Quiz 4 **Solutions**, Math 309 (Vinroot)

(1): Calculate the following determinant by expanding along the third row, where the entries are from $\mathbb{Z}_5$ (so arithmetic is modulo 5):

Solution: $\det \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix} = (-1)^{1+3}4(2\cdot 2 - 1\cdot 1) + (-1)^{2+3}0(1\cdot 2 - 3\cdot 1) + (-1)^{3+3}1(1\cdot 1 - 3\cdot 2).$

Recalling that the arithmetic is modulo 5, we have that this is equal to $4(4-1)+(1-6) = 4(3) = 2$.

(2): Calculate the following determinant after applying a single row operation, where the field of scalars is $\mathbb{C}$:

Solution: $\det \begin{bmatrix} 1 & i & 3 & -2+i \\ 0 & i & 7 & -3i \\ 0 & 0 & 1 & 2i \\ 0 & 0 & -i & i \end{bmatrix} = \det \begin{bmatrix} 1 & i & 3 & -2+i \\ 0 & i & 7 & -3i \\ 0 & 0 & 1 & 2i \\ 0 & 0 & 0 & -2+i \end{bmatrix}$, where we have added i times row 3 to row 4, which does not change the determinant. Since this matrix is upper triangular, the determinant is the product of the diagonal entries, and so is equal to $1 \cdot i \cdot 1 \cdot (-2+i) = -1-2i$.

(3): Either prove (by giving a brief proof) or disprove (by giving a counterexample) the following statement: If A and B are n-by-n matrices over the field F, then $\det(A) + \det(B) = \det(A+B)$.

Solution: The statement is false. A counterexample is given by $A = B = I_2$, with $F = \mathbb{R}$, so that $\det(A) = \det(B) = 1$, and $\det(I_n) + \det(I_n) = 2$. But $A + B = 2I_2$, which has 2's on the diagonal, so $\det(A+B) = 2 \cdot 2 = 4 \neq \det(A) + \det(B)$.

(4): Suppose that V and W are finite dimensional, with ordered bases α and β, respectively. If $T : V \to W$ is an invertible linear transformation, and $A = [T]_\alpha^\beta$, then $\det(A) \neq 0$.

TRUE FALSE

Solution: Since T is an invertible linear transformation, then it is an isomorphism, and we must have $\dim(V) = \dim(W)$. So A is a square matrix, and since T is invertible, then A is invertible, and we have seen that this is equivalent to $\det(A) \neq 0$.

(5): There exists a matrix $A \in M_{3\times 3}(\mathbb{Q})$ such that $\det(A^2) = 2$.

TRUE **FALSE**

Solution: If there is such a matrix A, then we know $\det(A) \in \mathbb{Q}$ since taking determinants only involves adding, subtracting, and multiplying entries of A, which are all in $\mathbb{Q}$. But then $\det(A^2) = \det(A)\det(A) = \det(A)^2$ from multiplicativity of the determinant. Then we have $\det(A)^2 = 2$, so $\det(A) = \pm\sqrt{2}$. However, we know that $\sqrt{2}$ (and so $-\sqrt{2}$) is irrational, which contradicts $\det(A) \in \mathbb{Q}$.