Quiz 2 **Solutions**, Math 302 (Vinroot)

(i): Find a particular solution to the differential equation $y'' + 4y' + 3y = 3t + 10$.

Solution: Since the forcing term is a linear polynomial, we guess a particular solution is of the form $y_p(t) = at + b$. Then $y_p'(t) = a$ and $y_p''(t) = 0$. Substituting these into the differential equation, we have

$$y_p'' + 4y_p' + 3y_p = 4a + 3at + 3b = 3t + 10.$$

Equating the coefficient of the linear terms (the t-term) on each side, we have $3a = 3$, and so $a = 1$. Equating the constant terms of each side gives $4a + 3b = 10$, and since $a = 1$, we have $4 + 3b = 10$, so $3b = 6$, and $b = 2$. Thus our particular solution is $y_p(t) = t + 2$.

(ii): Find the general solution to the differential equation from part **(i)**.

Solution: The general solution is of the form $y(t) = y_h + y_p$, where y_p is the particular solution from part **(i)**, and y_h is the general solution to the associated homogeneous differential equation. The homogeneous equation is $y'' + 4y' + 3y = 0$, which has characteristic equation $\lambda^2 + 4\lambda + 3 = 0$. Since $\lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1)$, we have the characteristic roots are $\lambda_1 = -3, \lambda_2 = -1$. The homogeneous general solution is then given by $y_h = C_1 e^{-3t} + C_2 e^{-t}$ for arbitrary constants C_1, C_2. The general solution to the original inhomogeneous differential equation is then

$$y(t) = y_h + y_p = C_1 e^{-3t} + C_2 e^{-t} + t + 2.$$

(iii): Using your solution from **(ii)**, find a solution to the initial value problem for the differential equation with $y(0) = 1$, $y'(0) = 5$.

Solution: If $y(t)$ is the function in the solution to **(ii)**, then we have $y'(t) = -3C_1 e^{-3t} - C_2 e^{-t} + 1$. The initial values then yield the equations

$$y(0) = C_1 e^{-3(0)} + C_2 e^{-0} + 0 + 2 = 1, \quad y'(0) = -3C_1 e^{-3(0)} - C_2 e^{-0} + 1 = 5.$$

These two equations give us the linear system

$$\begin{array}{rcrcr} C_1 & + & C_2 & = & -1 \\ -3C_1 & - & C_2 & = & 4. \end{array}$$

Adding the two equations above gives $-2C_1 = 3$, so $C_1 = -3/2$. Substituting this into the first equation gives $-3/2 + C_2 = -1$, so $C_2 = 1/2$. The solution is thus $y(t) = -\frac{3}{2}e^{-3t} + \frac{1}{2}e^{-t} + t + 2$.