This is an optional homework. You may turn it in any time before the Final Exam, and it will go towards your homework/quiz grade for the course. There is no penalty for not doing these problems.

Given any group $G$, and a fixed element $a \in G$, we define, like on the first midterm, the rationalizer of $a$, denoted $R(a)$, to be the set

$$R(a) = \{ g \in G \mid gag^{-1} = a^k \text{ for some } k \in \mathbb{Z} \}.$$ 

The purpose of this set of problems is to establish that, if we assume that $|a|$ is finite, then we indeed have $R(a) \leq G$, however, if $|a|$ is infinite, there are examples when $R(a)$ is not a subgroup of $G$.

1. Prove (without any assumptions on $|a|$) that $R(a)$ is nonempty, and is closed under the operation of $G$. That is, if $x, y \in R(a)$, then $xy \in R(a)$. Most of you did this on the midterm, but I am including it here for completeness and review.

2. Do problem 5 on page 92 on the text. That is, if $G$ is any group, and $x, a \in G$, then $|xax^{-1}| = |a|$. Be sure to prove the statement for $a$ having finite or infinite order. (Some of you did this problem as a review problem for the first midterm, but you can do it again here and turn it in for credit).

3. Now assume that $|a| = n$ is finite, and assume $g \in R(a)$, so $gag^{-1} = a^k$ for some $k \in \mathbb{Z}$. Prove that we must have gcd($n, k$) = 1. Use Problem 2 above, along with results from Chapter 4 of the text.

4. Continue to assume that $|a| = n$ is finite. Recall that you have shown (on the first Homework) that if gcd($n, k$) = 1, then there exists an integer $m$ such that $km \equiv 1 \pmod{n}$. Use this to prove that if $g \in R(a)$, then $g^{-1} \in R(a)$. This proves that if $|a|$ is finite, then $R(a) \leq G$.

5. Find an example of a group $G$, and an element $a \in G$ such that $a$ has infinite order (so $G$ is infinite necessarily), and such that there is some $g \in R(a)$, but $g^{-1} \notin R(a)$. Hint: Consider letting $a$ be a specific element of $\text{GL}(2, \mathbb{R})$ which we have seen before in Homework, which we know has infinite order.