Math 307, Extra homework (not required)

This is an optional homework. You may turn it in any time before the Final Exam, and it will go towards your homework/quiz grade for the course. There is no penalty for not doing these problems.

Given any group G, and a fixed element $a \in G$, we define, like on the first midterm, the *rationalizer* of a, denoted $R(a)$, to be the set

$$R(a) = \{g \in G \mid gag^{-1} = a^k \text{ for some } k \in \mathbb{Z}\}.$$

The purpose of this set of problems is to establish that, if we assume that $|a|$ is finite, then we indeed have $R(a) \leq G$, however, if $|a|$ is infinite, there are examples when $R(a)$ is not a subgroup of G.

1. Prove (without any assumptions on $|a|$) that $R(a)$ is nonempty, and is closed under the operation of G. That is, if $x, y \in R(a)$, then $xy \in R(a)$. Most of you did this on the midterm, but I am including it here for completeness and review.

2. Do problem 5 on page 92 on the text. That is, if G is any group, and $x, a \in G$, then $|xax^{-1}| = |a|$. Be sure to prove the statement for a having finite or infinite order. (Some of you did this problem as a review problem for the first midterm, but you can do it again here and turn it in for credit).

3. Now assume that $|a| = n$ is finite, and assume $g \in R(a)$, so $gag^{-1} = a^k$ for some $k \in \mathbb{Z}$. Prove that we must have $\gcd(n, k) = 1$. Use Problem **2** above, along with results from Chapter 4 of the text.

4. Continue to assume that $|a| = n$ is finite. Recall that you have shown (on the first Homework) that if $\gcd(n, k) = 1$, then there exists an integer m such that $km \equiv 1 (\text{mod } n)$. Use this to prove that if $g \in R(a)$, then $g^{-1} \in R(a)$. This proves that if $|a|$ is finite, then $R(a) \leq G$.

5. Find an example of a group G, and an element $a \in G$ such that a has infinite order (so G is infinite necessarily), and such that there is some $g \in R(a)$, but $g^{-1} \notin R(a)$. Hint: Consider letting a be a specific element of $\mathrm{GL}(2, \mathbb{R})$ which we have seen before in Homework, which we know has infinite order.