Conjugacy Classes of the Symmetric Groups

Math 430 - Spring 2009

Let $G$ be any group. If $g, x \in G$, we define the conjugate of $g$ by $x$ to be the element $xgx^{-1}$. (Note: some texts define the conjugate of $g$ by $x$ to be $x^{-1}gx$. By our definition, this would be the conjugate of $g$ by $x^{-1}$.) If $g, h \in G$, and there is some $x \in G$ such that $xgx^{-1} = h$, we say that $g$ and $h$ are conjugate in $G$. For the group $G$, define the relation $\sim$ by $g \sim h$ if $g$ and $h$ are conjugate in $G$.

**Proposition 1** Let $G$ be a group, and define the relation $\sim$ on $G$ by $g \sim h$ if $g$ and $h$ are conjugate in $G$. Then $\sim$ is an equivalence relation on $G$.

**Proof.** We need to check that $\sim$ satisfies the three defining properties of an equivalence relation. First, for any $g \in G$, we have $g \sim g$ since $ege^{-1} = g$, so the reflexive property holds. Now suppose $gh$. Then there is some $x \in G$ such that $xgx^{-1} = h$. Then we obtain $g = x^{-1}hx$. So we may conjugate $h$ by $x^{-1}$ to get $g$, so $h \sim g$ and the reflexive property holds. Now suppose $g, h, k \in G$, where $g \sim h$ and $h \sim k$. Then there are $y, z \in G$ such that $ygy^{-1} = h$ and $zhz^{-1} = k$. Substituting the former expression for $h$ into the latter, we obtain $zygy^{-1}z^{-1} = k$, or $(zy)g(zy)^{-1} = k$. So, we may conjugate $g$ by $zy$ to get $k$, so $g \sim k$ and the transitive property holds. Thus $\sim$ is an equivalence relation on $G$. □

Since $\sim$ is an equivalence relation on $G$, its equivalence classes partition $G$. The equivalence classes under this relation are called the conjugacy classes of $G$. So, the conjugacy class of $g \in G$ is

$$[g] = \{xgx^{-1} \mid x \in G\}.$$ 

**Exercise 1.** Let $G$ be any group, and let $x, g_1, g_2, \ldots, g_n \in G$. Show that for any $n$, the conjugate of $g_1g_2\cdots g_n$ by $x$ is the product of the conjugates
by \( x \) of \( g_1, g_2, \ldots, g_n \).

**Exercise 2.** Let \( G \) be an Abelian group. Show that for any \( a \in G \), the conjugacy class of \( a \) is the singleton set \( \{a\} \).

When \( G \) is non-Abelian, understanding the conjugacy classes of \( G \) is an important part of understanding the group structure of \( G \). Conjugacy classes play a key role in a subject called representation theory, which is one of the main applications of group theory to chemistry and physics.

We now determine the conjugacy classes of the symmetric group \( S_n \). We begin by noticing that any conjugate of a \( k \)-cycle is again a \( k \)-cycle.

**Lemma 1** Let \( \alpha, \tau \in S_n \), where \( \alpha \) is the \( k \)-cycle \( (a_1 a_2 \cdots a_k) \). Then

\[
\tau \alpha \tau^{-1} = (\tau(a_1) \tau(a_2) \cdots \tau(a_k)).
\]

*Proof.* Consider \( \tau(a_i) \) such that \( 1 \leq i \leq k \). Then we have \( \tau^{-1}(a_i) = a_i \), and \( \alpha(a_i) = a_{i+1 \mod k} \). We now have \( \tau \alpha \tau^{-1}(\tau(a_i)) = \tau(a_{i+1 \mod k}) \). Now take any \( j \) such that \( j \in \{1, 2, 3, \ldots, n\} \), but \( j \neq a_i \) for any \( i \). Then \( \alpha(j) = j \) since \( j \) is not in the \( k \)-cycle defining \( \alpha \). So, \( \tau \alpha \tau^{-1}(\tau(j)) = \tau(j) \). We now see that \( \tau \alpha \tau^{-1} \) fixes any number which is not of the form \( \tau(a_i) \) for some \( i \), and we have

\[
\tau \alpha \tau^{-1} = (\tau(a_1) \tau(a_2) \cdots \tau(a_k)). \quad \square
\]

For any permutation \( \alpha \in S_n \), we know we can write \( \alpha \) as a product of disjoint cycles. Suppose we write \( \alpha \) in this way, and \( \alpha \) has cycles of length \( k_1, k_2, k_3, \ldots, k_t \), where \( k_1 \geq k_2 \geq k_3 \geq \ldots \geq k_t \), and where we include 1’s in this list for fixed points. We call the sequence \( (k_1, k_2, k_3, \ldots, k_t) \) the cycle type of \( \alpha \). Note that \( \sum_{i=1}^{t} k_i = n \) since every element in \( \{1, 2, \ldots, n\} \) is either fixed or appears in some cycle.

