset K$, and let $F = K \setminus V$. If $F = \varnothing$, then V is open and compact, and we are done. The set F is closed, and $\{x\}$ is a connected component of X since X is totally disconnected, and so $\{x\}$ is a connected component of the compact subset K. Since $F \cap \{x\} = \varnothing$, then by Lemma 4.5, there is a compact open set W containing x such that $F \cap W = \varnothing$. That is, $W \subset V \subset U$, and W is a compact open neighborhood of x.

($\Leftarrow$): First, since every point $x \in X$ has a compact neighborhood, then X is locally compact. Now let S be the connected component of $x \in X$. Suppose that $y \neq x$ and $y \in S$. Since X is Hausdorff, it is T_1, and so x has an open neighborhood W such that $y \notin W$. Let U be a compact open neighborhood of x which is contained in W. Since X is Hausdorff, U is closed. So, $U' = U \cap S$ is closed and open in S. But $y \notin U'$, and so U' is a proper nonempty clopen subset of S, contradicting the fact that S is connected. Thus, $S = \{x\}$, and X is totally disconnected. □

Corollary 4.1. *Let G be a locally compact totally disconnected group, and H a subgroup of G. Then H is closed if and only if H is a locally compact totally disconnected group, if and only if G/H is a locally compact totally disconnected Hausdorff space.*

Proof. First, suppose H is closed, and let $x \in H$, and U any neighborhood of x in H. Then $U = H \cap V$, where V is a neighborhood of x in G. By Theorem 4.1, V contains a compact open neighborhood of x, say F. By definition, $H \cap F$ is an open neighborhood of x in H, and it is contained in U. Since F is compact and G is Hausdorff, then F is closed, and so $F \cap H$ is closed in G. Moreover, $F \cap H$ is compact in G since it is closed and contained in F, which is compact. Thus $F \cap H$ is compact in H. Now $F \cap H$ is a compact open neighborhood of x in H which is contained in U, and H is locally compact and totally disconnected by Theorem 4.1. Conversely, if H is locally compact in the subspace topology, then it is automatically closed by Proposition 4.1.

For the second part, suppose that H is closed. Since $p : G \to G/H$ is an open map by part (2) of Proposition 3.1, and is continuous by definition, then the image under p of compact open sets of G are compact open sets of G/H. If U is an open neighborhood of $xH \in G/H$, then $p^{-1}(U)$ is an open neighborhood of $x \in G$, which contains a compact open neighborhood K of x, by Theorem 4.1. Then $p(K)$ is a compact open neighborhood of xH contained in $p(p^{-1}(U)) = U$. Thus G/H is locally compact and totally disconnected by Theorem 4.1. Since H is assumed to be closed, then G/H is Hausdorff by part (4) of Proposition 3.1. Conversely, if G/H is Hausdorff, then H is automatically closed also by part (4) of Proposition 3.1. □

Theorem 4.2. *Let G be a locally compact totally disconnected group. Every neighborhood of 1 contains a compact open subgroup of G. If G is a compact totally disconnected group, then every neighborhood of 1 contains a compact open normal subgroup of G.*

Proof. Since G is a locally compact totally disconnected group, each neighborhood of 1 contains a compact open neighborhood V of 1, from Theorem 4.1. Let us denote $V^n = VV^{n-1}$ for $n \geq 2$. Let $F = (G \setminus V) \cap V^2$. Since V is open, $G \setminus V$ is closed, and

since G is Hausdorff and V is compact, V is closed and so V^2 is closed, by Proposition 1.5. Thus F is closed.

We have $V \cap F = \varnothing$, where V is compact and F is closed. By Lemma 1.1 and Proposition 1.1, there is an open symmetric neighborhood W of 1, $W \subset V$, such that $VW \cap F = \varnothing$. Since $W \subset V$, then $VW \subset V^2$. Because $F = (G \setminus V) \cap V^2$, and $VW \cap F = \varnothing$, then we must have $VW \subset V$. Now we have

$$VW^2 \subset VW \subset V,$$

and by induction we must have $VW^n \subset V$ for every integer $n \geq 0$. Since W was chosen to be symmetric, then in fact $VW^n \subset V$ for every integer n. In particular, since $1 \in V$, we have

$$\bigcup_{n \in \mathbb{Z}} W^n \subset V.$$

Now, $H = \cup_{n \in \mathbb{Z}} W^n$ is a subgroup of G contained in V. Since each W^n is open, then H is an open subgroup, and is thus closed by Proposition 1.2. Since $H \subset V$ and V is compact and H is closed, then H must be compact. Thus, H is a compact open subgroup of G.

Suppose now that G is compact and totally disconnected. Since G is locally compact and totally disconnected, then any neighborhood of 1 contains a compact open subgroup H', as we have just shown. Now consider

$$H = \bigcap_{x \in G} xH'x^{-1}.$$

By Lemma 4.3, there is a neighborhood U of 1 such that $U \subset xH'x^{-1}$ for every $x \in G$ (since G is compact). In other words, H contains an open neighborhood of 1, and is thus open. H is a subgroup, since it is the intersection of subgroups, and is normal by construction. Since $xH'x^{-1}$ is closed (since it is compact) for every $x \in G$, then H is closed, and is thus compact since G is compact. So, H is a compact open normal subgroup of G. □

The following characterization of locally compact totally disconnected groups follows immediately from Theorems 4.1 and 4.2.

Corollary 4.2. *A Hausdorff topological group G is locally compact and totally disconnected if and only if every neighborhood of 1 contains a compact open subgroup.*

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