Let $E$ be a local field, and $\mathcal{O}_E$ its ring of integers. A lattice in $E^n$ is a finitely generated $\mathcal{O}_E$-submodule $\Lambda$ such that $\Lambda \otimes_{\mathcal{O}_E} E = E^n$; equivalently, an $\mathcal{O}_E$-submodule such that $\Lambda \cong \mathcal{O}_E^n$. We will make use of, but not prove, the equivalence of these two characterizations of lattices in $E^n$.

**Lemma 0.1.** If $K$ is a compact group acting continuously and linearly on $E^n$, then there exists a lattice $\Lambda$ stabilized by $K$.

**Proof.** Let $\Lambda'$ be any lattice in $E^n$. The set $U = \{ g \in K : g\Lambda' \subset \Lambda' \}$ is open; to see this, note for instance that if $v_1, \ldots, v_n$ is a basis of $\Lambda'$ and $f_i : K \to E^n$ is the map $g \mapsto gv_i$, then $U = \bigcap_i \{ g : gv_i \in \Lambda' \} = \bigcap_i f_i^{-1}(\Lambda')$. The sets $gU$ for $g \in K$ are open covers of $K$, hence they have a finite subcover $g_1U, \ldots, g_rU$ for $i = 1, \ldots, r$. Set $\Lambda = g_1\Lambda' + \cdots + g_r\Lambda'$; by the first characterization of lattices, this is again a lattice.

To prove that $K$ stabilizes $\Lambda$ we need to check that $g(g_iu) \in \Lambda$ for each $g \in K$, $v \in \Lambda'$, and $i = 1, \ldots, r$. Write $gg_i = g_ju$ with $u \in U$. Then $g(g_iu) = g_j(uv) \in g_j\Lambda'$, as desired. \hfill $\square$

Write $G = \text{GL}_n(E)$ and let $G$ act naturally on $E^n$, which contains the standard lattice $\mathcal{O}_E^n$.

**Lemma 0.2.** The stabilizer of $\mathcal{O}_E^n$ under this action is $\text{GL}_n(\mathcal{O}_E)$.

**Proof.** It is clear that $\text{GL}_n(\mathcal{O}_E)$ stabilizes $\mathcal{O}_E^n$. On the other hand if $g \in G \setminus \text{GL}_n(\mathcal{O}_E)$, suppose that the $(i, j)$-entry of $g$ is not contained in $\mathcal{O}_E$. If $e_j$ is the $j$th standard basis vector of $E^n$ then $ge_j \notin \mathcal{O}_E^n$, so $g$ does not stabilize $\mathcal{O}_E^n$. \hfill $\square$

Now we are ready to prove the following.

**Theorem 0.3.** $\text{GL}_n(\mathcal{O}_E)$ is a maximal compact subgroup of $\text{GL}_n(E)$.

**Proof.** Let $\pi \in \mathcal{O}_E$ be a uniformizer. To see that $\text{GL}_n(\mathcal{O}_E)$ is compact, note that the kernel of the homomorphism $\text{GL}_n(\mathcal{O}_E) \to \text{GL}_n(\mathcal{O}_E/\pi)$ is $I + \pi M_n(\mathcal{O}_E)$, which is certainly compact (it is homeomorphic to $\mathcal{O}_E^n$). Since $\text{GL}_n(\mathcal{O}_E/\pi)$ is finite, $\text{GL}_n(\mathcal{O}_E)$ is the (disjoint) union of finitely many translates of a compact set.

Now suppose that $K$ is a compact subgroup of $\text{GL}_n(E)$ containing $\text{GL}_n(\mathcal{O}_E)$. By Lemma 0.1 there is a lattice $\Lambda$ stabilized by $K$. Noting that $K$ also stabilizes $c\Lambda$ for any $c \in E^\times$, we may replace $\Lambda$ with $\pi\Lambda$ so that $\Lambda \subset \mathcal{O}_E^n$ but $\Lambda \not\subset \mathcal{O}_E^n$. Choose $v \in \Lambda$ so that one of the coordinates of $v$ is a unit. It is easily checked that the $\text{GL}_n(\mathcal{O}_E)$-orbit of $v$ is all of $\mathcal{O}_E^n$. Since the $K$-orbit of $v$ is at least as large, we must have $\mathcal{O}_E^n \subset \Lambda$, hence $\Lambda = \mathcal{O}_E^n$. Thus $K$ stabilizes $\mathcal{O}_E^n$, and Lemma 0.2 gives the reverse inequality $K \subset \text{GL}_n(\mathcal{O}_E)$. \hfill $\square$
We can reinterpret this result as follows.

**Corollary 0.4.** The following are equivalent:

1. $K$ is a maximal compact subgroup of $GL_n(E)$;
2. $K$ is the stabilizer of some lattice $\Lambda$;
3. $K$ is conjugate to $GL_n(O_E)$.

**Proof.** The second and third statements are trivially seen to be equivalent (the group $A GL_n(O_E) A^{-1}$ is the stabilizer of $A \cdot O^n_E$). Any conjugate of a maximal compact subgroup is again a maximal compact subgroup, so the third statement together with Theorem 0.3 implies the first.

Finally, suppose $K$ is a maximal compact subgroup of $GL_n(E)$. By Lemma 0.1, $K$ stabilizes a lattice $\Lambda$. Then $K$ is contained in the stabilizer of $\Lambda$, which is conjugate to $GL_n(O_E)$, hence compact; by maximality, $K$ is equal to the stabilizer of $\Lambda$. \qed