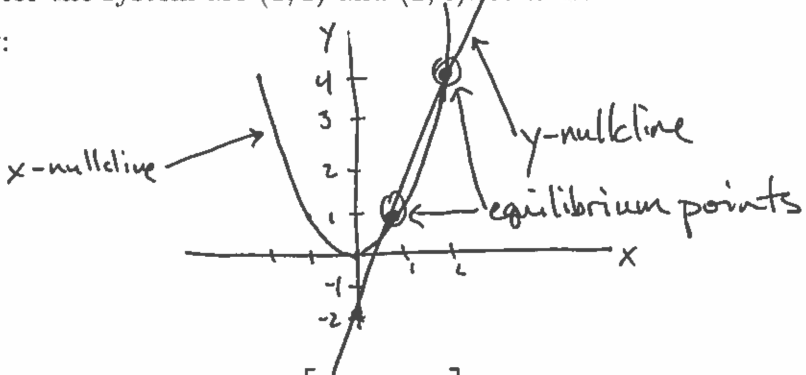
Quiz 3 **Solutions**, Math 302 (Vinroot)

(1): Find the x- and y-nullclines and the equilibrium points, and sketch these in the xy-plane, of the following system of differential equations:

$$x' = x^2 - y$$
$$y' = 3x - 2 - y$$

Solution: The x-nullcline occurs when $x' = 0$, so when $x^2 - y = 0$, and so this is exactly the parabola $y = x^2$. The y-nullcline occurs when $y' = 0$, so when $3x - 2 - y = 0$. This is given by the line $y = 3x - 2$. The equilibrium points occur when both $x' = 0$ and $y' = 0$, so we need $y = x^2$ and $y = 3x - 2$, which gives $x^2 = 3x - 2$, or $x^2 - 3x + 2 = 0$. This yields $(x-2)(x-1) = 0$, so $x = 2$ or $x = 1$. Plugging these back in for y-values gives $y = 4$ when $x = 2$ and $y = 1$ when $x = 1$. Thus the two equilibrium points for the system are $(1, 1)$ and $(2, 4)$. A sketch of the nullclines and equilibrium points is given below:



(2): Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} -3 & -1 \\ 1 & -2 \end{bmatrix}$. Determine what type of equilibrium point the origin is for this system, and sketch the solutions of the system in the phase plane.

Solution: We can determine the type of equilibrium point at the origin by either finding the eigenvalues of A directly, or by finding the trace and determinant of A. In the first method, we have the characteristic equation is given by $\det(A - \lambda I) = (-3-\lambda)(-2-\lambda) + 1 = \lambda^2 + 5\lambda + 7 = 0$. The eigenvalues are then given by $(-5 \pm \sqrt{-3})/2$. So, we have complex eigenvalues with negative real part, which means the equilibrium point is a spiral sink. Using trace and determinant, we can also see that $T = \mathrm{tr}(A) = -5 < 0$, $D = \det(A) = 7 > 0$, and $T^2 - 4D = -3 < 0$, also indicating that there is a spiral sink. To sketch a graph in the phase plane, we find the tangent vectors at $(1, 0)^\top$ and $(0, 1)^\top$, which are also given by the first and second columns of A. A sketch is given below.