Conjugacy Classes of the Symmetric Groups

Math 430 - Spring 2009

Let G be any group. If $g, x \in G$, we define the *conjugate of g by x* to be the element xgx^{-1}. (Note: some texts define the conjugate of g by x to be $x^{-1}gx$. By our definition, this would be the conjugate of g by x^{-1}.) If $g, h \in G$, and there is some $x \in G$ such that $xgx^{-1} = h$, we say that g and h are *conjugate* in G. For the group G, define the relation $\sim$ by $g \sim h$ if g and h are conjugate in G.

Proposition 1 *Let G be a group, and define the relation $\sim$ on G by $g \sim h$ if g and h are conjugate in G. Then $\sim$ is an equivalence relation on G.*

Proof. We need to check that $\sim$ satisfies the three defining properties of an equivalence relation. First, for any $g \in G$, we have $g \sim g$ since $ege^{-1} = g$, so the reflexive property holds. Now suppose $g\tilde{h}$. Then there is some $x \in G$ such that $xgx^{-1} = h$. Then we obtain $g = x^{-1}hx$. So we may conjugate h by x^{-1} to get g, so $h \sim g$ and the reflexive property holds. Now suppose $g, h, k \in G$, where $g \sim h$ and $h \sim k$. Then there are $y, z \in G$ such that $ygy^{-1} = h$ and $zhz^{-1} = k$. Substituting the former expression for h into the latter, we obtain $zygy^{-1}z^{-1} = k$, or $(zy)g(zy)^{-1} = k$. So, we may conjugate g by zy to get k, so $g \sim k$ and the transitive property holds. Thus $\sim$ is an equivalence relation on G. $\square$

Since $\sim$ is an equivalence relation on G, its equivalence classes partition G. The equivalence classes under this relation are called the *conjugacy classes* of G. So, the conjugacy class of $g \in G$ is

$$[g] = \{xgx^{-1} \mid x \in G\}.$$

Exercise 1. Let G be any group, and let $x, g_1, g_2, \ldots, g_n \in G$. Show that for any n, the conjugate of $g_1 g_2 \cdots g_n$ by x is the product of the conjugates by x of $g_1, g_2, \dots, g_n$.

Exercise 2. Let G be an Abelian group. Show that for any $a \in G$, the conjugacy class of a is the singleton set $\{a\}$.

When G is non-Abelian, understanding the conjugacy classes of G is an important part of understanding the group structure of G. Conjugacy classes play a key role in a subject called *representation theory*, which is one of the main applications of group theory to chemistry and physics.

We now determine the conjugacy classes of the symmetric group S_n. We begin by noticing that any conjugate of a k-cycle is again a k-cycle.

Lemma 1 *Let $\alpha, \tau \in S_n$, where α is the k-cycle $(a_1\ a_2\ \cdots\ a_k)$. Then*

$$\tau\alpha\tau^{-1} = (\tau(a_1)\ \ \tau(a_2)\ \ \cdots\ \ \tau(a_k)).$$

Proof. Consider $\tau(a_i)$ such that $1 \le i \le k$. Then we have $\tau^{-1}\tau(a_i) = a_i$, and $\alpha(a_i) = a_{i+1 \bmod k}$. We now have $\tau\alpha\tau^{-1}(\tau(a_i)) = \tau(a_{i+1 \bmod k})$. Now take any j such that $j \in \{1, 2, 3, \dots, n\}$, but $j \neq a_i$ for any i. Then $\alpha(j) = j$ since j is not in the k-cycle defining α. So, $\tau\alpha\tau^{-1}(\tau(j)) = \tau(j)$. We now see that $\tau\alpha\tau^{-1}$ fixes any number which is not of the form $\tau(a_i)$ for some i, and we have

$$\tau\alpha\tau^{-1} = (\tau(a_1)\ \ \tau(a_2)\ \ \cdots\ \ \tau(a_k)). \quad \square$$

For any permutation $\alpha \in S_n$, we know we can write α as a product of disjoint cycles. Suppose we write α in this way, and α has cycles of length k_1, k_2, k_3, …, k_ℓ, where $k_1 \ge k_2 \ge k_3 \ge \ldots \ge k_\ell$, and where we include 1's in this list for fixed points. We call the sequence $(k_1, k_2, k_3, \dots, k_\ell)$ the *cycle type* of α. Note that $\sum_{i=1}^{\ell} k_i = n$ since every element in $\{1, 2, \dots, n\}$ is either fixed or appears in some cycle.

