Homework #6 Problem

Math 430 - Spring 2013

1. Let $R = \mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right] = \{a + b\left(\frac{1+\sqrt{-19}}{2}\right) \mid a, b \in \mathbb{Z}\}$, and define $N : R \to \mathbb{Z}$ by

$$N\left(a + b(1+\sqrt{-19})/2\right) = a^2 + ab + 5b^2 = (a + b/2)^2 + (19/4)b^2.$$

On Homework #5, you showed that R is an integral domain, that N is a multiplicative norm on R, and that the units of R are $R^\times = \{1, -1\}$. On the take-home midterm, you showed that any Euclidean domain D which is not a field must have a universal side divisor, that is, an element s which is nonzero and not a unit, such that for any $\alpha \in D$, either $s|\alpha$ or $s|\alpha - w$ for some unit $w \in D^\times$.

You will use all of these facts to show that R is not a Euclidean domain.

(a): Show that if $a, b \in \mathbb{Z}$ with $b \neq 0$, then $a^2 + ab + 5b^2 \geq 5$, and that the smallest nonzero values which N can take on R are 1, for the units $1, -1$, and 4, for the elements $2, -2$.

(b): Using the multiplicative norm N and **(a)**, prove that the only divisors of 2 in R are $1, -1, 2$ and -2, and that the only divisors of 3 in R are $1, -1, 3$, and -3.

(c): Prove that R is not a Euclidean domain by way of contradiction as follows. Suppose R is a Euclidean domain, and $s \in R$ is a universal side divisor. Taking $\alpha = 2$ in the definition of universal side divisor, then, s must divide either 2, $2-1 = 1$, or $2+1 = 3$. Conclude from **(b)** that we must then have $s = 2, -2, 3$, or -3. Then take $\alpha = (1+\sqrt{-19})/2 \in R$ in the definition of universal side divisor. Use the multiplicative norm N to show that none of $2, -2, 3$, or -3 can divide α, $\alpha - 1$, or $\alpha + 1$ in R. This contradicts the assumption that s is a universal side divisor of R.