Homework #7 Problem

Math 430 - Spring 2013

Let $R = \mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right] = \{a + b\left(\frac{1+\sqrt{-19}}{2}\right) \mid a, b \in \mathbb{Z}\}$, and define $N : R \to \mathbb{Z}$ by

$$N\left(a + b(1+\sqrt{-19})/2\right) = a^2 + ab + 5b^2 = (a + b/2)^2 + (19/4)b^2.$$

On Homeworks #5 and #6, you showed that R is an integral domain, but is not a Euclidean domain. You also showed, on Homework #5, that if an integral domain has a Dedekind-Hasse norm, then it must be a PID. Between this exercise and an exercise in the last homework, you will show that N is a Dedekind-Hasse norm for R, thus showing that R is a PID which is not a Euclidean domain.

You have already shown, when you showed that N is multiplicative, that $N(\alpha) = 0$ if and only if $\alpha = 0$. So, to show that N is a Dedekind-Hasse norm for R, you must show that given nonzero $\alpha, \beta \in R$, either $\beta | \alpha$ in R, or there are $s, t \in R$ such that $0 < N(s\alpha - t\beta) < N(\beta)$.

(a): Let $F = \{x + y(1+\sqrt{-19})/2 \mid x, y \in \mathbb{Q}\}$. Prove that $F = \mathbb{Q}(\sqrt{-19}) = \{u + v\sqrt{-19} \mid u, v \in \mathbb{Q}\}$, which we know is a field.

Now, we can extend the definition of N to $F = \mathbb{Q}(\sqrt{-19})$ by

$$N\left(x + y(1+\sqrt{-19})/2\right) = x^2 + xy + 5y^2 = (x + y/2)^2 + (19/4)y^2.$$

Note that your proof that N was multiplicative on R carries over directly to a proof that N as defined above is multiplicative on $F = \mathbb{Q}(\sqrt{-19})$. So, we can use this fact.

(b): Let $\alpha, \beta \in R$ be nonzero, and assume β does not divide α in R, so $\alpha/\beta \notin R$. Show that $\alpha/\beta \in \mathbb{Q}(\sqrt{-19})$. Explain why, to conclude R is a

PID, it is enough to find $s, t \in R$ such that

$$0 < N\left(\frac{\alpha}{\beta}s - t\right) < 1.$$

(c): Take $\alpha, \beta \in R$ nonzero as in **(b)**, so that $\alpha/\beta \in \mathbb{Q}(\sqrt{-19})$. So, there are integers a, b, c, such that $c > 1$ (otherwise $\alpha/\beta \in R$), the only integers dividing a, b, and c are 1 or -1 (that is, $\gcd(a, b, c) = 1$), and such that $\dfrac{\alpha}{\beta} = \dfrac{a + b\sqrt{-19}}{c}$. Since $\gcd(a, b, c) = 1$, there are integers x, y, z such that $ax + by + cz = 1$ (since in $\mathbb{Z}$, the ideal $\langle a, b, c\rangle$ is principal, but any generator must divide all of a, b, and c in $\mathbb{Z}$, so in particular, $1 \in \langle a, b, c\rangle$).

Suppose that $c \geq 5$. Explain why we can write $ay - 19bx = cq + r$ with $q, r \in \mathbb{Z}$ and $|r| \leq c/2$.

Let $s = y + x\sqrt{-19}$ and $t = q - z\sqrt{-19}$. Show that

$$0 < N\left(\frac{\alpha}{\beta}s - t\right) = \frac{(ay - 19bx - cq)^2 + 19(ax + by + cz)^2}{c^2} \leq \frac{1}{4} + \frac{19}{c^2} < 1,$$

if $c > 5$, and if $c = 5$, then

$$0 < N\left(\frac{\alpha}{\beta}s - t\right) = \frac{(ay - 19bx - cq)^2 + 19(ax + by + cz)^2}{c^2} \leq \frac{23}{25} < 1.$$

Next week, for the last part, we have to show how to choose $s, t \in R$ in the cases that $c = 2, 3$, or 4, and then we'll be done.