Quiz 5 **Solutions**, Math 309 (Vinroot)

(1): Suppose $\dim(V) = n$, and $m > n$, with m an integer. If δ is an m-linear alternating form on V, then $\delta(v_1, \ldots, v_m) = 0$ for any $v_1, \ldots, v_m \in V$.

TRUE FALSE

Solution: We know that if δ is m-linear and alternating, and we input any collection of linearly dependent vectors, then the output is 0 (which we proved in class). Since $\dim(V) = n$ and $m > n$, then we know that any collection of m vectors from V is linearly dependent. So δ must give 0 no matter what the input.

(2): Suppose δ is a 3-linear form on the F-vector space V, and fix some vector $w \in V$. Then the function $H : V \times V \to F$ defined by $H(x, y) = \delta(x, y, w)$ is a bilinear form.

TRUE FALSE

Solution: Since δ is linear in each of it's variables, then H being linear in its two variables follows. This statement can be generalized by defining an n-linear form from an m-linear form, where $n < m$, by fixing some $m - n$ vectors from V.

(3): Let $V = \mathbb{C}^2$, and define $B : V \times V \to \mathbb{C}$ by $B(x, y) = x_1y_1 + ix_1y_2 - x_2y_1 + ix_2y_2$, for $x, y \in V$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. You may assume that B is a bilinear form. Find the matrix A, for B with respect to the ordered basis α of V (which you may assume is a basis), where α is given by $\alpha = \left(\begin{bmatrix} i \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right)$.

Solution: Writing $\alpha = (v_1, v_2)$, we must compute $B(v_i, v_j)$ for each pair of basis vectors. From the definition of B given, we have $B(v_1, v_1) = (i)(i) + i(i)(-1) - (-1)(i) + i(-1)(-1) = 2i$, $B(v_1, v_2) = (i)(0)+i(i)(2)-(-1)(0)+i(-1)(2) = -2-2i$, $B(v_2, v_1) = (0)(i)+i(0)(-1)-(2)(i)+i(2)(-1) = -4i$, and $B(v_2, v_2) = (0)(0) + i(0)(2) - (2)(0) + i(2)(2) = 4i$. Our matrix A is thus given by

$$A = \begin{pmatrix} 2i & -2-2i \\ -4i & 4i \end{pmatrix}.$$

(4): Continuing from **(3)**, let $v = \begin{bmatrix} i \\ 1 \end{bmatrix}, w = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \in V$. First, calculate $B(v, w)$ by definition. Then, given $[v]_\alpha = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $[w]_\alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, calculate $[v]_\alpha^t A [w]_\alpha$, where A is your answer from **(3)**.

Solution: From the definition of B, we have $B(v, w) = (i)(0) + i(i)(2) - (1)(0) + i(1)(2) = -2 + 2i$.

Using our A from the last problem, we also have

$$B(v,w) = [v]_\alpha^t A [w]_\alpha = (1 \;\; 1) \begin{pmatrix} 2i & -2-2i \\ -4i & 4i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1 \;\; 1) \begin{pmatrix} -2-2i \\ 4i \end{pmatrix} = -2+2i.$$