Quiz 8 **Solutions**, Math 112, Section 2 (Vinroot)

Determine whether each of the following series converges or diverges. State the test you are using and why it works.

(a): $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(n+1)!}$.

Solution: The presence of the $(n+1)!$ should point you towards using the Ratio Test. We have $a_n = 4^n/(n+1)!$ and $a_{n+1} = 4^{n+1}/(n+1+1)! = 4^{n+1}/(n+2)!$. Then we compute:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{4^{n+1}/(n+2)!}{4^n/(n+1)!}\right| = \frac{4^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{4^n} = 4 \cdot \frac{(n+1)!}{(n+2)!} = \frac{4}{n+2}.$$

So, we now find the following limit:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{4}{n+2} = 0.$$

By the Ratio Test, since $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, then the series absolutely converges, and so converges.

(b): $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n+5^n}$.

Solution: We note that when n gets large, then we should expect the 5^n in the denominator to guide the behavior of the terms. That is, we should think of comparing this to a geometric series. Since $n + 5^n > 5^n$ for any n, we have

$$\frac{2}{n+5^n} < \frac{2}{5^n} = \frac{2}{5}\left(\frac{1}{5}\right)^{n-1}.$$

Now, $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5}\left(\frac{1}{5}\right)^{n-1}$ is a geometric series with $r = 1/5$, so $|r| < 1$ and the series converges. Since all terms of both series are positive, we may use the Comparison Test and conclude that the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n+5^n}$ also converges.