The Multiplicative Group of a Finite Field

Math 430

The purpose of these notes is to give a proof that the multiplicative group of a finite field is cyclic, without using the classification of finite abelian groups. We need the following lemma, the proof of which we omitted from class.

Lemma 1. *Suppose G is an abelian group, $x, y \in G$, and $|x| = r$ and $|y| = s$ are finite orders. Then there exists an element of G which has order* $\mathrm{lcm}(r, s)$.

Proof. Suppose first that $\gcd(r, s) = 1$, so that $\mathrm{lcm}(r, s) = rs$. Given $x, y \in G$ such that $|x| = r$ and $|y| = s$, consider $z = xy \in G$. Since $z^{rs} = x^{rs}y^{rs} = e$, then $|z| \leq rs$. If $|z| = m$, then $z^m = e$, so $e = e^s = z^{ms} = x^{ms}y^{ms} = x^{ms}$, since $y^s = e$. Since $x^{ms} = e$ and $|x| = r$, then $r|ms$, and $\gcd(r, s) = 1$, so $r|m$. Also $e = e^r = z^{mr} = x^{mr}y^{mr} = y^{mr}$, since $x^r = e$. Then since $|y| = s$ and $y^{mr} = e$, then $s|mr$, so $s|m$. Now $r|m$ and $s|m$ implies $rs|m$ since $\gcd(r, s) = 1$. So $|z| = m \geq rs$. Now $|z| = rs = \mathrm{lcm}(r, s)$.

We now consider the general case, where $\mathrm{lcm}(r, s)$ is not necessarily rs. Given $|x| = r$ and $|y| = s$ in the abelian group G, it is not true in general that xy will have order $\mathrm{lcm}(r, s)$ (try to find a counterexample). We decompose the positive integer r as a product $r = r_1r_2r_3r_4$ as follows:

$r_1 =$ the product of all prime factors of r which are not prime factors of s,

$r_2 =$ the product of all prime factors which occur with equal powers in r and s,

$r_3 =$ the product of all prime factors of r which occur in r and s, but in r with higher powers,

$r_4 =$ the product of all prime factors of r which occur in r and s, but in s with higher powers.

Define $s = s_1s_2s_3s_4$ analogously, with r and s in exchanged roles. Note that this means $s_2 = r_2$, and $\mathrm{lcm}(r, s) = r_1r_2r_3s_1s_3$. If we define $\tilde{r} = r_1r_2r_3$ and

$\tilde{s} = s_1 s_3$, then $\gcd(\tilde{r}, \tilde{s}) = 1$, and $\mathrm{lcm}(\tilde{r}, \tilde{s}) = \tilde{r}\tilde{s} = r_1 r_2 r_3 s_1 s_3 = \mathrm{lcm}(r, s)$. For example, if $r = 2^7 3^5 5^4 7^4$ and $s = 2^6 3^7 5^4 11^4$, then $r_1 = 7^4$, $r_2 = s_2 = 5^4$, $r_3 = 2^7$, $r_4 = 3^5$, $s_1 = 11^4$, $s_3 = 3^7$, $s_4 = 2^6$, and so $\tilde{r} = 7^4 5^4 2^7$ and $\tilde{s} = 11^4 3^7$.

Now $|x^{r_4}| = r/r_4 = r_1 r_2 r_3 = \tilde{r}$ and $|y^{s_2 s_4}| = s/s_2 s_4 = s_1 s_3 = \tilde{s}$. If we take $\tilde{x} = x^{r_4}$ and $\tilde{y} = y^{s_2 s_4}$, then $|\tilde{x}| = \tilde{r}$ and $|\tilde{y}| = \tilde{s}$, where $\gcd(\tilde{r}, \tilde{s}) = 1$, so by the first part of the proof, $|\tilde{x}\tilde{y}| = \tilde{r}\tilde{s}$. That is, taking $\tilde{z} = \tilde{x}\tilde{y}$, we have $\tilde{z} \in G$ with $|\tilde{z}| = \tilde{r}\tilde{s} = \mathrm{lcm}(r, s)$. □

The above lemma is enough to prove the desired statement.

Theorem 1. *Suppose F is a finite field. Then $F^\times = F \setminus \{0\}$ is a cyclic group under multiplication.*

Proof. Let $|F^\times| = m$. Suppose $\alpha \in F^\times$ has maximal possible order under multiplication over all elements of $F^\times$, and call this order $|\alpha| = k$. By Lagrange's Theorem, $k|m$, so in particular $k \leq m$.

Let $\beta \in F^\times$ be any element of $F^\times$. If $|\beta| = r$, then by Lemma **??**, $F^\times$ has some element of order $\mathrm{lcm}(r, k) \geq k$. Since k is the maximal order of all elements in $F^\times$, then we must have $\mathrm{lcm}(r, k) = k$, which implies $r|k$. Since $|\beta| = r$ and $r|k$, then we have $\beta^k = 1$. Since β was arbitrary, this means every element of $F^\times$ is a zero of the polynomial $x^k - 1 \in F[x]$, that is, $x^k - 1$ has m roots in F. However, we've shown that a polynomial of degree d over some field has at most d roots in that field. That is, we must have $m \leq k$. That is, we have $m = k$.

Now $|\alpha| = m = |F^\times|$. Thus $\langle \alpha \rangle = F^\times$ and $F^\times$ is a cyclic group under multiplication. □