(a): If $V$ is a vector space, give the conditions which define a subset $W$ to be a subspace of $V$.

Solution: $W$ is a subspace of $V$ if: (i) the zero vector $0$ is in $W$, (ii) whenever $u$ and $v$ are in $W$, then $u + v$ is in $W$, and (iii) whenever $u$ is in $W$ and $c$ is a scalar in $\mathbb{R}$, then $cu$ is in $W$.

(b): Consider the following collection of vectors in $\mathbb{R}^2$:

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_2 \geq 0 \right\},$$

so that $H$ is the collection of all vectors in the upper-half of the plane, including the $x_1$-axis. Show that $H$ satisfies two of the three conditions listed in (a), but does not satisfy one of them, and conclude that $H$ is not a subspace of $\mathbb{R}^2$.

Solution: First, $0$ is in $H$ since $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has second coordinate non-negative. Second, if $u = \begin{bmatrix} x \\ y \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$ are in $H$, so that $y \geq 0$ and $b \geq 0$, then

$$u + v = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \end{bmatrix}$$

is also in $H$, since $y + b \geq 0$. So, the first two conditions listed in (a) hold. However, if we consider, say $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is in $H$ since its second coordinate is $1 \geq 0$, and we take $c = -1$, then

$$cw = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

which is not an element of $H$. So, the third condition listed in (a) does not hold for $H$, and so $H$ is not a subspace of $\mathbb{R}^2$.  