Optional Homework #1

Math 410 - Galois Theory - Fall 2019

1. Let $(X, \leq)$ be a well-ordered poset. Let x_0 be the minimal element of X, which exists since X is well-ordered. If $S(x)$ is a statement about elements $x \in X$, then *transfinite induction* is the method of proving $S(x)$ for all $x \in X$ by first proving that $S(x_0)$ is true, and then showing that if $S(y)$ is true for all $y < x$, then $S(x)$ is true. In this problem you will apply transfinite induction and the well-ordering principle to prove an interesting statement as follows.
(a): Let $\mathcal{L}$ be the collection of all lines in the real plane $\mathbb{R}^2$. Use the well-ordering principle to show that there is a partial order on $\mathcal{L}$ so that every line has only countably many predecessors.
(b): Use transfinite induction on the well-ordered poset from **(a)** to show that there exists a subset S of $\mathbb{R}^2$ such that S intersects every line of $\mathbb{R}^2$ in exactly two points.

2. Let $(X, \leq)$ be a poset. A subset $C \subseteq X$ of X is called *cofinal* if, for every $x \in X$ there exists some $c \in C$ such that $x \leq c$. The purpose of this problem is to show that our proof that the Axiom of Choice implies Zorn's Lemma did not actually prove a stronger version of Zorn's Lemma which allows you to assume that only well-ordered subsets of a poset have upper bounds, because this is the same as Zorn's Lemma.
(a): Prove, using Zorn's Lemma (applied to well-ordered subsets), that if $(X, \leq)$ is a chain, then X contains a well-ordered cofinal subset.
(b): Prove that for any poset $(X, \leq)$, every well-ordered subset in X has an upper bound in X if and only if every chain in X has an upper bound in X.