Let \( R = \mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right] = \{a + b\left(\frac{1+\sqrt{-19}}{2}\right) \mid a, b \in \mathbb{Z}\} \), and define \( N : R \to \mathbb{Z} \) by
\[
N(a + b(1 + \sqrt{-19})/2) = a^2 + ab + 5b^2 = (a + b/2)^2 + (19/4)b^2.
\]

In this final set of exercises, you will finish showing that \( N \) is a Dedekind-Hasse norm for \( R \), thus proving that \( R \) is a PID which is not a Euclidean domain.

You have proven that it is enough to show that, for any nonzero \( \alpha, \beta \in R \), that if \( \beta \) does not divide \( \alpha \) in \( R \), then there are \( s, t \in R \) such that
\[
0 < N\left(\frac{\alpha}{\beta}s - t\right) < 1. \tag{1}
\]

Let \( \alpha, \beta \in R \) be nonzero, such that \( \beta \) does not divide \( \alpha \) in \( R \), so that \( \alpha/\beta \in \mathbb{Q}(\sqrt{-19}) \). Then from the last handout, there are \( a, b, c \in \mathbb{Z} \), with \( c > 1 \), such that \( \gcd(a, b, c) = 1 \), and such that \( \frac{\alpha}{\beta} = \frac{a + b\sqrt{-19}}{c} \). You proved that if \( c \geq 5 \), then there exist \( s, t \in R \) which satisfy (1). The final steps are to take care of the cases \( c = 2, 3, \) or 4.

(a): Suppose \( c = 2 \). Since \( \gcd(a, b, c) = 1 \), then \( a \) and \( b \) are of different parity. Prove that \( s = 1 \) and \( t = \frac{(a - 1) + b\sqrt{-19}}{2} \) are elements of \( R \) which satisfy (1).

(b): Suppose \( c = 3 \). Prove that \( a^2 + 19b^2 \) cannot be divisible by 3, by considering that it is \( a^2 + b^2 \) modulo 3, and the fact that 3 cannot divide all of \( a, b, \) and \( c \). Let \( a^2 + b^2 = 3q + r \) where \( q, r \in \mathbb{Z} \) and \( r = 1 \) or 2. Prove that \( s = a - b\sqrt{-19} \) and \( t = q \) are elements of \( R \) which satisfy (1).
(c): Finally, suppose $c = 4$. Then $a, b$ cannot both be even since $\gcd(a, b, c) = 1$. If one of $a, b$ is even and the other is odd, show that $a^2 + 19b^2$ is odd, so we can write $a^2 + 19b^2 = 4q + r$ for some $q, r \in \mathbb{Z}$ and $r = 1$ or $3$. Show that $s = a - b\sqrt{-19}$ and $t = q$ are elements of $R$ satisfying (1). If both $a$ and $b$ are odd, show that $a^2 + 19b^2 \equiv 4 \pmod{8}$, so $a^2 + 19b^2 = 8q + 4$ for some $q \in \mathbb{Z}$. Show that $s = \frac{a - b\sqrt{-19}}{2}$ and $t = q$ are elements of $R$ which satisfy (1).