Example 1. If $\sigma \in S_{10}$ and $\sigma = (1\ 3\ 4\ 5)(2\ 7\ 8\ 9)$, then σ has cycle type $(4, 4, 1, 1)$.

Example 2. If α is a k-cycle in S_n, where $k \le n$, then the cycle type of α is $(k, 1, \dots, 1)$, where there are $n - k$ 1's in the sequence.

We may now describe the conjugacy classes of the symmetric groups.

Theorem 1 *The conjugacy classes of any S_n are determined by cycle type. That is, if σ has cycle type $(k_1, k_2, \ldots, k_\ell)$, then any conjugate of σ has cycle type $(k_1, k_2, \ldots, k_\ell)$, and if ρ is any other element of S_n with cycle type $(k_1, k_2, \ldots, k_\ell)$, then σ is conjugate to ρ.*

Proof. Suppose that σ has cycle type $(k_1, k_2, \ldots, k_\ell)$, so that σ can be written as a product of disjoint cycles as $\sigma = \alpha_1 \alpha_2 \cdots \alpha_\ell$, where α_i is a k_i-cycle. Let $\tau \in S_n$, then by Exercise 1 we have

$$\tau\sigma\tau^{-1} = \tau\alpha_1\alpha_2\cdots\alpha_\ell\tau^{-1} = (\tau\alpha_1\tau^{-1})(\tau\alpha_2\tau^{-1})\cdots(\tau\alpha_\ell\tau^{-1}). \tag{1}$$

Now, for each i such that $1 \leq i \leq \ell$, we have α_i is a k_i-cycle. From Lemma 1, we know that $\tau\alpha_i\tau^{-1}$ is also a k_i-cycle. For any $i, j \in \{1, 2, \ldots, \ell\}$ such that $i \neq j$, we know that α_i and α_j are disjoint, and so $\tau\alpha_i\tau^{-1}$ and $\tau\alpha_j\tau^{-1}$ must be disjoint since τ is a one-to-one function. So, the product in (1) above is $\tau\sigma\tau^{-1}$ written as a product of disjoint cycles, and $\tau\alpha_i\tau^{-1}$ is a k_i-cycle. Now we see that any conjugate of σ has cycle type $(k_1, k_2, \ldots, k_\ell)$.

Now let $\sigma, \rho \in S_n$ both be of cycle type $(k_1, k_2, \ldots, k_\ell)$, and we show that σ and ρ are conjugate in S_n. Let σ and τ be written as products of disjoint cycles as

$$\sigma = \alpha_1\alpha_2\cdots\alpha_\ell \quad \text{and} \quad \rho = \beta_1\beta_2\cdots\beta_\ell,$$

where α_i and β_i are k_i-cycles. For each i, let us write

$$\alpha_i = (a_{i1}\ a_{i2}\ \cdots\ a_{ik_i}) \quad \text{and} \quad \beta_i = (b_{i1}\ b_{i2}\ \cdots\ b_{ik_i}).$$

Now define τ by $\tau(a_{ij}) = b_{ij}$ for every i, j such that $1 \leq i \leq \ell$ and $1 \leq j \leq k_i$. From Lemma 1, we have $\tau\alpha_i\tau^{-1} = \beta_i$. So, from Exercise 1, we have

$$\tau\sigma\tau^{-1} = (\tau\alpha_1\tau^{-1})(\tau\alpha_2\tau^{-1})\cdots(\tau\alpha_\ell\tau^{-1}) = \beta_1\beta_2\cdots\beta_\ell = \rho.$$

So, any two elements of S_n with the same cycle type are in the same conjugacy class. □

If n is a positive integer, a sequence of positive integers $(k_1, k_2, \ldots, k_\ell)$ such that $k_1 \geq k_2 \geq \cdots \geq k_\ell$ and $\sum_{i=1}^{\ell} k_i = n$ is called a *partition* of n. From Theorem 1, the partitions of n are in one-to-one correspondence with the conjugacy classes of S_n. The number of partitions of a positive number n is often denoted $p(n)$, called the *partition function*, and we have $p(n)$ is the number of conjugacy classes of S_n. The partition function and its properties are of great interest in number theory. There is no known closed formula for $p(n)$ in terms of n, but there are several known modular arithmetic equivalences for the function. For example, if m is any non-negative integer, then it is known that $p(5m + 4) \equiv 0 \bmod 5$.