Quiz 1 **Solutions**, Math 309 (Vinroot)

(1): The set of integers $\mathbb{Z}$, with standard addition and multiplication, is a field.

TRUE **FALSE**

Solution: Elements of $\mathbb{Z}$ other than ± 1 have no multiplicative inverse, but in a field, all nonzero elements have multiplicative inverses.

(2): If F is a field, then F is a vector space over itself (with the vector operations given by the field operations of F).

TRUE FALSE

Solution: Given a field F, if we take $V = F$, then the properties of the field F translate into V being an F-vector space

(3): The integers modulo 6, $\mathbb{Z}_6$, with addition and multiplication modulo 6, form a field.

TRUE **FALSE**

Solution: Since $2 \cdot 3 = 0$ in $\mathbb{Z}_6$, but also $2 \cdot 0 = 0$, then if $\mathbb{Z}_6$ were a field, we have $2 \cdot 3 = 2 \cdot 0$. Cancellation would give us $3 = 0$, a contradiction. The same argument can be adapted to show that whenever n is composite, $\mathbb{Z}_n$ is not a field.

(4): If V is a vector space over F, with $a, b \in F$ and $x \in V$ with $x \neq \mathbf{0}$, then $ax = bx$ implies $a = b$.

TRUE FALSE

Solution: From $ax = bx$, we can add $-(bx) = -bx$ to both sides, and since $bx + (-bx) = \mathbf{0}$, we have $ax + (-bx) = \mathbf{0}$. Then $(a + (-b))x = \mathbf{0}$. If $a \neq b$, then $a + (-b) \neq 0$ since otherwise by adding b to both sides and using properties of fields we would have $a = b$. So $a + (-b)$ has some multiplicative inverse in F, say c. Multiplying the left side by c gives, using properties of vector spaces and fields, $c(a + (-b))x = (c(a + (-b)))x = 1x = x$, while the right side becomes $c\mathbf{0} = \mathbf{0}$. We thus have $x = \mathbf{0}$. This is just one proof, but others may work fine as well.

(5): The set $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x + y \geq 0, x, y \in \mathbb{Q} \right\}$, with standard vector addition and scalar multiplication, is a vector space over $\mathbb{Q}$.

TRUE **FALSE**

Solution: We have, for example, that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an element of this set. But, the only possible additive inverse of this element is $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, which is not in the set since $-1 + (-1) < 0$. Since the set does not contain additive inverses of elements, in cannot be a vector space.

(6): The set $P(\mathbb{C})$ of polynomials with complex coefficients, with standard polynomial addition and scalar multiplication, is a $\mathbb{C}$-vector space.

TRUE FALSE

Solution: If F is any field, then $P(F)$ is an important example of an F-vector space. The vector space properties are exactly how we define arithmetic of polynomials. See Example 4 in Section 1.2.

(7): Give a brief proof of the following (show all necessary steps, but no need to quote axioms used): If V is an F-vector space, with $x, y \in V$, $a \in F$, and $a \neq 0$, then $ax = ay$ implies $x = y$.

Solution: Since $a \neq 0$, then we know a has a multiplicative inverse $b \in F$, so $ba = 1$. Multiplying both sides of $ax = ay$ by b, we have, applying properties of vector spaces and fields,

$$b(ax) = b(ay), \text{ so } (ba)x = (ba)y, \text{ so } 1x = 1y,$$

from which we can conclude $x = y$.