Quiz 3 **Solutions**, Math 309 (Vinroot)

(1): Suppose $T : V \to W$ is a linear transformation such that $\ker(T) = \{\mathbf{0}\}$, and $\dim(V) = \dim(W) = n$ (finite). Then T is invertible.

TRUE FALSE

Solution: Since $\ker(T) = \{\mathbf{0}\}$, then T is injective. Since $\dim(V) = \dim(W) = n$, then T is also surjective (by rank-nullity). Since T is injective (one-to-one) and surjective (onto), then T is bijective, and so it is invertible.

(2): If $T : V \to W$ is an injective linear transformation, and β is a basis for V, then $T(\beta)$ is a basis for $R(T)$.

TRUE FALSE

Solution: Since T is injective (one-to-one), then from homework we know T takes linearly independent subsets of V to linearly independent subsets of W. So $T(\beta)$ is linearly independent. We also know that $T(\mathrm{span}(\beta)) = \mathrm{span}(T(\beta))$, and since $\mathrm{span}(\beta) = V$, then $\mathrm{span}(T(\beta)) = T(V) = R(T)$. Now $T(\beta)$ is linearly independent and spans $R(T)$, so $T(\beta)$ is a basis for $R(T)$.

(3): If V and W are finite dimensional with ordered bases β and γ, and $T : V \to W$ is a surjective linear transformation, then the matrix $[T]_\beta^\gamma$ is invertible.

TRUE **FALSE**

Solution: We know that $[T]_\beta^\gamma$ is invertible if and only if T is an invertible linear transformation, so if and only if T is both injective and surjective. Since we only know T is surjective, we cannot say that T is invertible. For example, $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x$ is surjective, but is not injective. No matter what we choose as our ordered bases β and γ, $[T]_\beta^\gamma$ will not be square, so it cannot be invertible.

(4): If A is an m-by-n matrix, and B is an n-by-p matrix, give the definition of the matrix product AB by giving the expression for the (i, j) position of this product, $(AB)_{ij}$.

Solution: $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$, for any i and j such that $1 \leq i \leq m$ and $1 \leq j \leq p$.

(5): If $V = P_1(\mathbb{Q})$ with ordered basis $\beta = (2, 1 + x)$, what is $[3 + 4x]_\beta$? Show your (brief) computation.

Solution: We have to write the vector $v = 3 + 4x$ in terms of the ordered basis β. Since the only x-term in the basis is in $1 + x$, then the coefficient of this must be 4 we then have

$$3 + 4x = -\frac{1}{2}(2) + 4(1 + x), \quad \text{so} \quad [3 + 4x]_\beta = \begin{pmatrix} -1/2 \\ 4 \end{pmatrix}.$$

(6): If V and W are finite dimensional, explain why $\dim(\mathcal{L}(V, W)) = \dim(\mathcal{L}(W, V))$, and explain why this means the vector spaces $\mathcal{L}(V, W)$ and $\mathcal{L}(W, V)$ are isomorphic.

Solution: We proved that if V and W are finite dimensional, say $\dim(V) = n$ and $\dim(W) = m$, then we have

$$\dim(\mathcal{L}(V, W)) = \dim(V)\dim(W) = nm,$$

which we showed by proving that $\mathcal{L}(V, W)$ is isomorphic to $M_{m\times n}(F)$. Then we also have

$$\dim(\mathcal{L}(W, V)) = \dim(W)\dim(V) = mn = nm.$$

So, we have $\dim(\mathcal{L}(V, W)) = \dim(\mathcal{L}(W, V))$. We have also shown that any two vector spaces with the same dimension are isomorphic, and so we have $\mathcal{L}(V, W)$ is isomorphic to $\mathcal{L}(W, V)$.