The Commutator Subgroup

Math 430 - Spring 2013

Let $G$ be any group. If $a, b \in G$, then the commutator of $a$ and $b$ is the element $aba^{-1}b^{-1}$. Of course, if $a$ and $b$ commute, then $aba^{-1}b^{-1} = e$. Now define $C$ to be the set

$$C = \{x_1 x_2 \cdots x_n \mid n \geq 1, \text{ each } x_i \text{ is a commutator in } G\}.$$  

In other words, $C$ is the collection of all finite products of commutators in $G$. Then we have

**Proposition 1.** If $G$ is any group, then $C \triangleleft G$.

**Proof.** First, we have $e = eee^{-1}e^{-1} \in C$, so $C$ is nonempty and contains the identity. If $c, d \in C$, then we have $c = x_1 x_2 \cdots x_n$ and $d = y_1 y_2 \cdots y_m$, where each $x_i$ and each $y_j$ is a commutator in $G$. Then

$$cd = x_1 x_2 \cdots x_n y_1 y_2 \cdots y_m \in C,$$

since this is just another finite product of commutators. We also have

$$d^{-1} = (x_1 x_2 \cdots x_n)^{-1} = x_n^{-1} \cdots x_2^{-1} x_1^{-1}.$$  

If $x_i = a_i b_i a_i^{-1} b_i^{-1}$, then $x_i^{-1} = b_i a_i^{-1} b_i^{-1}$, which is also a commutator. Thus $c^{-1} \in C$, and $C \leq G$.

To prove $C$ is a normal subgroup of $G$, let $g \in G$, and $c = x_1 x_2 \cdots x_n \in C$. Then we have

$$gcg^{-1} = gx_1 x_2 \cdots x_n g^{-1} = (gx_1 g^{-1})(gx_2 g^{-1}) \cdots (gx_n g^{-1}),$$  \hspace{1cm} (1)

where we have just inserted $gg^{-1} = e$ between $x_i$ and $x_{i+1}$ for each $i < n$. Now, if $x_i = a_i b_i a_i^{-1} b_i^{-1}$, then we have

$$gx_i g^{-1} = ga_i b_i a_i^{-1} b_i^{-1} g^{-1} = (ga_i g^{-1})(gb_i g^{-1})(ga_i^{-1} g^{-1})(gb_i^{-1} g^{-1}).$$
Now note that \((ga_i g^{-1})^{-1} = (g^{-1})^{-1}a_i^{-1}g^{-1} = ga_i^{-1}g^{-1}\), and we have the analogous statement if we replace \(a_i\) by \(b_i\). So, we have
\[
g a_i g^{-1} = (ga_i g^{-1})(gb_i g^{-1})(ga_i g^{-1})^{-1}(gb_i g^{-1})^{-1},
\]
which is a commutator. Now, from (1), we have \(gcg^{-1}\) is a product of commutators, and so \(gcg^{-1} \in C\). Thus \(C \triangleleft G\).

The subgroup \(C\) of \(G\) is called the **commutator subgroup** of \(G\), and it general, it is also denoted by \(C = G'\) or \(C = [G, G]\), and is also called the **derived subgroup** of \(G\). If \(G\) is Abelian, then we have \(C = \{e\}\), so in one sense the commutator subgroup may be used as one measure of how far a group is from being Abelian. Specifically, we have the following result.

**Theorem 1.** Let \(G\) be a group, and let \(C\) be its commutator subgroup. Suppose that \(N \triangleleft G\). Then \(G/N\) is Abelian if and only if \(C \subseteq N\). In particular, \(G/C\) is Abelian.

**Proof.** First assume that \(G/N\) is Abelian. Let \(a, b \in G\). Since we are assuming that \(G/N\) is Abelian, then we have \((aN)(bN) = (bN)(aN)\), and so \(abN = baN\) by the definition of coset multiplication in the factor group. Now, we know \(abN = baN\) implies \(ab[ba]^{-1} \in N\), where \(ab[ba]^{-1} = aba^{-1}b^{-1}\), and so \(aba^{-1}b^{-1} \in N\). Since \(a\) and \(b\) were arbitrary, any commutator in \(G\) is an element of \(N\), and since \(N\) is a subgroup of \(G\), then any finite product of commutators in \(G\) is an element of \(N\). Thus \(C \subseteq N\).

Now suppose that \(C \subseteq N\), and let \(a, b \in G\). Then \(aba^{-1}b^{-1} \in N\), and so \(ab[ba]^{-1} \in N\). This implies \(abN = baN\), or that \((aN)(bN) = (bN)(aN)\). Since \(a\) and \(b\) were arbitrary, this holds for any elements \(aN, bN \in G/N\), and thus \(G/N\) is Abelian.

