MATH 519 - PROBLEM SET 2

1. Let R be a ring and $(I, \preceq)$ a directed set. Let $\{M_i\}_{i\in I}$ be a directed system of R-modules, with $\{f_j^i\}_{i,j\in I, i\preceq j}$ the corresponding directed system of R-homomorphisms, where $f_j^i : M_i \to M_j$ when $i \preceq j$. Let $(M, (f^i)_{i\in I})$ be the direct limit of this system, so that

$$M = \varinjlim_i M_i,$$

and $f^i : M_i \to M$ is such that if $i \preceq j$, then $f^j \circ f_j^i = f^i$. Prove that if $x_i \in M_i$ is such that $f^i(x_i) = 0$, then $f_j^i(x_i) = 0$ for some $j \neq i$, $i \preceq j$.

2. Prove that $\mathbb{Z}_p \cong \mathbb{Z}[[t]]/\langle t-p\rangle$ as rings, where $\langle t-p\rangle$ is the ideal generated by $t-p$ in $\mathbb{Z}[[t]]$.

3. Let K be a field with absolute value $|\cdot|_v$, and give K the metric topology induced by $|\cdot|_v$. Prove that K is a topological field with this topology (that is, show that addition, multiplication, and additive and multiplicative inverses are all continuous maps).

4. Prove that there is only one absolute value which extends the standard archimedean absolute value on $\mathbb{R}$ to $\mathbb{C}$.

5. (a): Let $|\cdot|_v$ be an absolute value on $\mathbb{F}_q(t)$, where q is a power of some prime p. Show that $|\cdot|_v$ is equivalent to either $|\cdot|_{f(t)}$ for some irreducible polynomial $f(t) \in \mathbb{F}_q[t]$, or to $|\cdot|_\infty$. (Hint: Show that there is some irreducible $f(t) \in \mathbb{F}_p[t]$ such that $|f(t)|_v \neq 1$. If $|f(t)|_v < 1$, proceed as in the case for non-archimedean absolute values of $\mathbb{Q}$. If $|f(t)|_v > 1$, show that also $|t|_v > 1$.)
(b): Show that the residue field of $\mathbb{F}_q(t)$ with respect to $|\cdot|_{f(t)}$ is isomorphic to $\mathbb{F}_{q^{d(f)}}$, where $d(f) = \deg(f)$, and with respect to $|\cdot|_\infty$ the residue field is isomorphic to $\mathbb{F}_q$.
(c): If $\mathbb{F}_q((1/t))$ has the topology from $|\cdot|_\infty$, and $\mathbb{F}_q((t))$ has the topology from $|\cdot|_t$, prove that $\mathbb{F}_q((1/t)) \cong \mathbb{F}_q((t))$ as topological fields. If $\mathbb{F}_{q^{d(f)}}((f(t)))$ has the topology from $|\cdot|_{f(t)}$ and $\mathbb{F}_{q^{d(f)}}((t))$ has the topology from $|\cdot|_t$, prove that $\mathbb{F}_{q^{d(f)}}((f(t))) \cong \mathbb{F}_{q^{d(f)}}((t))$ as topological fields.

6. Let F be a non-archimedean local field with ring of integers $\mathcal{O}$ and uniformizer π. Let $M_n(\mathcal{O})$ be the set of n-by-n matrices with entries from $\mathcal{O}$, and let I be the identity matrix. Show that for every $m \geq 1$, $K_m = I + \pi^m M_n(\mathcal{O})$ is a compact open subgroup of $\mathrm{GL}(n, F)$, and is a compact open normal subgroup of $\mathrm{GL}(n, \mathcal{O})$ (realize it as the kernel of a homomorphism). Also show that for every neighborhood U of I in $\mathrm{GL}(n, F)$, there is some $m \geq 1$ such that $K_m \subset U$.