Homework #8 Problem (Optional)

Math 430 - Spring 2013

Let $R = \mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right] = \{a + b\left(\frac{1+\sqrt{-19}}{2}\right) \mid a, b \in \mathbb{Z}\}$, and define $N : R \to \mathbb{Z}$ by

$$N\left(a + b(1 + \sqrt{-19})/2\right) = a^2 + ab + 5b^2 = (a + b/2)^2 + (19/4)b^2.$$

In this final set of exercises, you will finish showing that N is a Dedekind-Hasse norm for R, thus proving that R is a PID which is not a Euclidean domain.

You have proven that it is enough to show that, for any nonzero $\alpha, \beta \in R$, that if β does not divide α in R, then there are $s, t \in R$ such that

$$0 < N\left(\frac{\alpha}{\beta}s - t\right) < 1. \tag{1}$$

Let $\alpha, \beta \in R$ be nonzero, such that β dows not divide α in R, so that $\alpha/\beta \in \mathbb{Q}(\sqrt{-19})$. Then from the last handout, there are $a, b, c \in \mathbb{Z}$, with $c > 1$, such that $\gcd(a, b, c) = 1$, and such that $\dfrac{\alpha}{\beta} = \dfrac{a + b\sqrt{-19}}{c}$. You proved that if $c \geq 5$, then there exist $s, t \in R$ which satisfy (1). The final steps are to take care of the cases $c = 2, 3$, or 4.

(a): Suppose $c = 2$. Since $\gcd(a, b, c) = 1$, then a and b are of different parity. Prove that $s = 1$ and $t = \dfrac{(a-1) + b\sqrt{-19}}{2}$ are elements of R which satisfy (1).

(b): Suppose $c = 3$. Prove that $a^2 + 19b^2$ cannot be divisible by 3, by considering that it is $a^2 + b^2$ modulo 3, and the fact that 3 cannot divide all of a, b, and c. Let $a^2 + b^2 = 3q + r$ where $q, r \in \mathbb{Z}$ and $r = 1$ or 2. Prove that $s = a - b\sqrt{-19}$ and $t = q$ are elements of R which satisfy (1).

(c): Finally, suppose $c = 4$. Then a, b cannot both be even since $\gcd(a, b, c) = 1$. If one of a, b is even and the other is odd, show that $a^2 + 19b^2$ is odd, so we can write $a^2 + 19b^2 = 4q + r$ for some $q, r \in \mathbb{Z}$ and $r = 1$ or 3. Show that $s = a - b\sqrt{-19}$ and $t = q$ are elements of R satisfying (1). If both a and b are odd, show that $a^2 + 19b^2 \equiv 4 (\bmod\ 8)$, so $a^2 + 19b^2 = 8q + 4$ for some $q \in \mathbb{Z}$. Show that $s = \dfrac{a - b\sqrt{-19}}{2}$ and $t = q$ are elements of R which satisfy (1).