Math 302 - Ordinary Differential Equations (Vinroot)

Extra Review Problems for Midterm 2

The following are review problems for some material we covered in class but is not in the book. The solutions are on the second page.

1. Write each of the following complex numbers in the form $a + bi$.

(a): $(\sqrt{2} + i)e^{-i\pi/4}$ **(b):** $(1 + i)(e^{3i\pi/2} + 2e^{5i\pi})$

For each of the following, find the general solution to the homogeneous linear differential equation with the given characteristic polynomial, and give the order of the associated differential equation:

2. $(\lambda - 1)^2(\lambda + 2)^2(\lambda + 4)$

3. $(\lambda^2 + 4)^2(\lambda - 1)(\lambda + 3)^2$

4. $(\lambda - 3)^3(\lambda^2 + 3\lambda + 17/2)$

Solutions

1. We use Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and then multiply and simplify.
(a): $e^{-\pi/4} = \cos(-\pi/4) + i\sin(-\pi/4) = \sqrt{2}/2 - i\sqrt{2}/2$. So now

$$(\sqrt{2}+i)e^{-i\pi/4} = (\sqrt{2}+i)(\sqrt{2}/2-i\sqrt{2}/2) = 2/2+i\sqrt{2}/2-i2/2+\sqrt{2}/2 = (1+\sqrt{2}/2)+(-1+\sqrt{2}/2)i.$$

So $(\sqrt{2}+i)e^{-i\pi/4} = (1+\sqrt{2}/2) + (-1+\sqrt{2}/2)i$.
(b): $e^{3i\pi/2} = \cos(3\pi/2) + i\sin(3\pi/2) = -i$ and $e^{5i\pi} = \cos(5\pi) + i\sin(5\pi) = -1$. Now we have

$$(1+i)(e^{3i\pi/2} + 2e^{5i\pi}) = (1+i)(-i-2) = -i-2-2i+(i)(-i) = -3i-2+1 = -1-3i$$

So $(1+i)(e^{3i\pi/2} + 2e^{5i\pi}) = -1-3i$.

In each of the following, we give the characteristic roots and their multiplicities, then find the order of the associated differential equation, and then the general solution.

2. The roots of the characteristic equation are $\lambda = 1$ with multiplicity 2, $\lambda = -2$ with multiplicity 2, and $\lambda = -4$ with multiplicity 1 (with no repetition). Since the total number of roots (including repetition) is 5, the differential equation has order 5. The general solution is given by

$$x(t) = C_1e^t + C_2te^t + C_3e^{-2t} + C_4te^{-2t} + C_5e^{-4t}.$$

3. Since $\lambda^2 + 4 = (\lambda - 2i)(\lambda + 2i)$, we have the characteristic equation has roots $\lambda = 2i, -2i$, which are complex conjugates, with multiplicity 2 each, $\lambda = 1$ with multiplcity 1 (no repetition), and $\lambda = -3$ with multiplicity 2. The total number of roots with multiplicity is 7, so the degree of the associated differential equation is degree 7. The general solution is given by

$$x(t) = C_1\cos(2t) + C_2\sin(2t) + C_3t\cos(2t) + C_4t\sin(2t) + C_5e^t + C_6e^{-3t} + C_7te^{-3t}.$$

4. We first find the roots of $\lambda^2 + 3\lambda + 17/2 = 0$ using the quadratic formula, and we get

$$\lambda = \frac{-3 \pm \sqrt{9 - 4(17/2)}}{2} = \frac{-3 \pm \sqrt{-25}}{2} = -\frac{3}{2} \pm \frac{5}{2}i.$$

Now all of the roots are the complex conjugate roots $\lambda = -3/2 \pm (5/2)i$ each with multiplicity 1 (no repetition), and $\lambda = 3$ with multiplicity 3. Since there are 5 total roots, counting multiplicity, the associated differential equation has degree 5. The general solution is given by

$$x(t) = C_1e^{(-3/2)t}\cos((5/2)t) + C_2e^{(-3/2)t}\sin((5/2)t) + C_3e^{3t} + C_4te^{3t} + C_5t^2e^{3t}.$$