**Example 1.** If \( \sigma \in S_{10} \) and \( \sigma = (1 \ 3 \ 4 \ 5)(2 \ 7 \ 8 \ 9) \), then \( \sigma \) has cycle type \( (4, 4, 1, 1) \).

**Example 2.** If \( \alpha \) is a \( k \)-cycle in \( S_n \), where \( k \leq n \), then the cycle type of \( \alpha \) is \( (k, 1, \ldots, 1) \), where there are \( n-k \) 1’s in the sequence.

We may now describe the conjugacy classes of the symmetric groups.
Theorem 1 The conjugacy classes of any $S_n$ are determined by cycle type. That is, if $\sigma$ has cycle type $(k_1, k_2, \ldots, k_\ell)$, then any conjugate of $\sigma$ has cycle type $(k_1, k_2, \ldots, k_\ell)$, and if $\rho$ is any other element of $S_n$ with cycle type $(k_1, k_2, \ldots, k_\ell)$, then $\sigma$ is conjugate to $\rho$.

Proof. Suppose that $\sigma$ has cycle type $(k_1, k_2, \ldots, k_\ell)$, so that $\sigma$ can be written as a product of disjoint cycles as $\sigma = \alpha_1 \alpha_2 \cdots \alpha_\ell$, where $\alpha_i$ is a $k_i$-cycle. Let $\tau \in S_n$, then by Exercise 1 we have

$$\tau \sigma \tau^{-1} = \tau \alpha_1 \alpha_2 \cdots \alpha_\ell \tau^{-1} = (\tau \alpha_1 \tau^{-1})(\tau \alpha_2 \tau^{-1}) \cdots (\tau \alpha_\ell \tau^{-1}).$$

(1)

Now, for each $i$ such that $1 \leq i \leq \ell$, we have $\alpha_i$ is a $k_i$-cycle. From Lemma 1, we know that $\tau \alpha_i \tau^{-1}$ is also a $k_i$-cycle. For any $i, j \in \{1, 2, \ldots, \ell\}$ such that $i \neq j$, we know that $\alpha_i$ and $\alpha_j$ are disjoint, and so $\tau \alpha_i \tau^{-1}$ and $\tau \alpha_j \tau^{-1}$ must be disjoint since $\tau$ is a one-to-one function. So, the product in (1) above is $\tau \sigma \tau^{-1}$ written as a product of disjoint cycles, and $\tau \alpha_i \tau^{-1}$ is a $k_i$-cycle. Now we see that any conjugate of $\sigma$ has cycle type $(k_1, k_2, \ldots, k_\ell)$.

Now let $\sigma, \rho \in S_n$ both be of cycle type $(k_1, k_2, \ldots, k_\ell)$, and we show that $\sigma$ and $\rho$ are conjugate in $S_n$. Let $\sigma$ and $\tau$ be written as products of disjoint cycles as

$$\sigma = \alpha_1 \alpha_2 \cdots \alpha_\ell \quad \text{and} \quad \rho = \beta_1 \beta_2 \cdots \beta_\ell,$$

where $\alpha_i$ and $\beta_i$ are $k_i$-cycles. For each $i$, let us write

$$\alpha_i = (a_{i1} a_{i2} \cdots a_{ik_i}) \quad \text{and} \quad \beta_i = (b_{i1} b_{i2} \cdots b_{ik_i}).$$

Now define $\tau$ by $\tau(a_{ij}) = b_{ij}$ for every $i, j$ such that $1 \leq i \leq \ell$ and $1 \leq j \leq k_i$. From Lemma 1, we have $\tau \alpha_i \tau^{-1} = \beta_i$. So, from Exercise 1, we have

$$\tau \sigma \tau^{-1} = (\tau \alpha_1 \tau^{-1})(\tau \alpha_2 \tau^{-1}) \cdots (\tau \alpha_\ell \tau^{-1}) = \beta_1 \beta_2 \cdots \beta_\ell = \rho.$$

So, any two elements of $S_n$ with the same cycle type are in the same conjugacy class. $\square$

If $n$ is a positive integer, a sequence of positive integers $(k_1, k_2, \ldots, k_\ell)$ such that $k_1 \geq k_2 \geq \cdots \geq k_\ell$ and $\sum_{i=1}^\ell k_i = n$ is called a partition of $n$. From Theorem 1, the partitions of $n$ are in one-to-one correspondence with the conjugacy classes of $S_n$. The number of partitions of a positive number $n$ is often denoted $p(n)$, called the partition function, and we have $p(n)$ is the number of conjugacy classes of $S_n$. The partition function and its properties
are of great interest in number theory. There is no known closed formula for $p(n)$ in terms of $n$, but there are several known modular arithmetic equivalences for the function. For example, if $m$ is any non-negative integer, then it is known that $p(5m + 4) \equiv 0 \mod 5$. 