These are optional problems for Math 215, which may be turned in on Wed., Apr. 12, with Homework #9.

1. Let $S : \mathbb{R}^3 \to \mathbb{R}^4$ and $T : \mathbb{R}^4 \to \mathbb{R}^2$ be linear transformations given by

$$S\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 + 2x_2 \\ x_1 + 3x_2 \\ x_2 - x_3 \\ x_1 + x_2 + x_3 \end{pmatrix} \quad \text{and}$$

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} x_1 - 2x_2 + x_3 \\ 2x_1 + x_4 \end{pmatrix}.$$

Find a matrix A such that $(T \circ S)(\mathbf{v}) = A\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^3$.

2. Find bases for $\mathrm{Im}(T \circ S)$ and $\mathrm{Ker}(T \circ S)$, where T and S are as above. Is $T \circ S$ injective? Is $T \circ S$ surjective?

3 a.) Show that

$$\mathcal{B} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and } \mathcal{B}' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

are both bases of $\mathbb{R}^3$. Find the change of basis matrix for the change $\mathcal{B} \to \mathcal{B}'$. That is, find a matrix C such that $C[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{B}'}$ for any $\mathbf{v} \in \mathbb{R}^3$.

b.) Let

$$\mathbf{v} = \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix}_{\mathcal{B}}.$$

Write $\mathbf{v}$ as a coordinate vector using the basis $\mathcal{B}'$.

4. Define a linear transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$, in terms of the standard ordered basis of $\mathbb{R}^3$, by

$$F\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 3x_1 - 2x_2 \\ x_1 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}.$$

Find the matrix associated with F in terms of the basis $\mathcal{B}$ in problem **3** above. That is, find a matrix A such that $[F(\mathbf{v})]_{\mathcal{B}} = A[\mathbf{v}]_{\mathcal{B}}$ for any $\mathbf{v} \in \mathbb{R}^3$.