Quiz 7 **Solutions**, Math 112, Section 1 (Vinroot)

Solve each of the following. Show or explain all steps clearly for full credit.

1. Find the exact value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n - 3^{n-1}}{5^{n-1}}$.

Solution: The series is the difference of two convergent geometric series (with ratios 2/5 and 3/5), and we may compute as follows:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^n - 3^{n-1}}{5^{n-1}} &= \sum_{n=1}^{\infty} \frac{2^n}{5^{n-1}} - \sum_{n=1}^{\infty} \frac{3^{n-1}}{5^{n-1}} \\ &= \sum_{n=1}^{\infty} 2\left(\frac{2}{5}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^{n-1} \\ &= \frac{2}{1-\frac{2}{5}} - \frac{1}{1-\frac{3}{5}} \\ &= \frac{2}{3/5} - \frac{1}{2/5} = \frac{10}{3} - \frac{5}{2} = \frac{5}{6}.\end{aligned}$$

2. Use the integral test (and check that the conditions hold) to determine if the following series converges or diverges: $\displaystyle\sum_{n=1}^{\infty} 2ne^{-n^2}$.

Solution: The function of interest is $f(x) = 2xe^{-x^2}$, which is continuous (for all x) and positive for $x > 0$. To check if $f(x)$ is decreasing, we compute $f'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2e^{-x^2}(1-2x^2)$. Then we see that

$$f'(x) = 2e^{-x^2}(1-2x^2) < 0 \quad \text{when} \quad 1 - 2x^2 < 0,$$

which occurs when $x > \sqrt{1/2}$ (or $x < -\sqrt{1/2}$, but we don't care about that part). In particular, $f'(x) < 0$ when $x \geq 1$, so $f(x)$ is decreasing when $x \geq 1$. So the Integral Test applies here. We compute the improper integral

$$\int_1^{\infty} 2xe^{-x^2}\,dx = \lim_{b\to\infty} \int_1^b 2xe^{-x^2}\,dx.$$

We first compute the needed antiderivative, where we let $u = x^2$, so $du = 2x\,dx$ in the integral. So we have $\int 2xe^{-x^2}\,dx = \int e^{-u}\,du = -e^{-u} + C = -e^{-x^2} + C$. Now we have

$$\int_1^{\infty} 2xe^{-x^2}\,dx = \lim_{b\to\infty} \int_1^b 2xe^{-x^2}\,dx = \lim_{b\to\infty} \left[-e^{-x^2}\right]_1^b = \lim_{b\to\infty} (-e^{-b^2} + e^{-1}) = e^{-1},$$

since $\displaystyle\lim_{b\to\infty}(-e^{-b^2}) = 0$. Since the integral converges, then by the Integral Test the series $\displaystyle\sum_{n=1}^{\infty} 2ne^{-n^2}$ also **converges**.