Quiz 2 **Solutions**, Math 309 (Vinroot)

(1): If $T : V \to W$ is a linear transformation, give the precise definition of the *kernel* of T.

Solution: The *kernel* of T is the set $N(T) = \ker(T) = \{v \in V \mid T(v) = \mathbf{0}\}$, that is, the set of all vectors in v sent to the zero vector of W by T.

(2): Let $T : V \to W$ be a linear transformation, and suppose the only vector such that $T(v) = \mathbf{0}$ is $v = \mathbf{0}$. Then for all $x, y \in V$, $T(x) = T(y)$ implies $x = y$.

TRUE FALSE

Solution: This is equivalent to the statement that $N(T) = \{\mathbf{0}\}$ (the assumption given) implies that T is injective, which we proved.

(3): If $T : V \to W$ is a linear transformation, and β is a basis for V, then $T(\beta)$ is a basis for W.

TRUE **FALSE**

Solution: We only know that $T(\beta)$ spans $R(T)$. A quick counterexample to the statement is if $T : V \to W$ is the zero transformation, so $T(v) = \mathbf{0}$ for all $v \in V$, and V and W both have nonzero dimension. So, if $V = \mathbb{R}^2 = W$, then any basis of V is sent to $\{\mathbf{0}\}$, which is certainly not a basis for W. If we know that T is surjective, we can say that $T(\beta)$ spans W, and if we know that T is injective, we can say that $T(\beta)$ is linearly independent, so if T is both, so bijective, we can say that $T(\beta)$ is a basis for W.

(4): Let $V = P_2(\mathbb{Q})$ and $W = P_3(\mathbb{Q})$, and define $T : V \to W$ by $T(ax^2 + bx + c) = a^2x^3 + b$. Show that T is *not* linear (remember, you just need one explicit counterexample).

Solution: There are (infinitely) many ways to choose a counterexample to T satisfying the definition of a linear transformation. One is if we choose $v = x^2 \in V$, and the scalar $d = 2 \in \mathbb{Q}$. Then by definition of T, $T(2v) = T(2x^2) = 4x^3$. But $2T(v) = 2T(x^2) = 2x^3$. Since $T(2v) \neq 2T(v)$, then T is not a linear transformation.

(5): Let F be a field, $V = M_{2\times 2}(F)$, and $W = P_2(F)$. Define $T : V \to W$ by
$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = bx^2 + ax + c$. Assuming that T is a linear transformation, find $N(T)$ and $R(T)$ and their dimensions, with a brief explanation.

Solution: You should prove that T is linear as an exercise. Given this, we know $N(T) = \ker(T) = \{v \in V \mid T(v) = \mathbf{0}\}$. For the image of a matrix in $M_{2\times 2}(F)$ to be the zero polynomial under T, all we need is for all entries other than the lower-right to be 0. That is,

$$N(T) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \mid d \in F \right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

In particular, $\dim(N(T)) = 1$ here. Next, we note that every polynomial in $P_2(F)$ is in the image of T, since given $b, a, c \in F$, we have $bx^2 + ax + c$ is the image of $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ under T. That is, $R(T) = P_2(F) = W$ in this case, so that T is surjective, and $\dim(R(T)) = \dim(P_2(F)) = 3$. Note that $\dim(V) = \dim(M_{2\times 2}(F)) = 4$, and we see that $\dim(N(T)) + \dim(R(T)) = \dim(V)$, as it should.