Given a group \(G\), and its derived subgroup \(G'\), we may then consider the derived subgroup of \(G'\), or \([G', G'] = (G')'\). This is often denoted as \(G^{(2)}\). For any integer \(i \geq 0\), define the \(i\)th **derived subgroup** of \(G\), denoted \(G^{(i)}\), recursively as follows. Let \(G^{(0)} = G\), and for \(i \geq 1\), define \(G^{(i)} = (G^{(i-1)})' = [G^{(i-1)}, G^{(i-1)}]\). By Theorem 1, note that we have \(G^{(i)} \triangleleft G^{(i-1)}\), and by Theorem 1, \(G^{(i-1)}/G^{(i)}\) is abelian, for all \(i \geq 1\).

Recall that, from the first Homework, we may define a finite group \(G\) to be **solvable** if there are subgroups \(H_0 = \{e\}, H_1, \ldots, H_k = G\), such that \(H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_k\), and \(H_i/H_{i-1}\) is abelian for each \(i = 1, \ldots, k\). As we see now, the commutator subgroups are the key for understanding whether a finite group \(G\) is solvable.
Theorem 2. Let $G$ be a finite group. Then $G$ is solvable if and only if there exists some integer $k \geq 0$ such that the $k^{th}$ derived subgroup of $G$ is trivial, that is, $G^{(k)} = \{e\}$.

Proof. Let $k \geq 0$ such that $G^{(k)} = \{e\}$. By Proposition 1, we have $G^{(i)} \triangleleft G^{(i-1)}$ for any $i \geq 1$, so

$$\{e\} = G^{(k)} \triangleleft G^{(k-1)} \triangleleft \cdots \triangleleft G^{(2)} \triangleleft G^{(1)} = G' \triangleleft G^{(0)} = G.$$ 

Since $G^{(i-1)}/G^{(i)}$ is abelian for $i \geq 1$ by Theorem 1, then the existence of this subnormal series implies that $G$ is solvable (taking $H_i = G^{(k-i)}$ in the definition).

Now assume that $G$ is solvable. Then there are subgroup $\{e\} = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_k = G$ such that $H_i/H_{i-1}$ is abelian for $i = 1, \ldots, k$. In particular, $H_k/H_{k-1} = G/H_{k-1}$ is abelian, and so $G' \subset H_{k-1}$ by Theorem 1. By induction, suppose that $G^{(i)} \subset H_{k-i}$ for an $i \geq 1$. Since $H_{k-i}/H_{k-i-1}$ is abelian, then $H'_{k-i} \subset H_{k-i-1}$ by Theorem 1. By the induction hypothesis, $G^{(i)} \subset H_{k-i}$, and so every commutator in $G^{(i)}$ is a commutator in $H_{k-i}$, which implies $(G^{(i)})' \subset H_{k-i}'. \quad $ Since $(G^{(i)})' = G^{(i+1)}$ and $H'_{k-i} \subset H_{k-i-1}$, we have $G^{(i+1)} \subset H_{k-i-1}$. By induction, we then have $G^{(k)} \subset H_0 = \{e\}$. Thus $G^{(k)} = \{e\}$. $\square$

Theorem 2 is extremely useful for proving facts about finite solvable groups. For example, let $G$ be any finite group, and suppose $H \leq G$. Then $H' \leq G'$ since every commutator of $H$ is a commutator of $G$, and by induction $H^{(i)} \leq G^{(i)}$ for every $i \geq 0$. If $G$ is solvable, then $G^{(k)} = \{e\}$ for some $k$. Since $H^{(k)} \leq G^{(k)}$, then $H^{(k)} = \{e\}$ and thus $H$ is also solvable. This statement is true for an arbitrary group as well, but the argument is a bit more subtle.

Proposition 2. Let $G$ be any group, and suppose $H \leq G$. If $G$ is solvable, then $H$ is solvable.

Proof. Recall that the definition of an arbitrary group being solvable (finite or not) in Fraleigh is that it has a decomposition series such that every decomposition factor group is abelian, and thus cyclic of prime order. So, suppose that

$$\{e\} = K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_{m-1} \triangleleft K_m = G,$$

such that $K_i/K_{i-1}$ is cyclic of prime order, for $i = 1, \ldots, m$. For each $i$, consider $H \cap K_i$. Recall the second isomorphism theorem of groups, which
states that if $L$ is a group, $N \lhd L$, and $M \leq N$, then $NM/N \cong M/(M \cap N)$. For $i = 1, \ldots, m$, apply this theorem to the case that $L = K_i$, $N = K_{i-1}$, and $M = H \cap K_i$. Then we have

$$K_{i-1}(H \cap K_i)/K_{i-1} \cong (H \cap K_i)/(H \cap K_{i-1}),$$

since $M \cap N = H \cap K_i \cap K_{i-1} = H \cap K_{i-1}$. We also have

$$K_{i-1}(H \cap K_i)/K_{i-1} \leq K_i/K_{i-1},$$

and since $K_i/K_{i-1}$ is cyclic of prime order, then we must have $K_{i-1}(H \cap K_i)/K_{i-1}$ is either trivial or cyclic of prime order. So the same must be true of $(H \cap K_i)/(H \cap K_{i-1})$. Therefore, we can build a decomposition series $H$ with decomposition factors being cyclic of prime order, by using the subgroups $H \cap K_i$ (and not using those that are repeated). Thus $H$ is solvable. 

$\Box$