Quiz 1, Math 302 (Vinroot) **Solutions**

(i): Show that the following differential equation is **not** exact:

$$\left(\frac{1}{x}\sin(y) + 2e^y\right) + \left(\cos(y) + xe^y + \frac{2y}{x}\right)\frac{dy}{dx} = 0$$

Solution: We have $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{x}\sin(y) + 2e^y\right) = \frac{1}{x}\cos(y) + 2e^y$. Then we have $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\cos(y) + xe^y + \frac{2y}{x}\right) = e^y - \frac{2y}{x^2}$. Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, then the equation is not exact.

(ii): Multiply the equation in **(i)** by the function $\mu(x) = x$, and show the resulting differential equation is exact.

Solution: After multiplying, the new equation is

$$(\sin(y) + 2xe^y) + \left(x\cos(y) + x^2e^y + 2y\right)\frac{dy}{dx}.$$

Now we have $\frac{\partial}{\partial y}(\mu P) = \cos(y) + 2xe^y$, and $\frac{\partial}{\partial x}(\mu Q) = \cos(y) + 2xe^y$. Since $\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q)$, then this equation is exact.

(iii): Find a general solution to the exact differential equation from **(ii)**.

Solution: We want the function F such that $\frac{\partial F}{\partial x} = \mu P = \sin(y) + 2xe^y$. Integrating both sides with respect to x, we have $F(x, y) = x\sin(y) + x^2e^y + \phi(y)$, where $\phi(y)$ is the integration constant, so a function in terms of only y. We can now take the derivative of this function F with respect to y, and take advantage of the fact that $\frac{\partial F}{\partial y} = \mu Q$, and get

$$x\cos(y) + x^2e^y + \phi'(y) = x\cos(y) + x^2e^y + 2y.$$

Then we must have $\phi'(y) = 2y$, and so we can take $\phi(y) = y^2$. Now our general solution is

$$x\sin(y) + x^2e^y + y^2 = C,$$

for an arbitrary constant C.