Problem 24, Chapter 12

Math 307 - Fall 2011

**Problem:** If $R_1, R_2, \ldots, R_n$ are commutative rings with unity, show that $U(R_1 \oplus R_2 \oplus \cdots \oplus R_n) = U(R_1) \oplus U(R_2) \oplus \cdots \oplus U(R_n)$.

**Solution:** Let 1 denote the unity element in each of the rings $R_i$ (although they may be different elements in each ring, this is convenient notation). So, $(1, 1, \ldots, 1)$ is the unity element in $R_1 \oplus R_2 \oplus \cdots \oplus R_n$, since if $(a_1, a_2, \ldots, a_n) \in R_1 \oplus R_2 \oplus \cdots \oplus R_n$, then

$$(a_1, a_2, \ldots, a_n)(1, 1, \ldots, 1) = (a_1 \cdot 1, a_2 \cdot 1, \ldots, a_n \cdot 1) = (a_1, a_2, \ldots, a_n),$$

and

$$(1, 1, \ldots, 1)(a_1, a_2, \ldots, a_n) = (1 \cdot a_1, 1 \cdot a_2, \ldots, 1 \cdot a_n) = (a_1, a_2, \ldots, a_n).$$

Now, suppose that $(u_1, u_2, \ldots, u_n) \in U(R_1 \oplus R_2 \oplus \cdots \oplus R_n)$. Then, since this element is a unit in the ring $R_1 \oplus R_2 \oplus \cdots \oplus R_n$, then by definition, this means there exists an element $(x_1, x_2, \ldots, x_n) \in R_1 \oplus R_2 \oplus \cdots \oplus R_n$ such that

$$(u_1, u_2, \ldots, u_n)(x_1, x_2, \ldots, x_n) = (1, 1, \ldots, 1).$$

But we have, by definition of multiplication in the direct sum of rings,

$$(u_1, u_2, \ldots, u_n)(x_1, x_2, \ldots, x_n) = (u_1 x_1, u_2 x_2, \ldots, u_n x_n).$$

That is, we have

$$(u_1, u_2, \ldots, u_n)(x_1, x_2, \ldots, x_n) = (1, 1, \ldots, 1).$$

So, for each $i$, $1 \leq i \leq n$, we have $u_i x_i = 1 = x_i u_i$. That is, $x_i \in R_i$ and $x_i$ is a multiplicative inverse to $u_i$. Thus $u_i \in U(R_i)$. Therefore, $(u_1, u_2, \ldots, u_n) \in U(R_1) \oplus U(R_2) \oplus \cdots \oplus U(R_n)$. Therefore, we have

$$U(R_1 \oplus R_2 \oplus \cdots \oplus R_n) \subseteq U(R_1) \oplus U(R_2) \oplus \cdots \oplus U(R_n).$$
To prove the other direction of containment, suppose that \((y_1, y_2, \ldots, y_n) \in U(R_1) \oplus U(R_2) \oplus \cdots \oplus U(R_n)\). This means \(y_i \in U(R_i)\). So, there exists a \(w_i \in R_i\) such that \(y_iw_i = 1 = w_iy_i\). Consider the element \((w_1, w_2, \ldots, w_n) \in R_1 \oplus R_2 \oplus \cdots \oplus R_n\). Then we have

\[
(y_1, y_2, \ldots, y_n)(w_1, w_2, \ldots, w_n) = (y_1w_1, y_2w_2, \ldots, y_nw_n) = (1, 1, \ldots, 1),
\]

and

\[
(w_1, w_2, \ldots, w_n)(y_1, y_2, \ldots, y_n) = (w_1y_1, w_2y_2, \ldots, w_ny_n) = (1, 1, \ldots, 1).
\]

Thus, \((w_1, w_2, \ldots, w_n)\) is a multiplicative inverse for the element \((y_1, y_2, \ldots, y_n)\) in \(R_1 \oplus R_2 \oplus \cdots \oplus R_n\). In particular, we have \((y_1, y_2, \ldots, y_n) \in U(R_1) \oplus U(R_2) \oplus \cdots \oplus U(R_n)\). Therefore,

\[
U(R_1) \oplus U(R_2) \oplus \cdots \oplus U(R_n) \subseteq U(R_1 \oplus R_2 \oplus \cdots \oplus R_n).
\]

Thus, we have now shown \(U(R_1 \oplus R_2 \oplus \cdots \oplus R_n) = U(R_1) \oplus U(R_2) \oplus \cdots \oplus U(R_n)\).