(1): Calculate the following determinant by expanding along the third row, where the entries are from \( \mathbb{Z}_5 \) (so arithmetic is modulo 5):

\[
\begin{bmatrix}
1 & 2 & 1 \\
3 & 1 & 2 \\
4 & 0 & 1
\end{bmatrix}
\]

Solution: 

\[
\det \begin{bmatrix}
1 & 2 & 1 \\
3 & 1 & 2 \\
4 & 0 & 1
\end{bmatrix} = (-1)^{1+3}4(2 \cdot 2 - 1 \cdot 1) + (-1)^{2+3}0(1 \cdot 2 - 3 \cdot 1) + (-1)^{3+3}1(1 \cdot 1 - 3 \cdot 2).
\]

Recalling that the arithmetic is modulo 5, we have that this is equal to \(4(4 - 1) + (1 - 6) = 4(3) = 2\).

(2): Calculate the following determinant after applying a single row operation, where the field of scalars is \( \mathbb{C} \):

\[
\begin{bmatrix}
1 & i & 3 & -2 + i \\
0 & i & 7 & -3i \\
0 & 0 & 1 & 2i \\
0 & 0 & -i & i
\end{bmatrix}
\]

Solution: 

\[
\det \begin{bmatrix}
1 & i & 3 & -2 + i \\
0 & i & 7 & -3i \\
0 & 0 & 1 & 2i \\
0 & 0 & -i & i
\end{bmatrix} = \det \begin{bmatrix}
1 & i & 3 & -2 + i \\
0 & i & 7 & -3i \\
0 & 0 & 1 & 2i \\
0 & 0 & 0 & -2 + i
\end{bmatrix},
\]

where we have added \( i \) times row 3 to row 4, which does not change the determinant. Since this matrix is upper triangular, the determinant is the product of the diagonal entries, and so is equal to \(1 \cdot i \cdot 1 \cdot (-2 + i) = -1 - 2i\).

(3): Either prove (by giving a brief proof) or disprove (by giving a counterexample) the following statement: If \( A \) and \( B \) are \( n \)-by-\( n \) matrices over the field \( F \), then \( \det(A) + \det(B) = \det(A + B) \).

Solution: The statement is false. A counterexample is given by \( A = B = I_2 \), with \( F = \mathbb{R} \), so that \( \det(A) = \det(B) = 1 \), and \( \det(I_2) + \det(I_2) = 2 \). But \( A + B = 2I_2 \), which has 2’s on the diagonal, so \( \det(A + B) = 2 \cdot 2 = 4 \neq \det(A) + \det(B) \).

(4): Suppose that \( V \) and \( W \) are finite dimensional, with ordered bases \( \alpha \) and \( \beta \), respectively. If \( T : V \to W \) is an invertible linear transformation, and \( A = [T]_\alpha^\beta \), then \( \det(A) \neq 0 \).

TRUE FALSE

Solution: Since \( T \) is an invertible linear transformation, then it is an isomorphism, and we must have \( \dim(V) = \dim(W) \). So \( A \) is a square matrix, and since \( T \) is invertible, then \( A \) is invertible, and we have seen that this is equivalent to \( \det(A) \neq 0 \).

(5): There exists a matrix \( A \in M_{3 \times 3}(\mathbb{Q}) \) such that \( \det(A^2) = 2 \).

TRUE FALSE

Solution: If there is such a matrix \( A \), then we know \( \det(A) \in \mathbb{Q} \) since taking determinants only involves adding, subtracting, and multiplying entries of \( A \), which are all in \( \mathbb{Q} \). But then \( \det(A^2) = \det(A) \det(A) = \det(A)^2 \) from multiplicativity of the determinant. Then we have \( \det(A)^2 = 2 \), so \( \det(A) = \pm \sqrt{2} \). However, we know that \( \sqrt{2} \) (and so \( -\sqrt{2} \)) is irrational, which contradicts \( \det(A) \in \mathbb{Q} \).