Homework #7 Problem

Math 430 - Spring 2013

Let \( R = \mathbb{Z} \left[ \frac{1+\sqrt{-19}}{2} \right] = \{ a + b \left( \frac{1+\sqrt{-19}}{2} \right) \mid a, b \in \mathbb{Z} \} \), and define \( N : R \to \mathbb{Z} \) by

\[
N \left( a + b \left( \frac{1+\sqrt{-19}}{2} \right) \right) = a^2 + ab + 5b^2 = (a + b/2)^2 + (19/4)b^2.
\]

On Homeworks #5 and #6, you showed that \( R \) is an integral domain, but is not a Euclidean domain. You also showed, on Homework #5, that if an integral domain has a Dedekind-Hasse norm, then it must be a PID. Between this exercise and an exercise in the last homework, you will show that \( N \) is a Dedekind-Hasse norm for \( R \), thus showing that \( R \) is a PID which is not a Euclidean domain.

You have already shown, when you showed that \( N \) is multiplicative, that \( N(\alpha) = 0 \) if and only if \( \alpha = 0 \). So, to show that \( N \) is a Dedekind-Hasse norm for \( R \), you must show that given nonzero \( \alpha, \beta \in R \), either \( \beta | \alpha \) in \( R \), or there are \( s, t \in R \) such that \( 0 < N(s\alpha - t\beta) < N(\beta) \).

\( \textbf{(a)}: \) Let \( F = \{ x + y(1 + \sqrt{-19})/2 \mid x, y \in \mathbb{Q} \} \). Prove that \( F = \mathbb{Q}(\sqrt{-19}) = \{ u + v\sqrt{-19} \mid u, v \in \mathbb{Q} \} \), which we know is a field.

Now, we can extend the definition of \( N \) to \( F = \mathbb{Q}(\sqrt{-19}) \) by

\[
N \left( x + y(1 + \sqrt{-19})/2 \right) = x^2 + xy + 5y^2 = (x + y/2)^2 + (19/4)y^2.
\]

Note that your proof that \( N \) was multiplicative on \( R \) carries over directly to a proof that \( N \) as defined above is multiplicative on \( F = \mathbb{Q}(\sqrt{-19}) \). So, we can use this fact.

\( \textbf{(b)}: \) Let \( \alpha, \beta \in R \) be nonzero, and assume \( \beta \) does not divide \( \alpha \) in \( R \), so \( \alpha/\beta \notin R \). Show that \( \alpha/\beta \in \mathbb{Q}(\sqrt{-19}) \). Explain why, to conclude \( R \) is a
PID, it is enough to find $s, t \in R$ such that

$$0 < N\left(\frac{\alpha}{\beta}s - t\right) < 1.$$ 

(c): Take $\alpha, \beta \in R$ nonzero as in (b), so that $\alpha/\beta \in \mathbb{Q}(\sqrt{-19})$. So, there are integers $a, b, c$, such that $c > 1$ (otherwise $\alpha/\beta \in R$), the only integers dividing $a, b$, and $c$ are 1 or $-1$ (that is, $\gcd(a, b, c) = 1$), and such that $\frac{\alpha}{\beta} = \frac{a + b\sqrt{-19}}{c}$. Since $\gcd(a, b, c) = 1$, there are integers $x, y, z$ such that $ax + by + cz = 1$ (since in $\mathbb{Z}$, the ideal $(a, b, c)$ is principal, but any generator must divide all of $a, b$, and $c$ in $\mathbb{Z}$, so in particular, $1 \in (a, b, c)$).

Suppose that $c \geq 5$. Explain why we can write $ay - 19bx = cq + r$ with $q, r \in \mathbb{Z}$ and $|r| \leq c/2$.

Let $s = y + x\sqrt{-19}$ and $t = q - z\sqrt{-19}$. Show that

$$0 < N\left(\frac{\alpha}{\beta}s - t\right) = \frac{(ay - 19bx - cq)^2 + 19(ax + by + cz)^2}{c^2} \leq \frac{1}{4} + \frac{19}{c^2} < 1,$$

if $c > 5$, and if $c = 5$, then

$$0 < N\left(\frac{\alpha}{\beta}s - t\right) = \frac{(ay - 19bx - cq)^2 + 19(ax + by + cz)^2}{c^2} \leq \frac{23}{25} < 1.$$ 

Next week, for the last part, we have to show how to choose $s, t \in R$ in the cases that $c = 2, 3, 4$, and then we’ll be done.